A Las Vegas algorithm to solve Elliptic Curve Discrete Logarithm Problem

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Indocrypt 2018
December 11, 2018
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The discrete logarithm problem

We assume that $E(\mathbb{F}_q)$ be a group of prime order $p$. Let $P$ and $Q$ be two non-identity points, such that, $Q = mP$ for $1 \leq m < p$. The discrete logarithm problem is to find the $m$. 

Attacks on DLP

There are two kinds of attack on DLP. One is generic attack. Other kind of attacks are the index-calculus kind of attacks on DLP.

Non-generic attack

Our attack is of the second kind.
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A theorem

Definition
An elliptic curve $E$ over a field $\mathbb{F}_q$ is a non-singular plane curve of degree 3 together with a point $O$.

Theorem
Let $E$ be an elliptic curve over $\mathbb{F}_q$ and $P_1, P_2, \ldots, P_k$ be points on that curve, where $k = 3n'$ for some positive integer $n'$. Then $\sum_{i=1}^{k} P_i = O$ if and only if there is a curve $C$ of degree $n'$ that passes through these points. Multiplicities are intersection multiplicities.
Choose \( k \) such that \( k = 3n' \) for some positive integer \( n' \).

Choose random non-identity points \( P_1, P_2, \ldots, P_s \) and \( Q_1, Q_2, \ldots, Q_t \) such that \( s + t = k \).

Check if there is a homogeneous curve of degree \( n' \) that passes through these points; where \( P_i = n_iP \) and \( Q_j = -n'_jQ \) for some integers \( n_i \) and \( n'_j \).

If there is a curve, the discrete logarithm problem is solved.

Otherwise repeat the process.

To choose these points \( P_i \) and \( Q_j \), we choose a random point \( n_i, n'_j \) and compute \( n_iP \) and \( -n'_jQ \).

We would choose \( n_i \) and \( n'_j \) to be distinct from the ones chosen before to give rise to distinct points \( P_i \) and \( Q_j \) on \( E \).
How do we know if there is a curve

Let $C = \sum_{i+j+k=n'} a_{ijk} x^i y^j z^k$ be a complete homogeneous curve of degree $n'$. An ordering of $i, j, k$ is fixed and $C$ is presented according to that ordering. By complete we mean that the curve has all the possible monomials of degree $n'$. We need to check if $P_i$, $i = 1, 2, \ldots, s$ and $Q_j$ for $j = 1, 2, \ldots, t$ satisfy the curve $C$. Note that, there is no need to compute the values of $a_{ijk}$, just mere existence will solve the discrete logarithm problem.
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Let \( P \) be a point on \( \mathcal{E} \). We denote by \( \overline{P} \) the value of \( C \) when the values of \( x, y, z \) in \( P \) is substituted in \( C \). Similarly for \( Qs \). We now form a matrix \( \mathcal{M} \) where the rows of \( \mathcal{M} \) are \( \overline{P}_i \) for \( i = 1, 2, \ldots, s \) and \( \overline{Q}_j \) for \( j = 1, 2, \ldots, t \). If this matrix has a non-zero left-kernel, we have solved the discrete logarithm problem. By left-kernel we mean the kernel of \( \mathcal{M}^T \), the transpose of \( \mathcal{M} \).
Why look at left-kernel

We use $\mathcal{K}$ for the left-kernel of $\mathcal{M}$ and $\mathcal{K}'$ as the (right) kernel.
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**Theorem**

The following are equivalent:

(A) $\mathcal{K} = 0$.

(B) $\mathcal{K}'$ only contain curves that are a multiple of $\mathcal{E}$. 

Take the number of points $k = 3n' + l$, then

**Theorem**

If $l \geq 1$, the dimension of the left kernel of $\mathcal{M}$ is $l$. 

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**Corollary**

Assume that $\mathcal{M}$ has $3n' + l$ rows, computed from the same number of points of the elliptic curve $\mathcal{E}$. If there is a curve $\mathcal{C}$ intersecting $\mathcal{E}$ non-trivially in $3n'$ points among $3n' + l$ points, then there is a vector $v$ in $\mathcal{K}$ with at least $l$ zeros. Conversely, if there is a vector $v$ in $\mathcal{K}$ with at least $l$ zeros, then there is a curve $\mathcal{C}$ passing through those $3n'$ points that correspond to the non-zero entries of $v$ in $\mathcal{M}$. 

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What is the advantage

In the exhaustive search we would have picked a random set of $3n'$ points and then checked to see if the sum of those points is $Q$. In the above algorithm we are taking a set of $3n' + l$ points and then checking all possible $3n'$ subsets of this set simultaneously. There are $\binom{3n'+l}{l}$ such subsets. This is one of the main advantage of our algorithm.
The algorithm

A sketch of the algorithm

Choose $k = 3n' + l$.

- Choose points from the elliptic curve.
- Construct the matrix $\mathcal{M}$.
- Compute the left-kernel $\mathcal{K}$ of $\mathcal{M}$.
- If the left kernel contains a vector with $l$ zeros we are done.
- If not, repeat.
**Problem L**

**Problem**

Let $W$ be a $l$-dimensional subspace of a $n$-dimensional vector space $V$. The vectors in the vector space are presented as linear sum of some fixed basis of $V$. The problem is to determine, if $W$ contains a vector with $l$ zeros. If there is one such vector, find that vector.

**Theorem**

When $p$ tends to infinity, the probability of success of the above algorithm is approximately $1 - \frac{1}{e} \approx 0.6321$. The number of rows of the matrix $M$ required to reach this probability is $O(\log p)$. This makes our algorithm polynomial in both time and space complexity.
Solving Problem L

Take $3n' = l$. Then $\mathcal{K}$ is two $l \times l$ matrix stacked sideways.

$$\mathcal{K} = \begin{bmatrix}
* & * & \ldots & \ldots & \ldots & 0 & \ldots & 0 & 1 \\
* & * & \ldots & \ldots & \ldots & 0 & \ldots & 1 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
* & * & \ldots & \ldots & \ldots & 1 & 0 & \ldots & 0 \\
\end{bmatrix}$$

- The matrix $\mathcal{K}$ is the basis for the left-kernel $\mathcal{K}$. The size of the matrix is $l \times 2l$. We look for a square sub-matrix with determinant zero from the left part.
- In particular, for the purpose of this talk we look at the $2 \times 2$ submatrix.
Some experimental results

- Number of iterations 40. The average number of kernels computed is the y-axis. The x-axis is the size of the field.
- Number of cores used 66.