De-Preferential Attachment Random Graphs

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(Joint work with Subhabrata Sen)

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   • A Toy Model for a Ecosystem/Food-Chain
   • De-Preferential Attachment Model

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   • Models for $m = 1$
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Introduction

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Main Results

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- Linear Case
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Open Problems
Suppose we model an *ecosystem/food-chain* starting with one species where every new species which arrives later is a predator to the existing ones.
A Simple Predator-Prey Ecosystem

- Suppose we model an ecosystem/food-chain starting with one species where every new species which arrives later is a predator to the existing ones.

- It is natural to believe that given a choice, a new predator will like to choose its food from the existing species which are not eaten by many.
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- Suppose we model an ecosystem/food-chain starting with one species where every new species which arrives later is a predator to the existing ones.

- It is natural to believe that given a choice, a new predator will like to choose its food from the existing species which are not eaten by many.

- In other words, a new predator will have less incentive or less preference to choose its prey from the existing species which have many predators.
If we now define a graph with vertices as the species and the edges/links between vertices through the predator – prey relation, then such a graph should be modeled by...
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- This is opposite of the usual “rich get richer model”, also known as, preferential attachment model [Barabási and Albert (1999)].

- We will call any such model a de-preferential attachment model.

- Our goal will be to study such a model rigorously and compare its properties with the preferential attachment model.
Like in the preferential attachment model we will start with an initial graph $G_1$ with possibly just one vertex.
De-Preferential Attachment Random Graphs

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At every (discrete) time $n + 1 \geq 2$, we will add one new vertex, say $v_{n+1}$ to the existing graph, say $G_n$, by letting it to join to the existing vertices $\{v_1, v_2, \ldots, v_n\}$. 
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The mechanism in which $v_{n+1}$ joins to the existing vertices will be random but with preference for vertices with lesser degree.
To make things rigorous we need to fix couple of issues:

- How many existing vertices are going to be joined with a new vertex?
  - We will initially consider the case when each new vertex will join only to one existing vertex. Note that this will lead to a tree (good for modeling food-chain network).
  - We will also consider the case when each new vertex is going to join to $m \geq 1$ existing vertices where $m$ will be a fixed positive integer. In this case we can have multiple edges and self-loops depending on the mechanism in which the $m$ new links will be formed. Also there can be formation of cycles. None of these are good for a food-chain network, as $A$ multiplies eats $B$ or $A$ eats itself or even $A$ eats $B$ which eats $C$ but $C$ eats $A$ are not suitable for such a network.
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- We will write \(d_i(n + 1, k)\), for \(k = 0, \ldots, m\), to denote the degree of the vertex \(v_i\), \(i = 1, \ldots, n\), after \(k\) half-edges of \(v_{n+1}\) have been attached.
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- We will write \(d_i(n+1, 0) = d_i(n)\) for any \(1 \leq i \leq n\) and note \(d_n(n) = m\).
- Let \(\{\mathcal{F}_{n,k} \mid 0 \leq k \leq m - 1, n \geq 1\}\) be the natural filtration of the random attachments.
- If \(m = 1\) then we will simply write the natural filtration as \(\{\mathcal{F}_n\}_{n \geq 1}\).
Models for $m = 1$

- We start with $G_1$ which consists of one vertex with one unattached *half-edge*. So $d_1(1) = 1$. 

\[
\begin{align*}
\text{Linear De-Preferential Model:} \\
P(v_{n+1} \rightarrow v_i | F_n) \propto (2^n - 1 - d_i(n)) \\
\text{Inverse De-Preferential Model:} \\
P(v_{n+1} \rightarrow v_i | F_n) \propto \frac{1}{d_i(n)} \\
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  \]

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  that is,

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  P\left(v_{n+1} \rightarrow v_i \mid F_n\right) = \frac{C}{d_i(n)},
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  where $C = D_n = \sum_{i=1}^{n} \frac{1}{d_i(n)}$. 

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  where $C_n^{-1} = D_n = \sum_{i=1}^{n} \frac{1}{d_i(n)}$.  

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Models for $m > 1$

- We start with $G_1$ which consists of one vertex with $m$ unattached half-edges. So $d_1(1) = m$. 
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- At time $n + 1$, the new vertex $v_{n+1}$ comes with $m$ half-edges, namely, $e_{n+1,1}, e_{n+1,2}, \ldots, e_{n+1,m}$, which are joined **sequentially by updating the degrees of the existing vertices** and are not allowed to join to $v_{n+1}$. 
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- This prevents the formation of the self-loops.

- We still have the possibility of having multiple edges between two vertices.
Models for $m > 1$

- **Linear De-Preferential Model:**

\[
P\left( e_{n+1,k+1} = \{v_j, v_{n+1}\} \mid \mathcal{F}_{n+1,k} \right) = \frac{1}{n-1} \left( 1 - \frac{d_j(n+1, k)}{k + (2n-1)m} \right)
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**Models for $m > 1$**

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- **Inverse De-Preferential Model:**

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  P \left( e_{n+1,k+1} = \{ v_j, v_{n+1} \} \mid \mathcal{F}_{n+1,k} \right) = C_{n+1,k} \frac{1}{d_j(n+1, k)}
  \]

  where \( C_{n+1,k}^{-1} =: D_{n+1,k} = \sum_{j=1}^{n} \frac{1}{d_j(n+1, k)}. \)
A somewhat similar, in fact a bit more general model was considered by Sevim and Rikvold (2006, 2008).
Some Earlier Work

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- They obtain by some intuitive arguments (not quite rigorous) the asymptotic degree distribution and validated their claims by simulation results.
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They obtain by some intuitive arguments (not quite rigorous) the asymptotic degree distribution and validated their claims by simulation results.

Our results support their observations.
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Main Results: Linear Case with $m = 1$

**Theorem 1 (WLLN for fixed vertex degree)**

Fix a vertex $i \geq 1$ then

$$\frac{d_i(n)}{\log n} \overset{P}{\longrightarrow} 1.$$
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$$\frac{d_i(n)}{\log n} \xrightarrow{P} 1.$$  

**Theorem 2 (CLT for fixed vertex degree)**
Fix a vertex $i \geq 1$ then

$$\frac{d_i(n) - \log n}{\sqrt{\log n}} \xrightarrow{d} \text{Normal} (0, 1).$$
Main Results: Linear Case with $m = 1$

**Theorem 3 (Asymptotic degree distribution)**

Let $P_k(n)$ be the proportion of vertices in $G_n$ with degree $k \geq 1$. Then for any $k \geq 1$,

$$P_k(n) \xrightarrow{} \frac{1}{2^k} \text{ a.s.}$$
Main Results: Linear Case with $m = 1$

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$$P_k(n) \to \frac{1}{2^k} \text{ a.s.}$$

**Remark:** The asymptotic degree distribution of $G_n$ is Geometric $(\frac{1}{2})$ which has mean 2, mode 1 and exponential tail.
Main Results: Linear Case with $m = 1$

**Theorem 4 (Asymptotic degree distribution of the chosen vertex)**

Let $U_{n+1}$ be the (random) selected vertex from $\{v_1, v_2, \ldots, v_n\}$ where the new vertex $v_{n+1}$ connects. Then for any $k \geq 1$,

$$
P(\text{degree}_{G_n}(U_{n+1}) = k) \rightarrow \frac{1}{2^k}.
$$
Main Results: Linear Case with $m \geq 1$

**Theorem 5 (WLLN for fixed vertex degree)**

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Fix a vertex $i \geq 1$ then

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**Theorem 6 (CLT for fixed vertex degree)**

Fix a vertex $i \geq 1$ then

$$\frac{d_i(n) - m \log n}{\sqrt{m \log n}} \xrightarrow{d} \text{Normal (0, 1)}.$$
Main Results: Inverse Case with $m = 1$

Theorem 7 (SLLN for fixed vertex degree)

Fix a vertex $i \geq 1$ then

$$\frac{d_i(n)}{\sqrt{\log n}} \rightarrow \sqrt{\frac{2}{\lambda^*}} \quad \text{a.s.,}$$

where $\lambda^* > 0$ is the unique positive solution of the equation

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{1}{1 + i\lambda^*} = 1.$$
Main Results: Inverse Case with $m = 1$

**Theorem 8 (Asymptotic degree distribution)**

Let $P_k(n)$ be the proportion of vertices in $G_n$ with degree $k \geq 1$. Then for any $k \geq 1$,

$$P_k(n) \rightarrow k\lambda^* \prod_{i=1}^{k} \frac{1}{i\lambda^* + 1} \text{ a.s.}$$

Remark: The asymptotic degree distribution of $G_n$ has mean 2, mode 1 and thin tail.
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Let $U_{n+1}$ be the (random) selected vertex from $\{v_1, v_2, \ldots, v_n\}$ where the new vertex $v_{n+1}$ connects. Then for any $k \geq 1$,

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P \left( \text{degree}_{G_n} (U_{n+1}) = k \right) \rightarrow \prod_{i=1}^{k} \frac{1}{i\lambda^* + 1}.$$

Main Results: Inverse Case with $m > 1$

Theorem 10 ("WLLN" for fixed vertex degree)

There exist constants $0 < C_1 < C_2 < \infty$ such that for any fixed vertex $i$,

$$
P \left( C_1 \leq \frac{d_i(n)}{m\sqrt{\log n}} \leq C_2 \right) \to 1,
$$

as $n \to \infty$. 

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Techniques Used for the Linear De-Preferential Case

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- In this case $m = 1$ and $m > 1$ are not much different.

- For the CLTs we use martingale CLT.
Techniques Used for the Inverse De-Preferential Case

- We used two different *embeddings/couplings* for this case.
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- One type of embedding for $m = 1$ and a different embedding for $m > 1$. 
Techniques Used for the Inverse De-Preferential with \( m = 1 \)

- We consider a continuous time age dependent branching process and keep all the statistics, that is, entire growing tree structure.
Techniques Used for the Inverse De-Preferential with $m = 1$

- We consider a continuous time age dependent branching process and keep all the statistics, that is, entire growing tree structure.

- Formally, let $\mathcal{G}$ be the set of all finite rooted tree. We consider a continuous time process $\{\Upsilon(t) : t \geq 0\}$ of randomly growing trees on $\mathcal{G}$. 
Techniques Used for the Inverse De- Preferential with $m = 1$

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- This process is an example of a Crump-Mode-Jagers (CMJ) branching process [Crump and Mode (1968) and Jagers (1969)].
Techniques Used for the Inverse De- Preferential with $m = 1$

**Embedding Theorem for $m = 1$**

Starting with $\tau_1 = 0$ consider the following sequence of stopping times

$$\tau_n := \inf\{t \geq \tau_{n-1} \mid |\mathcal{Y}(t)| = n\}.$$  

For $m = 1$, the sequence of random graphs $\{G_n\}_{n=1}^\infty$ have the same distribution as the sequence of random trees $\{\mathcal{Y}(\tau_n)\}_{n=1}^\infty$. 

Remarks:
(i) This is immediate from the construction of the CMJ branching process.
(ii) For studying preferential attachment model with non-linear weights a similar observation was made by Rudas and Tóth (2007).
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Techniques Used in the Proofs  
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Techniques Used for the Inverse De-Preferential with $m = 1$

- Let $\hat{\rho}(\lambda)$ be the expected Laplace transform of the pure birth process $(\xi(t))_{t \geq 0}$. 

$\hat{\rho}(\lambda) = \sum_{n=1}^{\infty} n \prod_{i=1}^{n} \frac{1}{i \lambda + 1}$.

Thus $\hat{\rho}(\lambda) = 1$ has a unique positive solution which we denote by $\lambda^* > 0$. $\lambda^*$ is called the Malthusian parameter for the (supercritical) CMJ process.
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**Theorem A of Nerman (1961)**

Suppose $\{\Upsilon(t) : t \geq 0\}$ is a (supercritical) CMJ process with Multhusian parameter $\lambda^*$ and let $\phi : G \rightarrow \mathbb{R}$ be bounded function. Then the following limit holds almost surely

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\lim_{t \to \infty} \frac{1}{|\Upsilon(t)|} \sum_{x \in \Upsilon(t)} \phi(\Upsilon(t)_{\downarrow x}) = \lambda^* \int_0^\infty \exp\{-\lambda^* t\} E(\phi(\Upsilon(t))) dt,
$$

where for a tree $T \in G$ and a vertex $x \in T$ we define $T_{\downarrow x}$ as the sub-tree rooted at $x$ consisting of all the descendants of $x$. 

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- For $i \geq 1$, let $(Z_i(t))_{t \geq 0}$ be i.i.d. copies of the pure birth process $(Z(t))_{t \geq 0}$.
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- We recursively define the following stopping times starting with $\tau_1 = 0$, 
  
  \[
  \begin{align*}
  \tau_2 &:= \inf \left\{ t \geq 0 \mid Z_1(t) - m = m \right\} \\
  \tau_3 &:= \inf \left\{ t \geq \tau_2 \mid Z_1(t) + Z_2(t - \tau_2) - 2m = m \right\} \\
  &\vdots \\
  \tau_{n+1} &:= \inf \left\{ t \geq \tau_n \mid Z_1(t) + Z_2(t - \tau_2) + \cdots + Z_n(t - \tau_n) - nm = m \right\}
  \end{align*}
  \]
Techniques Used for the Inverse De-Preferential with $m > 1$

**Embedding Theorem for $m > 1$**

For $m \geq 1$, the two sequence of random variables, namely, $\{ (d_i(n))_{i=1}^n \mid n \geq 1 \}$ and $\{ (Z_i(\tau_n - \tau_i))_{i=1}^n \mid n \geq 1 \}$ has the same distribution.
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WLLN for the Pure Birth Process

Let $\{Z(t) : t \geq 0\}$ be a pure birth process with $P(Z(0) = m) = 1$ and birth rates $\lambda_i = \frac{1}{i}$. Then

$$\frac{Z(t)}{\sqrt{t}} \xrightarrow{P} \sqrt{2}.$$
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2 Model Description
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But it is necessary assumption for the results on inverse case which we prove using the embedding to CMJ branching process.
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For $m = 1$ case one should remove the dependency on the initial configuration but it seems it is a technically very difficult problem!
Thank You