INTERNET TRAFFIC MODELS
WITH
RANDOM TRANSMISSION RATES

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This work describes fluid approximations to internet traffic models. Research done over approximately last ten years has firmly established the presence of concepts like self-similarity, heavy tails. A standard model for analyzing such approximations is the $M/G/\infty$ model. The transmission begins according to a Poisson process. Generally, the models consider the transmissions to continue for a random amount of time at a (both temporally and stochastically) constant rate, which, for the purpose of normalization, is taken to be unit. Obviously this assumption has its practical limitation.

There have been some attempts in the literature to consider time-varying but non-random rates. Chapter 2 considers a model where the transmission continues for a random length of time at a rate which is also random, but remains unchanged over the duration of the transmission. Both the variables are considered to be heavy tailed with infinite variance, but finite mean. To describe the joint distribution of the transmission length and rate, we modify the usual concept of asymptotic independence. The usual definition is too broad to characterize the behavior of the

product of two asymptotically independent random variables, in our case, the length and the rate of transmission. Under such a model, the cumulative input traffic is approximated by a stable Lévy process for large time scales.

The above model considers the transmission rate and the transmission time instead of the transmitted file size. It fails to model the temporal variability of the rate. Also further empirical analysis of internet and WAN traffic shows multifractal behavior of the cumulative input traffic at small time scales. Chapter 3 deals with these observations. Here each transmission consists of a random file size, which is heavy-tailed with infinite variance but finite mean, and a transmission schedule. The dependence structure and the distribution of the schedule is described in details. At small time scale the multifractal behavior of the cumulative traffic is shown to be inherited from individual transmission schedules. At large time scales, the approximation continues to be stable Lévy motion.

However, empirical research shows self-similar Gaussian approximation for large time scale. The model of Chapter 3 is modified in the final chapter to consider a family of models to account for this behavior.

Biographical Sketch

Krishanu Maulik was born on February 25, 1975 in Calcutta (presently Kolkata), India as the only child of Atin and Anima Maulik. He began his schooling at the local Greenwood Kindergarten. He completed high school from St. Lawrence High School in 1993.

After that, Krishanu joined Indian Statistical Institute, where he was introduced to the beautiful world of probability and statistics. After 5 years of training, mostly in Calcutta, and for a month in Delhi, he obtained his Baccalaureate and Masters degrees in Statistics in 1996 and 1998 respectively.

Then he joined Department of Statistical Science in Cornell University, Ithaca, New York, USA. He received a M.S. degree in 2001 and completed his Ph.D. degree in 2002.

Upon completion of his degree, Krishanu will join EURANDOM in Eindhoven, The Netherlands, as a post-doctoral fellow.

আমার অনুপ্রেরণা, মাকে

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বাবার স্মৃতির উদ্দেশ্যে

To Ma, my inspiration

and

In memory of Baba

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Finally and most importantly, I recall the support of my parents: the quiet encouragement of my father and the active inspiration - which can be very nagging at times - of my mother, who also happened to teach me my first alphabets. Thank you, Baba and Ma.

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Chapter 1

Introduction

1.1 Overview

Long range dependence, self-similarity and heavy tails are established concepts required for modeling broadband networks. Leland et al. (1994) gave the first empirical evidence in the context of LAN traffic. Further studies by Erramilli et al. (1996), Resnick (1997), Willinger et al. (1997) documents the inadequacy of the finite variance model.

Network traffic models generally contain many sources transmitting data. Transmissions can be modelled by the infinite source Poisson model, sometimes called the $M/G/\infty$ input model (cf. Guerin et al., 1999, Heath et al., 1999, Jelenković and Lazar, 1999, Jelenković and Lazar, 1996, Mikosch et al., 2002, Resnick and Rootzén, 2000, Resnick and van den Berg, 2000). The times between the starts of transmissions are modeled as i.i.d. exponentially distributed random variables.

Thus, to account for the long range dependence and self-similar nature of the traffic, it becomes important to consider transmission times to be heavy tailed (Willinger et al., 1997). The sources start transmitting at times $\{\Gamma_k\}$; we assume $\{\Gamma_k\}$ is a sequence strictly increasing to ∞. Each transmission consists of a file of size J_k, a duration of transmission L_k and a transmission schedule $\{\mathcal{A}_k(t), t \geqslant 0\}$, all chosen at random according to some distribution to be specified. We assume J_k, L_k are positive and $\mathcal{A}_k(t)$ denotes the cumulative amount of data transmitted in time t after the transmission has begun. $\mathcal{A}_k$ is a non-decreasing càdlàg function starting from 0 and growing to J_k in finite time, which vanishes on the negative half-line. Clearly the quantities J_k, L_k and $\mathcal{A}_k$ are related. We have $\mathcal{A}_k(\infty) = J_k$. We also have

$$L_k = \inf\{t : \mathcal{A}_k(t) = J_k\}.$$

Also observe that, $\mathcal{A}_k(t) = J_k$, for $t \geqslant L_k$. The quantity of interest is the traffic process which results from aggregating cumulative traffic from all sources in $[0, t]$, and which is defined as

$$X(t) = \sum_{k=1}^{\infty} \mathcal{A}_k(t - \Gamma_k).$$

Most of the existing research assumes the rate of transmission to be constant and non-random, which for the sake of normalization, we take as unit. This gives us the special case, where $\mathcal{A}_k$ is a linear function with unit slope, or more precisely, $\mathcal{A}_k(t) = t$, for $0 \leqslant t \leqslant L_k$. In this case, we have $J_k = L_k$. Konstantopoulos and Lin (1998) replaced the constant, non-random rate by a deterministic rate function which is regularly varying. They showed that the cumulative input process at a large time scale is approximated by a stable Lévy motion. Resnick and van den Berg

(2000) extended the convergence to hold on the space $D[0, \infty)$ of càdlàg functions with Skorohod's M_1 topology (cf. Skorohod, 1956, Whitt, 1999a,b,c, 2002).

A recent empirical study on several internet traffic data sets by Guerin et al. (1999) shows that the infinite source Poisson model with unit transmission rate often gives an indequate fit to data. This study suggests that the transmission rate is also a random variable with heavy tail. There have been few studies of this aspect of the internet traffic data modeling. In a series of recent papers, Levy, Pipiras and Taqqu (cf. Levy and Taqqu, 2000, Pipiras and Taqqu, 2000, Pipiras et al., 2000) consider the case where the transmission rate is also random for a superposition of renewal reward processes. They show that the limiting behavior for large time scale and large number of superpositioned models can either be a stable Lévy process with stationary, *independent* increments or symmetric stable process with stationary, but *dependent* increments, depending on the relative rate of growth of the time scale and number of models. Their results parallel the results of Mikosch et al. (2002) for the infinite source Poisson model who also obtain two different limits depending on the growth rate time scale relative to the intensity of the Poisson process.

The Levy, Pipiras, Taqqu papers mentioned above consider the renewal-reward model and assume the transmission rate to be independent of the length of transmission. It is difficult to conclude from evidence in measured data that rate and the length of the transmission are always independent, but in certain cases we may reasonably assume that the rate and the length of the transmission are at least asymptotically independent. Chapter 2 explains this issue in detail with an example.

The above analysis encourages us to consider, in Chapter 2, a linear transmission schedule with the rate of transmission for the k-th transmission being R_k. In that case, $\mathcal{A}_k(t) = R_k t$, for $0 \leqslant t \leqslant L_k$ and $J_k = L_k R_k$. Both the transmission length and the transmission rate have marginal distributions with heavy tails. It is further reasonable to assume that their bivariate distribution has a bivariate regularly varying tail, which is asymptotically independent in the sense used in extreme value theory. However, it is shown in Chapter 2, this definition is too broad to conclude anything interesting about the tail behavior of product of two asymptotically independent random variables. Note that the size of the transmitted file is the product of the length and the rate of the transmission! In Chapter 2, an alternative notion of asymptotic independence is given and equivalent formulations are discussed. This definition of asymptotic independence is then used to model the joint distribution of L_k and R_k. In this chapter, we finally show an approximation to X by a stable Lévy motion at large time scale.

However, even the random transmission rate does not completely capture the essence of a random transmission schedule. Also, we base our model in Chapter 2 on the transmission rate and length, whereas a more natural choice will be the size of the transmitted file and the transmission schedule. In Chapter 3, we address this issue. For k-th transmission, we consider a random process A_k, which is a non-decreasing càdlàg function starting from 0 and increasing to ∞, and vanishing on the negative half-line. In this case, we have $\mathcal{A}_k(t) = A_k(t) \wedge J_k$. Then we show a stable Lévy process approximation to the input process at large time scale.

Also, usual approximation results show self-similarity, which is consistent with the macroscopic analysis of the network traffic data at a time scale of a few hundred milliseconds or larger. However, these models were posed without considering the complicated multifractal behavior of the WAN traffic observed at fine time scales below a few hundred milliseconds. Paxson and Floyd (1995) observed the limitations of the usual model in their study. Later Riedi and Lévy Véhel (1997) and Mannersalo and Norros (1997) analyzed different WAN traces to empirically observe the multifractal behavior of ATM WAN traces. These observations stimulated researchers to look for a model which could explain both the microscopic as well as the macroscopic behaviors. In Chapter 3, we explain the fine time scale behavior by assuming individual transmission schedules exhibit multifractality. This results in multifractal behavior for the cumulative taffic process at the microscopic level and still gives a stable Lévy motion as the macroscopic approximation. Chapter 3 also gives a quick review of multifractals and weak convergence in the space of càdlàg functions.

Though the model in Chapter 3 succeeds in modeling the transmission schedule reasonably generally and provide a unified model for microscopic and macroscopic analysis, it suffers from one limitation. The approximation at large time scale is a stable process and hence does not have finite variance. Among other sources, Riedi and Willinger (2000), Willinger et al. (1997) argue for a Gaussian approximation both from the empirical point of view as well as heuristically. This encourages to search for a Gaussian approximation in Chapter 4. Mikosch et al. (2002) has considered a family of infinite source Poisson input model with increasing traffic input rate, where the transmission schedule is linear with unit slope. They show a stable Lévy approximation for a slow growth of input rate and a fractional Brownian motion approximation when it is fast. We generalize this result in Chapter 4 where we consider the transmission schedule to be a random process. We succeed in showing multifractal approximation for small time scale. For large time scale, the approximation is stable Lévy motion if the input rate grows slowly and it is self-similar Gaussian process in case of a fast growth.

1.2 Notations

We end this chapter with a quick look at the notations used in this dissertation. The vectors are denoted by bold letters and operations on vectors are always interpreted component by component. We denote the vectors $(0, 0)$ and (∞, ∞) by $\mathbf{0}$ and $\boldsymbol{\infty}$ respectively. The two dimensional boxes are denoted by their lower left and upper right corners, for example, $(\mathbf{a}, \mathbf{b}]$ stands for $(a_1, b_1] \times (a_2, b_2]$ and we can similarly define the boxes $(\mathbf{a}, \mathbf{b})$, $[\mathbf{a}, \mathbf{b})$ and $[\mathbf{a}, \mathbf{b}]$.

For a non-decreasing function x, we define its left continuous inverse as

$$x^{\leftarrow}(t) = \inf\{u : x(u) \geqslant t\} \tag{1.1}$$

and its right continuous inverse as

$$x^{\rightarrow}(t) = \inf\{u : x(u) > t\} \tag{1.2}$$

For a non-negative random variable U, we denote its distribution function by F_U, i.e., $F_U(u) = \mathrm{P}[U \leqslant u]$. Let $\bar{F}_U(u) = 1 - F_U(u)$. We define the quantile function

$\tilde{b}_U$ as

$$\tilde{b}_U(T) = \inf\left\{u : \bar{F}_U(u) \leqslant \frac{1}{T}\right\} = \left(\frac{1}{\bar{F}_U}\right)^{\leftarrow}(T). \tag{1.3}$$

Recall that a function ϕ is *regularly varying* of index α and is denoted by $\phi \in RV_\alpha$ (cf. Resnick, 1987, Section 0.4), if for all $u > 0$,

$$\lim_{t\to\infty} \frac{\phi(tu)}{\phi(t)} = u^\alpha.$$

We say that U has a tail of index $\alpha_U > 0$, if $\bar{F}_U \in RV_{-\alpha_U}$. In such a case (cf. Resnick, 1987, Proposition 0.8(v)), $\tilde{b}_U \in RV_{\alpha_U^{-1}}$ and also we have

$$\lim_{T\to\infty} T\,\mathrm{P}[U_1 > \tilde{b}_U(T)u] = u^{-\alpha_U}. \tag{1.4}$$

Conversely, if (1.4) holds, then $\tilde{b}_U \in RV_{\alpha_U^{-1}}$ and $\bar{F}_U \in RV_{-\alpha_U}$. In either of these cases, we can choose a strictly increasing, absolutely continuous function b_U, such that $\tilde{b}_U \sim b_U$, i.e.,

$$\lim_{T\to\infty} \frac{\tilde{b}_U(T)}{b_U(T)} = 1$$

(cf. Resnick, 1987, Proposition 0.8(vii)). We can further say that

$$\lim_{T\to\infty} T\,\mathrm{P}[U_1 > b_U(T)u] = u^{-\alpha_U}. \tag{1.5}$$

Chapter 2

Asymptotic Independence and a Network Traffic Model

2.1 Introduction

There have been various attempts made in the literature to weaken the condition of independence of two random variables. One such notion is that of asymptotic independence used in the context of the extreme value theory (cf. Resnick, 1987, Chapter 5). The joint distribution of two random variables is asymptotically independent if the coordinatewise maximum of n i.i.d. observations from that distribution under suitable scaling has a non-degenerate limit which is a product measure, as n increases to ∞. However, as shown with examples in Section 2.2, this concept is too weak to conclude anything meaningful about the product of the random variables. So we introduce a new concept of asymptotic independence, which is stronger than the asymptotic independence used in extreme value theory, but weaker than independence. This concept of asymptotic independence is not symmetric. We study this concept and its equivalent formulation in Section 2.2. Under additional moment conditions, we study the behavior of the product of two asymptotically independent random variables in Section 2.3 and give some illuminating examples. Then we use this concept of asymptotic independence to study a network traffic model. This network traffic model motivated our study of asymptotic independence and products since the product of transmission rate and transmission duration yields the quantity transmitted.

It is difficult to conclude from evidence in measured data that rate and the length of the transmission are always independent, but in certain cases we may reasonably assume that the rate and the length of the transmission are at least asymptotically independent. As an example, we consider the BUburst dataset considered by Guerin et al. (1999). This is data processed from the original 1995 Boston University data described in the report by Cunha et al. (1995) and cataloged at the Internet Traffic Archive (ITA) web site `www.acm.org/sigcomm/ITA/`. A plot of the transmission length against the transmission rate, (see Figure 2.1) shows that most of the data pairs hug the axes, which suggests the variables are at least asymptotically independent. However, if we plot the data in the log scale on both the axes, then a weak linear dependence is observable and the correlation coefficient between the two variables after log transform is approximately -0.379, which argues against an independence assumption. We consider the log transform to make the variables have finite second moment, so that the correlation coefficient becomes meaningful.

The Hill estimates obtained for the transmission length, the transmission rate and the size of the transmitted file are 1.407, 1.138 and 1.157 respectively. These estimates are consistent with the observations made in Guerin et al. (1999). The corresponding Hill plots are given in Figure 2.2. For each of the variables, the plots in the first column, named Hill plot, give plots of $\{(k, \hat{\alpha}_{k,n}) : 1 \leqslant k \leqslant n\}$, where $\hat{\alpha}_{k,n}$ is the Hill estimator of α based on k upper order statistics. The plots in the second column, named AltHill plot (cf. Resnick and Stărică, 1997), give the Hill estimates in an alternative scale and plot $\left\{\left(\theta, \hat{\alpha}_{\lceil n^{\theta} \rceil, n}\right) : 0 \leqslant \theta \leqslant 1\right\}$ (cf. Resnick and Stărică, 1997). This plot expands the original Hill plot on the left side and helps looking at that part more closely. The third plot, named the Stărică plot, is an exploratory device suggested by Stărică (1999, Section 7) to decide on the number of upper order statistics to be used. It uses the fact that for a random variable X with Pareto tail of parameter α, we have

$$\lim_{T \to \infty} T\,\mathrm{P}\left[\frac{X}{T^{\frac{1}{\alpha}}} > r\right] = r^{-\alpha}.$$

For every k, we estimate the left hand side by

$$\hat{\nu}_{n,k}\big((r, \infty]\big) = \frac{1}{k} \sum_{i=1}^{n} \mathbf{I}\left[\frac{X_i}{(n/k)^{1/\hat{\alpha}_{n,k}}} > r\right].$$

We expect the ratio of $\hat{\nu}_{n,k}\big((r, \infty]\big)$ and $r^{-\hat{\alpha}_{n,k}}$, called the scaling ratio, to be approximately 1, at least for values of r in a neighborhood of 1, if we have made the correct choice of k. In the Starica plot, we plot the above scaling ratio against the scaling constant r, and choose k so that the graph hugs the horizontal line of height 1. The interesting point to be noted is the fact that the rate of the transmission has a much heavier tail than the length of transmission. This justifies the study of a model with a random rate with heavy tails. The tail of the size of the transmitted file, which is the product of the rate and the time of transmission, is comparable to the rate of the transmission, the heavier one between time and rate. This is in agreement with Theorem 2.3.1.

Since both the transmission length and the transmission rate have marginal distributions with heavy tails, it is further reasonable to assume that their bivariate distribution has a bivariate regularly varying tail, which is asymptotically independent in the sense used in extreme value theory. However, as described in Sections 2.2 and 2.3, the usual notion of asymptotic independence from extreme value theory is not sufficient for meaningful analysis. So we assume that the transmission and the transmission length are assumed to be asymptotically independent in the sense described in Section 2.2. Section 2.4 outlines the network model and states the limit theorem, which is proved in Section 2.5. Section 2.6 comments on the appropriateness of the model for the data and suggests possible improvements.

2.2 Asymptotic independence

Consider i.i.d. random vectors $(X, Y), (X_1, Y_1), (X_2, Y_2), \ldots$. In extreme value theory, X and Y are considered asymptotically independent, if the coordinatewise sample maxima, $(\vee_{i=1}^n X_i, \vee_{i=1}^n Y_i)$, under suitable centering and scaling, converges weakly to a product measure. When both X and Y have regularly varying tail probabilities, this is equivalent to the existence of regularly varying functions b_X and b_Y, such that,

$$T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}, \frac{Y}{b_Y(T)}\right) \in \cdot\right] \stackrel{\mathrm{v}}{\to} \nu(\cdot) \text{ on } [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}, \tag{2.1}$$

where ν is a measure satisfying $\nu((\mathbf{0}, \boldsymbol{\infty}]) = 0$. The convergence above is vague convergence. This means that ν concentrates on the axes $\{0\} \times (0, \infty]$ and $(0, \infty] \times \{0\}$ (cf. Resnick, 1987, Chapter 5). There is an equivalent formulation of the above concept where the variables are transformed so as to have the similar tails (cf. de Haan and de Ronde, 1998, Section 4), which states:

$$T\,\mathrm{P}\left[\frac{(b_X^{\leftarrow}(X), b_Y^{\leftarrow}(Y))}{T} \in \cdot\right] \stackrel{\mathrm{v}}{\to} \tilde{\nu}(\cdot) \text{ on } [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}, \tag{2.2}$$

where $\tilde{\nu}$ satisfies $\tilde{\nu}((\mathbf{0}, \boldsymbol{\infty}]) = 0$, and $\tilde{\nu}$ is also homogeneous of index -1. Thus if we define

$$\Phi(\theta) = \tilde{\nu}\left\{(s,t) : s \vee t > 1, \frac{t}{s} \leqslant \tan\theta\right\}, 0 \leq \theta \leq \frac{\pi}{2} \tag{2.3}$$

then the asymptotic independence is equivalent to the fact that Φ is supported on$\{0, \frac{\pi}{2}\}$. Motivated by the network modelling problem, we are interested in understanding how the distribution tail behavior of random variables affects tail behavior of the products of random variables. The class of distributions possessing classical asymptotic independence is too broad a class for the study of the products as is clear from the following examples.

Example 2.2.1. Let U and V be random variables with regularly varying tails of indices $-\alpha_U$ and$-\alpha_V$, with $1 < \alpha_U, \alpha_V < 2$. Let b_U and b_V be the corresponding quantile functions, defined as in (1.3). Let B be a Bernoulli random variable with probability of success 0.5, independent of U and V. Then define

$$(X,Y) = B(U,0) + (1-B)(0,V).$$

Then we have

$$T\,\mathrm{P}\left[\left(\frac{X}{b_U(T)}, \frac{Y}{b_V(T)}\right) \in \cdot\right] = \frac{1}{2}T\,\mathrm{P}\left[\left(\frac{U}{b_U(T)}, 0\right) \in \cdot\right] + \frac{1}{2}T\,\mathrm{P}\left[\left(0, \frac{V}{b_V(T)}\right) \in \cdot\right]$$
$$\xrightarrow{\mathrm{v}} \frac{1}{2}\nu_{\alpha_U} \times \varepsilon_0(\cdot) + \frac{1}{2}\varepsilon_0 \times \nu_{\alpha_V}(\cdot) \quad \text{on } [\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\},$$

where ε_0 is the Dirac measure at 0. Thus the limiting measure is concentrated on the axes; i. e., on the set $\big(\{0\} \times (0,\infty]\big) \cup \big((0,\infty] \times \{0\}\big)$. We conclude that (2.1) holds, but no interesting product behavior is possible since $XY \equiv 0$.

In the next example, (2.1) again holds. The product XY is not degenerate, but still we cannot draw any interesting conclusion about the tail behavior of the product.

Example 2.2.2. Let U, V and B be as in the previous example. Define

$$(X,Y) = B(U,U^p) + (1-B)(V^p,V),$$

where $0 < p \leqslant \frac{1}{2}$. Suppose $\alpha_V < \alpha_U$, so that V has a heavier tail. Now observe that $\alpha_U < 2 < 2\alpha_V$, since $\alpha_V > 1$, and similarly also $\alpha_V < 2 < 2\alpha_U$. So we have $b_U(T)^p \leqslant \sqrt{b_U(T)} = o(b_V(T))$ and $b_V(T)^p \leqslant \sqrt{b_V(T)} = o(b_U(T))$ as $T \to \infty$. Then

$$T\,\mathrm{P}\left[\left(\frac{X}{b_U(T)}, \frac{Y}{b_V(T)}\right) \in \cdot\right]$$
$$= \frac{T}{2}\,\mathrm{P}\left[\left(\frac{U}{b_U(T)}, \frac{U^p}{b_V(T)}\right) \in \cdot\right] + \frac{T}{2}\,\mathrm{P}\left[\left(\frac{V^p}{b_U(T)}, \frac{V}{b_V(T)}\right) \in \cdot\right]$$

$$\xrightarrow{\text{v}} \frac{1}{2}\nu_{\alpha_U} \times \varepsilon_0(\cdot) + \frac{1}{2}\varepsilon_0 \times \nu_{\alpha_V}(\cdot) \quad \text{on } [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\},$$

and the limiting measure is concentrated on the axes.

Also $XY = BU^{1+p} + (1-B)V^{1+p}$. Then, since $\alpha_V < \alpha_U$, we have,

$$\mathrm{P}[XY > x] \sim \frac{1}{2}\,\mathrm{P}[V^{1+p} > x],$$

which is regularly varying of index $-\frac{\alpha_V}{1+p}$. Since $\mathrm{P}[X > \cdot] \in RV_{-\alpha_U}$, $\mathrm{P}[Y > \cdot] \in RV_{-\alpha_V}$, the tail behavior of XY cannot be concluded from the tail behavior of the factors even though (2.1) holds.

These examples reinforce the idea that the classical notion of asymptotic independence from extreme value theory contains little information about the tail behavior of the product. So we need to strengthen the concept.

In the following we write ν_β for the measure on $(0,\infty]$ satisfying $\nu_\beta((x,\infty]) = x^{-\beta}$, $x > 0$, $\beta > 0$.

Definition 2.2.1. For two strictly positive random variables X and Y, we say Y is *asymptotically independent* of X, if

$$T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}, Y\right) \in \cdot\right] \xrightarrow{\text{v}} \nu_{\alpha_X} \times G(\cdot) \text{ on } D := (0,\infty] \times [0,\infty], \tag{2.4}$$

where G is a probability measure with $G\big((0,\infty)\big) = 1$.

The definition is not symmetric in X and Y. Also, if Y is asymptotically independent of X, then considering vague convergence on the set $(x,\infty] \times [0,\infty]$, we obtain, for all $x > 0$,

$$t\,\mathrm{P}\left[\frac{X}{b_X(T)} > x\right] \to x^{-\alpha_X}$$

and hence X has a regularly varying tail of index $-\alpha_X$. It should be clear that if $\overline{F}_X \in RV_{-\alpha}$, and Y is independent of X, then (2.4) holds.

We consider the cone $D = (0, \infty] \times [0, \infty]$ instead of the more natural choice of $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$ for the simple reason that a characterization of (2.4) by means of multivariate regular variation on the larger set is not possible without further assumptions. See Theorem 2.2.1 and Lemma 2.2.2.

Condition (2.4) in the definition of asymptotic independence is stronger than the usual concept of asymptotic independence in extreme value theory.

Lemma 2.2.1. *Assume both the conditions* (2.4) *and* (2.1) *hold. Then* ν *satisfies* $\nu\big((\mathbf{0}, \boldsymbol{\infty}]\big) = 0$.

Proof. Fix $x > 0$. Let us define $\mathbf{x} = (x, x)$. Since $b_X(T) \to \infty$, we have, for all $K > 0$, $b_X(T)\, x > K$, for sufficiently large T. Hence we have, for all $K > 0$,

$$\nu\big((x, \infty] \times (x, \infty]\big) = \nu\big((\mathbf{x}, \boldsymbol{\infty}]\big) = \lim_{T\to\infty} T\, \mathrm{P}[X > b_L(T)x, Y > b_R(T)x] \tag{2.5}$$

$$\leqslant \lim_{T\to\infty} T\, \mathrm{P}[X > K, Y > b_R(T)x] \tag{2.6}$$

$$= \bar{G}(K)\nu_{\alpha_R}(x, \infty]. \tag{2.7}$$

Then letting $K \to \infty$, we get $\nu\big((\mathbf{x}, \boldsymbol{\infty}]\big) = 0$, for all $x > 0$. □

We now characterize asymptotic independence in terms of standard multivariate regular variation on the cone D (cf. Resnick, 1987, Chapter 5), in the spirit of the characterization of multivariate regular variation using a polar coordinate transformation (cf. Basrak, 2000).

Theorem 2.2.1. *Assume X and Y are strictly positive, finite random variables. Suppose*

$$T\,\mathrm{P}[X > b_X(T)] \to 1. \tag{2.8}$$

Then the following are equivalent:

(i) $T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}, Y\right) \in \cdot\right] \xrightarrow{\mathrm{v}} (\nu_\alpha \times G)(\cdot)$ *on D*
for some $\alpha > 0$, and G a probability measure satisfying $G((0,\infty)) = 1$.

(ii) $T\,\mathrm{P}\left[\frac{(X, XY)}{b_X(T)} \in \cdot\right] \xrightarrow{\mathrm{v}} \nu(\cdot)$ *on D,*
where $\nu(\{(x,y) : x > u\}) > 0$ for all $u > 0$.

In fact, ν is homogeneous of order $-\alpha$; i.e., $\nu(u\cdot) = u^{-\alpha}\nu(\cdot)$ on D, and is given by

$$\nu = \begin{cases} (\nu_\alpha \times G) \circ \theta^{-1} & \text{on} \quad (0,\infty) \times [0,\infty) \\ 0 & \text{on} \quad D \setminus ((0,\infty) \times [0,\infty)) \end{cases}, \tag{2.9}$$

where $\theta(x,y) = (x, xy)$, if $(x,y) \in D \setminus \{(\infty, 0)\}$ and $\theta(\infty, 0)$ is defined arbitrarily.

Remarks. The function θ as defined above is Borel-measurable, irrespective of its value at the point $(\infty, 0)$, since the singleton subset $\{(\infty, 0)\}$ is a measurable subset of D.

Condition (2.8) holds, for example, when X has a regularly varying tail.

The measure ν as defined above is Radon. To see this, note that the relatively compact sets in D are contained in $[a,\infty] \times [0,\infty]$ and

$$\nu([a,\infty] \times [0,\infty]) = (\nu_\alpha \times G)([a,\infty] \times [0,\infty]) = a^{-\alpha} < \infty.$$

Proof of Theorem 2.2.1. **(i) ⇒ (ii):** Let $0 < s < \infty$ and $S \in \mathcal{B}([0,\infty])$. Define

$$V_{s,S} = \{(x, xy) \in D : s < x < \infty, y \in S\}.$$

Now $V_{s,[t_1,t_2]}$ is relatively compact in D for all $0 \leqslant t_1 \leqslant t_2 \leqslant \infty$. Also if $G(\{t_1, t_2\}) = 0$, then we have,

$$T\,\mathrm{P}\left[\frac{(X, XY)}{b_X(T)} \in V_{s,[t_1,t_2]}\right] = T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}, Y\right) \in (s,\infty) \times [t_1, t_2]\right]$$
$$\to (\nu_\alpha \times G)((s,\infty) \times [t_1, t_2]) = \nu\left(V_{s,[t_1,t_2]}\right),$$

where ν is as defined in (2.9). Now, fix $s_0 \in (0,\infty)$. Note that

$$T\,\mathrm{P}\left[\frac{(X, XY)}{b_X(T)} \in V_{s_0,[0,\infty]}\right] \to \nu\left(V_{s_0,[0,\infty]}\right) = (\nu_\alpha \times G)((s_0,\infty) \times [0,\infty]) = s_0^{-\alpha},$$

which is also strictly positive and finite. Hence $T\,\mathrm{P}\left[\frac{(X,XY)}{b_X(T)} \in V_{s_0,[0,\infty]}\right]$ is strictly positive and finite for all large T. So we can define probability measures $Q_T(\cdot)$ and $Q(\cdot)$ on $(s_0,\infty) \times [0,\infty]$ for all large T, by

$$Q_T(\cdot) = \frac{\mathrm{P}\left[\frac{(X,XY)}{b_X(T)} \in \cdot\right]}{\mathrm{P}\left[\frac{(X,XY)}{b_X(T)} \in V_{s_0,[0,\infty]}\right]} \text{ and } Q(\cdot) = \frac{\nu(\cdot)}{\nu\left(V_{s_0,[0,\infty]}\right)}.$$

Then

$$Q_T(V_{s,[t_1,t_2]}) \to Q(V_{s,[t_1,t_2]}) \quad \forall s \in (s_0,\infty), 0 \leqslant t_1 \leqslant t_2 \leqslant \infty \text{ with } G(\{t_1, t_2\}) = 0. \tag{2.10}$$

Let

$$\mathcal{P} = \{V_{s_1,[t_1,t_2]} \setminus V_{s_2,[t_1,t_2]} : s_0 < s_1 < s_2 < \infty, 0 \leqslant t_1 \leq t_2 \leqslant \infty\}.$$

Observe $B \in \mathcal{P}$ is a Q-continuity set iff $G(\{t_1, t_2\}) = 0$. So by (2.10), $Q_T(B) \to Q(B)$ for all Q-continuity sets $B \in \mathcal{P}$. Also, clearly for every $\mathbf{x}$ in $(s_0,\infty) \times [0,\infty]$

and positive ε, there is an A in $\mathcal{P}$, for which $\mathbf{x} \in A^\circ \subseteq A \subseteq B(\mathbf{x}, \varepsilon)$, where A° is the interior of A and $B(\mathbf{x}, \varepsilon)$ is the ball of radius ε around $\mathbf{x}$. Now $\mathcal{P}$ is a π-system. Then, by Theorem 2.3 of Billingsley (1999), we have

$$Q_T \Rightarrow Q \text{ on } (s_0, \infty) \times [0, \infty].$$

Thus $Q_T(B) \to Q(B)$ for all Borel sets B of $(s_0, \infty) \times [0, \infty]$ with boundary in $(s_0, \infty) \times [0, \infty]$ having zero Q-measure, for all $s_0 > 0$. Hence the same result holds with Q_T, Q replaced by $T\,\mathrm{P}\left[\frac{(X,XY)}{b_X(T)} \in \cdot\right]$, ν respectively.

Let K be relatively compact in D with $\nu(\partial_D K) = 0$. Then there exists $s_0 > 0$, such that $K \subset (s_0, \infty] \times [0, \infty]$. Define $B = K \cap \big((s_0, \infty) \times [0, \infty]\big)$. Then B is Borel in $(s_0, \infty) \times [0, \infty]$ and $\nu\left(\partial_{(s_0,\infty)\times[0,\infty]} B\right) = 0$. We have, by definition of ν in (2.9),

$$T\,\mathrm{P}\left[\frac{(X, XY)}{b_X(T)} \in K\right] = T\,\mathrm{P}\left[\frac{(X, XY)}{b_X(T)} \in B\right] \to \nu(B) = \nu(K).$$

Therefore

$$T\,\mathrm{P}\left[\frac{(X, XY)}{b_X(T)} \in \cdot\right] \xrightarrow{\mathrm{v}} \nu(\cdot) \text{ on } D,$$

where ν is defined as in (2.9). Thus,

$$\nu(\{(x, y) : x > u\}) = (\nu_\alpha \times G)(\{(x, y) : x > u\}) = u^{-\alpha} > 0, \quad \forall u > 0.$$

(ii)⇒ (i): Define $U = \{(x, y) \in D : x > 1\}$.

Choose integer n_v such that $b(n_v) \leqslant v < b(n_v + 1)$. Fix $0 < s \leqslant \infty$, $0 \leqslant t \leqslant \infty$, so that $\nu(\partial([s, \infty] \times [t, \infty])) = 0$. Then

$$\frac{n_v}{n_v + 1} \cdot \frac{(n_v + 1)\,\mathrm{P}\left[\frac{(X,XY)}{b_X(n_v+1)} \in [s, \infty] \times [t, \infty]\right]}{n_v\,\mathrm{P}\left[\frac{(X,XY)}{b_X(n_v)} \in U\right]}$$

$$\leqslant \frac{\mathrm{P}\left[v^{-1}(X,XY)\in[s,\infty]\times[t,\infty]\right]}{\mathrm{P}\left[v^{-1}(X,XY)\in U\right]}$$

$$\leqslant \frac{n_v+1}{n_v}\cdot\frac{n_v\,\mathrm{P}\left[\frac{(X,XY)}{b_X(n_v)}\in[s,\infty]\times[t,\infty]\right]}{(n_v+1)\,\mathrm{P}\left[\frac{(X,XY)}{b_X(n_v+1)}\in U\right]},$$

and taking the limit as $v\to\infty$, we find, for $0<s\leqslant\infty$ and $0<t\leqslant\infty$, that

$$\frac{\mathrm{P}\left[v^{-1}(X,XY)\in[s,\infty]\times[t,\infty]\right]}{\mathrm{P}\left[v^{-1}(X,XY)\in U\right]}\to\frac{\nu\left([s,\infty]\times[t,\infty]\right)}{\nu(U)}.$$

Arguing as before, normalizing to probability measures and so on, we have

$$\frac{\mathrm{P}[(X,XY)\in v\cdot]}{\mathrm{P}[(X,XY)\in vU]}\stackrel{\mathrm{v}}{\to}\frac{\nu(\cdot)}{\nu(U)}\text{ on }D.$$

Then, by the usual argument, (cf. Resnick, 1987), $\frac{\nu(\cdot)}{\nu(U)}$ is homogeneous on D of order $-\alpha$, for some $\alpha>0$, and hence this is true for $\nu(\cdot)$; i. e., $\nu(s\cdot)=s^{-\alpha}\nu(\cdot)$ on D.

Now, by (2.8) and the fact that X and Y are strictly positive and finite random variables, we have,

$$1=\lim_{T\to\infty}T\,\mathrm{P}\left[\frac{X}{b_X(T)}>1\right]=\lim_{T\to\infty}T\,\mathrm{P}\left[\frac{X}{b_X(T)}>1,Y\in(0,\infty)\right]=\nu(V_{1,(0,\infty)}).$$

Thus $G(\cdot):=\nu(V_{1,\cdot})$ is a probability measure on $[0,\infty]$ with $G(\mathbb{R}_+)=1$.

Also, for all $s>0$, $S\in\mathcal{B}([0,\infty])$,

$$\nu(V_{s,S})=(\nu_\alpha\times G)\big((s,\infty]\times S\big)=(\nu_\alpha\times G)(\theta^{-1}V_{s,S}),$$

and thus ν has a form as defined in (2.9).

Again, $V_{s,[t_1,t_2]}$ is a ν-continuity set iff $(s,\infty]\times[t_1,t_2]$ is a $(\nu_\alpha\times G)$-continuity set iff $G(\{t_1,t_2\})=0$. Thus for all $s>0$, all $0\leqslant t_1\leqslant t_2\leqslant\infty$ with $G(\{t_1,t_2\})=0$, i. e., $V_{s,[t_1,t_2]}$ a ν-continuity set, we have, for all $s>0$, $S\in\mathcal{B}([0,\infty])$,

$$T\,\mathrm{P}\left[\frac{X}{b_X(T)}>s,Y\in[t_1,t_2]\right]=T\,\mathrm{P}\left[\frac{(X,XY)}{b_X(T)}\in V_{s,[t_1,t_2]}\right]$$

$$\to \nu(V_{s,[t_1,t_2]}) = (\nu_\alpha \times G)\big((s,\infty] \times [t_1,t_2]\big).$$

Then, arguing by normalizing to probability measures as before, we get,

$$T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}, Y\right) \in \cdot\right] \xrightarrow{\mathrm{v}} (\nu_\alpha \times G)(\cdot) \text{ on } D.$$

□

Remark. Observe that in the above theorem, (i) implies (ii), even without the assumption (2.8) made separately, since (i) implies X has a regular varying tail and hence (2.8) holds.

Now we extend the characterization of asymptotic independence in terms of multivariate regular variation to the larger set $[\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\}$.

Lemma 2.2.2. *As in Theorem 2.2.1, assume X and Y are strictly positive, finite random variables satisfying* (2.4)*. Further assume two moment conditions that G has finite α-th moment and*

$$\lim_{\varepsilon \downarrow 0} \limsup_{T\to\infty} T\,\mathrm{E}\left[\left(\frac{XY}{b_X(T)}\right)^{\delta} \mathbf{1}_{\left[\frac{X}{b_X(T)} \leqslant \varepsilon\right]}\right] = 0 \textit{ for some } \delta > 0. \tag{2.11}$$

Then

$$T\,\mathrm{P}\left[\frac{(X, XY)}{b_X(T)} \in \cdot\right] \xrightarrow{\mathrm{v}} \tilde{\nu} \textit{ on } [\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\}, \tag{2.12}$$

where

$$\tilde{\nu} = \begin{cases} \nu & \textit{on } D \\ 0 & \textit{on } \{0\} \times (0,\infty] \end{cases}, \tag{2.13}$$

with ν defined as in (2.9).

Remark. As defined in the lemma, $\tilde{\nu}$ is Radon on $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$, if α-th moment of G is finite. To see this, note that a relatively compact set in $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$ is contained in $[\mathbf{0}, \boldsymbol{\infty}] \setminus [\mathbf{0}, \mathbf{a}]$ for some $a > 0$, where $\mathbf{a} = (a, a)$. So it is enough to check the finiteness of $\tilde{\nu}([\mathbf{0}, \boldsymbol{\infty}] \setminus [\mathbf{0}, \mathbf{a}])$. We consider the set $[\mathbf{0}, \boldsymbol{\infty}] \setminus [\mathbf{0}, \mathbf{a}]$ in two disjoint components, namely, $(a, \infty] \times [0, \infty]$ and $[0, a] \times (a, \infty]$. Now

$$\tilde{\nu}((a, \infty] \times [0, \infty]) = (\nu_\alpha \times G)((a, \infty] \times [0, \infty]) = a^{-\alpha} < \infty,$$

and

$$\begin{aligned}\tilde{\nu}([0, a] \times (a, \infty]) = \nu((0, a] \times (a, \infty]) &= (\nu_\alpha \times G)(\{\mathbf{x} : 0 < x_1 \leqslant a, x_1 x_2 > a\}) \\ = \int\limits_{(1,\infty)} \left(\left(\frac{a}{x_2} \right)^{-\alpha} - a^{-\alpha} \right) G(dx_2) &= a^{-\alpha} \int\limits_{(1,\infty)} x_2^\alpha \, G(dx_2) - G((1, \infty)).\end{aligned}$$

Thus $\tilde{\nu}$ is Radon iff $\int_{(1,\infty)} x_2^\alpha \, G(dx_2) < \infty$ iff G has finite α-th moment, which has been assumed.

Proof of Lemma 2.2.2. We have already seen in Theorem 2.2.1 that, (i) of Theorem 2.2.1 implies vague convergence in (2.12) on D. Let K be relatively compact in $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$ with $\tilde{\nu}(\partial K) = 0$. Choose $\varepsilon_k \downarrow 0$ such that $K_{\varepsilon_k} := K \cap ([\varepsilon_k, \infty] \times [0, \infty])$ satisfy $\tilde{\nu}(\partial K_{\varepsilon_k}) = 0$. Then

$$\liminf_{T \to \infty} T \, \mathrm{P} \left[\frac{(X, XY)}{b_X(T)} \in K \right] \geqslant \lim_{T \to \infty} T \, \mathrm{P} \left[\frac{(X, XY)}{b_X(T)} \in K_{\varepsilon_k} \right] = \nu(K_{\varepsilon_k}) = \tilde{\nu}(K_{\varepsilon_k}).$$

Letting $k \to \infty$, and using the definition of $\tilde{\nu}$,

$$\liminf_{T \to \infty} T \, \mathrm{P} \left[\frac{(X, XY)}{b_X(T)} \in K \right] \geqslant \tilde{\nu}(K \cap D) = \tilde{\nu}(K).$$

Since K is relatively compact in $[\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\}$, there exists $s\in(0,\infty)$ such that $K\subseteq[\mathbf{0},\mathbf{s}]^c$, where $\mathbf{s}=(s,s)$. Therefore,

$$\begin{aligned}\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{(X,XY)}{b_X(T)}\in K\right] &\leqslant \lim_{T\to\infty} T\,\mathrm{P}\left[\frac{(X,XY)}{b_X(T)}\in K_{\varepsilon_k}\right]\\ &\quad+\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{(X,XY)}{b_X(T)}\in K\cap([0,\varepsilon_k)\times(s,\infty])\right]\\ &\leqslant \tilde{\nu}(K_{\varepsilon_k})+\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{X}{b_X(T)}<\varepsilon_k,\frac{XY}{b_X(T)}>s\right]\\ &\leqslant \tilde{\nu}(K_{\varepsilon_k})+s^{-\delta}\limsup_{T\to\infty} T\,\mathrm{E}\left[\left(\frac{XY}{b_X(T)}\right)^{\delta}\mathbf{1}_{\left[\frac{X}{b_X(T)}<\varepsilon_k\right]}\right].\end{aligned}$$

Letting $k\to\infty$, by (ii) of Theorem 2.2.1,

$$\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{(X,XY)}{b(T)}\in K\right]\leqslant\tilde{\nu}(K).$$

Hence,

$$T\,\mathrm{P}\left[\frac{(X,XY)}{b(T)}\in\cdot\right]\xrightarrow{\mathrm{v}}\tilde{\nu}(\cdot)\text{ on }[\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\}.$$

□

In fact, it is easily seen from Theorem 2.2.1 and Lemma 2.2.2 that the converse also holds, which is summarized in the following corollary.

Corollary 2.2.1. *If X and Y are strictly positive, finite random variables with $T\,\mathrm{P}[X_1>b(T)]\to 1$. Also assume the truncated moment condition* (2.11)*. Then Y is asymptotically independent of X, i.e.,*

$$T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)},Y\right)\in\cdot\right]\xrightarrow{\mathrm{v}}\nu_\alpha\times G(\cdot)\ \text{ on } D=(0,\infty]\times[0,\infty]$$

for some $\alpha>0$, and some probability measure G on $[0,\infty]$ with $G((0,\infty))=1$ and finite α-th moment iff

$$T\,\mathrm{P}\left[\frac{(X,XY)}{b_X(T)}\in\cdot\right]\xrightarrow{\mathrm{v}}\tilde{\nu}(\cdot)\ \text{ on } [\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\}$$

for some Radon measure $\tilde{\nu}$ satisfying

$$\tilde{\nu}(\{(x,y) : x > u\}) > 0 \text{ for some } u > 0. \tag{2.14}$$

In fact, $\tilde{\nu}$ is homogeneous of order $-\alpha$ and is given as in (2.9) *and* (2.13).

Another equivalent formulation of asymptotic independence justifies the nomenclature and loosely says that Y is asymptotically independent of X, if the conditional distribution of Y is same as the marginal distribution, when X is large.

Lemma 2.2.3. *Assume X and Y are strictly positive, finite random variables. Then*

$$T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}, Y\right) \in \cdot\right] \xrightarrow{\mathrm{v}} \nu_{\alpha_X} \times G(\cdot) \text{ on } D \tag{2.15}$$

implies

$$\mathrm{P}[Y \in \cdot | X > x] \Rightarrow G(\cdot) \text{ as } x \to \infty, \tag{2.16}$$

where the second convergence is the usual weak convergence. If the marginal distribution of X has a regularly varying tail of index $-\alpha_X$, then the converse also holds.

Proof. Observe that (2.15) holds iff for all $x > 0$ and $y \geqslant 0$, we have

$$T\,\mathrm{P}\left[\frac{X}{b_X(T)} > x, Y \leqslant y\right] \to x^{-\alpha_X} G(y). \tag{2.17}$$

However,

$$T\,\mathrm{P}\left[\frac{X}{b_X(T)} > x, Y \leqslant y\right] = T\,\mathrm{P}\left[\frac{X}{b_X(T)} > x\right] \mathrm{P}[Y \leqslant y | X > b_X(T)x],$$

and, by the assumption about the nature of the marginal distribution of X or by asymptotic independence, we have,

$$T\,\mathrm{P}\left[\frac{X}{b_X(T)} > x\right] \to x^{-\alpha_L}.$$

Thus (2.17) holds iff

$$\mathrm{P}[Y \leqslant y | X > b_X(T)x] \to G(y)$$

as $T \to \infty$, for all $y \geqslant 0$. This is equivalent to (2.16), since $b_X(T) \to \infty$. □

The required truncated moment condition (2.11) holds, when X and Y are independent random variables with X having regularly varying tails of index $-\alpha_X$ and Y having all moments less than α_Y finite, where $\alpha_X < \alpha_Y$.

Lemma 2.2.4. *Let X and Y be independent random variables taking values in $(0, \infty)$ with respective marginal distributions F_X and F_Y and quantile functions b_X and b_Y. Let the index of regular variation of $\overline{F}_X$ be $-\alpha_X$ and $\mathrm{E}[Y^\delta] < \infty$ for $\delta < \alpha_Y$ with $\alpha_X < \alpha_Y$. Then* (2.11) *holds for $\delta \in (\alpha_X, \alpha_Y)$.*

Proof. We have by independence and $\delta < \alpha_Y$ and Karamata's theorem that as $T \to \infty$,

$$T\,\mathrm{E}\left[\left(\frac{X}{b_X(T)}Y\right)^\delta \mathbf{1}_{\left[\frac{X}{b_X(T)} \leqslant \varepsilon\right]}\right] = \mathrm{E}(Y^\delta)\frac{T}{(b_X(T))^\delta}\,\mathrm{E}\left[X^\delta \mathbf{1}_{\left[\frac{X}{b_X(T)} \leqslant \varepsilon\right]}\right]$$
$$\sim \frac{\alpha_X}{\delta - \alpha_X}\,\mathrm{E}(Y^\delta)\varepsilon^\delta T\overline{F}_L(b_X(T)\varepsilon) \sim \frac{\alpha_X}{\delta - \alpha_X}\,\mathrm{E}(Y^\delta)\varepsilon^{\delta - \alpha_X},$$

which goes to 0 as $\varepsilon \to 0$, since $\delta > \alpha_X$. Thus, for all $\delta \in (\alpha_X, \alpha_Y)$,

$$\lim_{\varepsilon \downarrow 0} \limsup_{T \to \infty} \mathrm{E}\left[\left(\frac{X}{b_X(T)}Y\right)^\delta \mathbf{1}_{\left[\frac{X}{b_X(T)} < \varepsilon\right]}\right] = 0.$$

□

2.3 Products and asymptotic independence

We are now ready to study the product of two asymptotically independent random variables.

Theorem 2.3.1. *Suppose X and Y are two strictly positive, finite random variables, which are asymptotically independent. Assume the moment conditions that* (2.4) *holds with the limit distribution function $\nu_{\alpha_X} \times G$, and G has α_X-th moment finite and* (2.11) *holds. Then*

$$T\,\mathrm{P}\left[\frac{XY}{b_X(T)} > z\right] \sim z^{-\alpha_X} \int_0^\infty u^{\alpha_X}\, G(du),$$

and hence the quantile function of $Z = XY$ satisfies

$$b_Z(T) \sim \left(\int_0^\infty u^{\alpha_X}\, G(du)\right)^{\frac{1}{\alpha_X}} b_X(T).$$

Proof. Let

$$A_\varepsilon = \{(x, y) : \varepsilon < x < \varepsilon^{-1}, xy > z\}.$$

Note A_ε is relatively compact in D and

$$\partial A_\varepsilon = \left(\{\varepsilon\} \times \left[\frac{z}{\varepsilon}, \infty\right]\right) \cup \left(\{\varepsilon^{-1}\} \times [z\varepsilon, \infty]\right) \cup \{(x, y) : \varepsilon < x < \varepsilon^{-1}, xy = z\}.$$

Choose a sequence $\varepsilon_k \downarrow 0$ such that $(\nu_{\alpha_L} \times G)(\partial A_{\varepsilon_k}) = 0$, for all k. Then

$$\liminf_{T\to\infty} T\,\mathrm{P}\left[\frac{X}{b_X(T)} Y > z\right] \geqslant \lim_{T\to\infty} T\,\mathrm{P}\left[\left(\frac{X}{b_X(n)}, Y\right) \in A_{\varepsilon_k}\right] = (\nu_{\alpha_L} \times G)(A_{\varepsilon_k})$$

by (IB). Taking the limit as $k \to \infty$,

$$\liminf_{T\to\infty} T\,\mathrm{P}\left[\frac{X}{b_X(T)} Y > z\right] \geqslant (\nu_{\alpha_L} \times G)(\{(x, y) : xy > z, 0 < x < \infty\})$$

$$= (\nu_{\alpha_L} \times G)(\{(x,y) : xy > z, 0 < x \leq \infty\}),$$

since $\nu_{\alpha_L}(\{\infty\}) = 0$. Also, since $\{(x,y) \in D : x \geq \varepsilon_k^{-1}\}$ is a $\nu_{\alpha_L} \times G$ continuity set, we have,

$$\begin{aligned}
&\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{X}{b_X(T)}Y > z\right] \\
\leqslant& \lim_{T\to\infty} T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}, Y\right) \in A_{\varepsilon_k}\right] + \lim_{T\to\infty} T\,\mathrm{P}\left[\frac{X}{b_X(T)} \geqslant \varepsilon_k^{-1}\right] \\
&+ \limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{X}{b_X(T)}Y > z, \frac{X}{b_X(T)} \leqslant \varepsilon_k\right] \\
=&(\nu_{\alpha_X} \times G)(A_{\varepsilon_k}) + \nu_{\alpha_X}\left(\left[\varepsilon_k^{-1}, \infty\right]\right) \\
&+ \limsup_{T\to\infty} T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}Y\right)\mathbf{1}_{\left[\frac{X}{b_X(T)} \leqslant \varepsilon_k\right]} > z\right] \\
\leqslant&(\nu_{\alpha_X} \times G)(A_{\varepsilon_k}) + \varepsilon_k^{\alpha_X} + z^{-\delta} \limsup_{T\to\infty} T\,\mathrm{E}\left[\left(\frac{X}{b_X(T)}Y\right)^{\delta}\mathbf{1}_{\left[\frac{X}{b_X(T)} \leqslant \varepsilon_k\right]}\right].
\end{aligned}$$

Taking limits as $k \to \infty$, and using (2.11) and the fact $\alpha_L > 0$

$$\begin{aligned}
\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{X}{b_X(T)}Y > z\right] &\leqslant (\nu_{\alpha_L} \times G)(\{(x,y) : xy > z, 0 < x < \infty\}) \\
&= (\nu_{\alpha_L} \times G)(\{(x,y) \in D : xy > z\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{X}{b_X(T)}Y > z\right] &\leqslant (\nu_{\alpha_Y} \times G)(\{(x,y) \in D : xy > z\} \\
&= \int_0^\infty \nu_{\alpha_X}\left(\left(\frac{z}{u}, \infty\right]\right) G(du) = z^{-\alpha_X} \int_0^\infty u^{\alpha_X}\, G(du).
\end{aligned}$$

Therefore

$$\lim_{T\to\infty} T\,\mathrm{P}\left[\frac{X}{b_X(T)}Y\left(\int_0^\infty u^{\alpha_X}\, G(du)\right)^{-\frac{1}{\alpha_L}} > z\right] = z^{-\alpha_X},$$

and hence

$$b_Z(T) \sim \left(\int_0^\infty u^{\alpha_X} \, G(du) \right)^{\frac{1}{\alpha_X}} b_X(T) \sim b_X \left(\int_0^\infty u^{\alpha_X} \, G(du) T \right).$$

□

Thus, under further moment conditions, the product of two asymptotically independent random variables with regularly varying tails again has a regularly varying tail, whose behavior is similar to that of the heavier one of the two factors. This is a generalization of Breiman (1965)'s result about the product of two independent random variables.

The product of two random variables, which are not asymptotically independent, but whose tails satisfy multivariate regular variation, offers contrasting behavior to the asymptotically independent case.

Proposition 2.3.1. *Suppose (X, Y) is multivariate regularly varying in the sense that there exists regularly varying functions b_X, b_Y, such that*

$$T \, \mathrm{P}\left[\left(\frac{X}{b_X(T)}, \frac{Y}{b_Y(T)} \right) \in \cdot \right] \xrightarrow{\mathrm{v}} \nu(\cdot) \not\equiv 0 \tag{2.18}$$

on $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$ *and* $\nu\Big((\{\infty\} \times (0, \infty]) \cup ((0, \infty] \times \{\infty\}) \Big) = 0$ *and* $\nu\big((\mathbf{0}, \boldsymbol{\infty}]\big) > 0$.

Then for some $\alpha_X > 0$, $\alpha_Y > 0$,we have

$$\mathrm{P}[X > \cdot] \in RV_{-\alpha_X},$$

$$\mathrm{P}[Y > \cdot] \in RV_{-\alpha_Y}$$

and

$$\mathrm{P}[XY > \cdot] \in RV_{-\frac{\alpha_X \alpha_Y}{\alpha_X + \alpha_Y}}.$$

Proof. Let b_X and b_Y be regularly varying with indices $1/\alpha_X$ and $1/\alpha_Y$ respectively. Now, for any $x > 0$, such that $(x, \infty] \times [0, \infty]$ is a ν-continuity set, we have,

$$T\,\mathrm{P}\Big[\frac{X}{b_X(T)} > x\Big] = T\,\mathrm{P}\Big[\Big(\frac{X}{b_X(T)}, \frac{Y}{b_Y(T)}\Big) \in (x, \infty] \times [0, \infty]\Big] \to \nu\big((x, \infty] \times [0, \infty]\big),$$

where we define the limit to be K_x, which is positive and finite for some $x_0 > 0$, since $\nu\big((\mathbf{0}, \boldsymbol{\infty}]\big) > 0$.

Now, we have, for any $x > 0$,

$$\begin{aligned} T\,\mathrm{P}\left[\frac{X}{b_X(T)} > x\right] &= T\,\mathrm{P}\left[\frac{X}{\frac{x}{x_0} b_X(T)} > x_0\right] \\ &\sim \left(\frac{x}{x_0}\right)^{-\alpha} \left(\frac{x}{x_0}\right)^{\alpha} T\,\mathrm{P}\left[\frac{X}{b_X\Big(\big(\frac{x}{x_0}\big)^{\alpha} T\Big)} > x_0\right] \\ &\to (x_0)^{\alpha} K_{x_0} x^{-\alpha} \end{aligned}$$

and hence $\mathrm{P}[X > \cdot] \in RV_{-\alpha_X}$. Similarly, we can check that $\mathrm{P}[Y > \cdot] \in RV_{-\alpha_Y}$.

Define for $z > 0$ and any positive number K,

$$A_{K,z} := \{(x, y) : xy > z, x \leqslant K, y \leqslant K\}.$$

Then, for any $z > 0$, we have,

$$T\,\mathrm{P}\left[\frac{XY}{b_X(T) b_Y(T)} > z\right] \geqslant T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}, \frac{Y}{b_Y(T)}\right) \in A_{K,z}\right].$$

Then, letting T got to ∞ first, and then letting K go to ∞ through a sequence so that $A_{K,z}$ is a ν-continuity set, we have

$$\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{XY}{b_X(T) b_Y(T)} > z\right] \geqslant \nu\big(\{(x, y) : xy > z\}\big).$$

On the other hand, we have,

$$\begin{aligned} T\,\mathrm{P}\left[\frac{XY}{b_X(T)b_Y(T)} > z\right] \;&\leqslant\; T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}, \frac{Y}{b_Y(T)}\right) \in A_{K,z}\right] \\ &\quad + T\,\mathrm{P}\left[\frac{X}{b_X(T)} > K\right] + T\,\mathrm{P}\left[\frac{X}{b_X(T)} > K\right] \end{aligned}$$

Now, by regularly varying tails of X and Y, the last two terms converge to $K^{-\alpha_X}$ and $K^{-\alpha_Y}$ respectively, as $T \to \infty$. Then letting K go to ∞ through a sequence so that $A_{K,z}$ is a ν-continuity set, they go to zero. Hence, we have

$$\liminf_{T\to\infty} T\,\mathrm{P}\left[\frac{XY}{b_X(T)b_Y(T)} > z\right] \leqslant \nu\big(\{(x,y) : xy > z\}\big).$$

Thus,

$$T\,\mathrm{P}\left[\frac{XY}{b_X(T)b_Y(T)} > z\right] \to \nu\big(\{(x,y) : xy > z\}\big).$$

Then, since $b_X b_Y$ is a regularly varying function of index $\frac{\alpha_X+\alpha_Y}{\alpha_X\alpha_Y}$, and $\nu\big(\{(x,y) : xy > z\}\big) > 0$ for some $z > 0$, arguing as in the case of X, we have the required result. □

In Proposition 2.3.1, if α_X and α_Y are between 1 and 2, i.e., X and Y have finite mean but infinite variance, then the product XY has a regularly varying tail of index $-\frac{\alpha_X\alpha_Y}{\alpha_X+\alpha_Y} \in (\frac{1}{2}, 1)$, i.e., the product has a much heavier tail with infinite mean. This result contrasts with Theorem 2.3.1, where the product of asymptotically independent random variables has tail behavior similar to the factor with the heavier tail.

The following example shows that asymptotic independence as defined here is not enough to conclude something meaningful about the product, and the truncated moment condition (2.11) cannot be dropped. Condition (2.4) holds, but condition

(2.11) fails and we are unable to conclude anything meaningful about the tail behavior of the product using the tail behavior of the factors.

Example 2.3.1. Suppose we have independent vectors (U_1, V_1), (U_2, V_2) which are independent of the Bernoulli random variable B with probability of success 0.5. We assume

(i) The random variables (U_1, V_1) are independent with

$$\mathrm{P}[U_1 > \cdot] \in RV_{-\alpha_1},\ \mathrm{P}[V_1 > \cdot] \in RV_{-\alpha_2}$$

with

$$1 < \alpha_2 < \alpha_1 < 2,$$

so that V_1 has the heavier tail. Let b_1 and b_2 be quantile functions of U_2 and V_2 respectively.

(ii) The random variables (U_2, V_2) are dependent with multivariate regularly varying distribution in the sense that there exists regularly varying functions b_3 and b_4 of indices $1/\alpha_3$ and $1/\alpha_4$ respectively, such that

$$T\,\mathrm{P}\left[\left(\frac{U_2}{b_3(T)}, \frac{V_2}{b_4(T)}\right) \in \cdot\right] \xrightarrow{\mathrm{v}} \nu^*(\cdot)$$

on $[\mathbf{0}, \boldsymbol{\infty}]\backslash\{\mathbf{0}\}$, where $\nu^*\big((\mathbf{0}, \boldsymbol{\infty}]\big) > 0$ but $\nu^*\big(\{\infty\}\times(0,\infty]\cup(0,\infty]\times\{\infty\}\big) = 0$ and $1 < \alpha_4 < \alpha_3 < 2$. Then by Proposition 2.3.1,

$$\mathrm{P}[U_2 > \cdot] \in RV_{-\alpha_3} \text{ and } \mathrm{P}[V_2 > \cdot] \in RV_{-\alpha_4},$$

and V_2 has a heavier tail.

(iii) Assume further that

$$\alpha_1 < \alpha_3,\ \alpha_2 < \alpha_4, \tag{2.19}$$

and define

$$(X, Y) = B(U_2, V_2) + (1 - B)(U_1, V_1).$$

We have the following conclusions.

1. We have

$$P[X > x] = \frac{1}{2}P[U_1 > x] + \frac{1}{2}P[U_2 > x] \in RV_{-\alpha_1},$$
$$P[Y > x] = \frac{1}{2}P[V_1 > x] + \frac{1}{2}P[V_2 > x] \in RV_{-\alpha_2}$$

so that Y has the heavier tail.

2. Define the measure $\nu_0 = \frac{1}{2}\varepsilon_0 \times \nu_{\alpha_2} + \frac{1}{2}\nu_{\alpha_1} \times \varepsilon_0$, and then on $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$

$$T\,\mathrm{P}\left[\left(\frac{X}{b_1(T)}, \frac{Y}{b_2(T)}\right) \in \cdot\right] \xrightarrow{\mathrm{v}} \nu_0.$$

To see this, note

$$T\,\mathrm{P}\left[\left(\frac{X}{b_1(T)}, \frac{Y}{b_2(T)}\right) \in \cdot\right] = \frac{T}{2}\,\mathrm{P}\left[\left(\frac{U_1}{b_1(T)}, \frac{V_1}{b_2(T)}\right) \in \cdot\right] + \frac{T}{2}\,\mathrm{P}\left[\left(\frac{U_2}{b_1(T)}, \frac{V_2}{b_2(T)}\right) \in \cdot\right],$$

and the second term goes to zero since $b_i \in RV_{\alpha_i^{-1}}$, $\quad i = 1, \ldots, 4$, and (2.19) imply

$$b_3(T) = o(b_1(T)), \quad b_4(T) = o(b_2(T)),$$

and the vague limit of the first term is ν_0 on $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$.

3. We obtain a different vague limit on the cone $(\mathbf{0},\boldsymbol{\infty}]$. For $x>0, y>0$, we have, using assumption (ii), that

$$\begin{aligned}
&T\,\mathrm{P}\left[\left(\frac{X}{b_3(T)},\frac{Y}{b_4(T)}\right)\in(x,\infty]\times(y,\infty]\right]\\
=&\frac{1}{2}T\,\mathrm{P}\left[\frac{U_2}{b_3(T)}>x,\frac{V_2}{b_4(T)}>y\right]+\frac{1}{2}T\,\mathrm{P}\left[\frac{U_1}{b_3(T)}>x\right]\mathrm{P}\left[\frac{V_1}{b_4(T)}>y\right]\\
\to&\frac{1}{2}\nu^*\big((x,\infty]\times(y,\infty]\big)+0
\end{aligned}$$

and thus the vague limit on $(\mathbf{0},\boldsymbol{\infty}]$ is $\frac{1}{2}\nu^*$. To verify the limit of 0 for the second term, note $b_i^{\leftarrow}\in RV_{\alpha_i}$, $i=1,\dots,4$ and as $T\to\infty$

$$\begin{aligned}
TP[U_1>b_3(T)x]P[V_1>b_4(T)y]\sim&\frac{T}{b_1^{\leftarrow}(b_3(T)x)b_2^{\leftarrow}(b_4(T)y)}\\
\sim&\frac{Tx^{-\alpha_1}y^{-\alpha_2}}{b_1^{\leftarrow}\circ b_3(T)b_2^{\leftarrow}\circ b_4(T)}
\end{aligned}$$

which as a function of T is regularly varying with index $1-\frac{\alpha_1}{\alpha_3}-\frac{\alpha_2}{\alpha_4}$. The result follows if we show $\frac{\alpha_1}{\alpha_3}+\frac{\alpha_2}{\alpha_4}>1$. However

$$\frac{\alpha_1}{\alpha_3}+\frac{\alpha_2}{\alpha_4}>\frac{1}{2}(\alpha_1+\alpha_2)>\frac{2}{2}=1,$$

since $1<\alpha_i<2$, for $i=1,\dots,4$.

4. We have that

$$\mathrm{P}[XY>\cdot]\in RV_{-\frac{\alpha_3\alpha_4}{\alpha_3+\alpha_4}},\ \text{and}\ \frac{1}{2}<\frac{\alpha_3\alpha_4}{\alpha_3+\alpha_4}<1.$$

Note

$$XY=B\,U_2V_2+(1-B)\,U_1V_1,$$

so that

$$\mathrm{P}[XY>\cdot]=\frac{1}{2}\,\mathrm{P}[U_2V_2>\cdot]+\frac{1}{2}\,\mathrm{P}[U_1V_1>\cdot].$$

From Breiman (1965) or Lemma 2.2.4 and Theorem 2.3.1, we have $\mathrm{P}[U_1V_1 > \cdot] \in RV_{-\alpha_2}$, and from Proposition 2.3.1, $\mathrm{P}[U_1V_1 > \cdot] \in RV_{-\frac{\alpha_3\alpha_4}{\alpha_3+\alpha_4}}$. But

$$\frac{\alpha_3\alpha_4}{\alpha_3+\alpha_4} < 1 < \alpha_2,$$

and thus

$$\mathrm{P}[XY > \cdot] \sim \frac{1}{2}\,\mathrm{P}[U_2V_2 > \cdot],$$

which is surprisingly heavy in view of the fact that

$$\mathrm{P}[X > \cdot] \in RV_{-\alpha_1} \text{ and } \mathrm{P}[Y > \cdot] \in RV_{-\alpha_2},$$

and $1 < \alpha_1,\ \alpha_2 < 2$.

We conclude that the tail of the product is hidden from a condition like (2.1) or knowledge of the marginal distributions.

5. We have Y is asymptotically independent of X, but the truncated moment condition (2.11) fails. For the asymptotic independence condition (2.4), we have,

$$\begin{aligned} T\,\mathrm{P}\left[\frac{X}{b_1(T)} > x, Y \leqslant y\right] =& \frac{T}{2}\,\mathrm{P}\left[\frac{U_1}{b_1(T)} > x, V_1 \leqslant y\right] \\ &+ \frac{T}{2}\,\mathrm{P}\left[\frac{U_2}{b_1(T)} > x\right]\mathrm{P}[V_2 \leqslant y]. \end{aligned}$$

Then the first term converges to $\left(\frac{1}{2}\nu_{\alpha_1} \times F_{V_1}\right)\left((x,\infty] \times [0,y]\right)$, where F_{V_1} denotes the distribution function of V_1. Also,

$$\frac{T}{2}\,\mathrm{P}\left[\frac{U_2}{b_1(T)} > x, V_2 \leqslant y\right] \leqslant \frac{T}{2}\,\mathrm{P}\left[\frac{U_2}{b_3(T)} > \frac{b_1(T)}{b_3(T)}x\right] \to 0.$$

since $b_3(T) = o(b_1(T))$. Thus if we define $b_X(T) = 2^{-1/\alpha_1} b_1(T)$, then

$$T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}, Y\right) \in \cdot\right] \xrightarrow{\mathrm{v}} \left(\nu_{\alpha_1} \times F_{V_1}\right)(\cdot)$$

on D.

For the truncated moment condition (2.11), observe that,

$$\begin{aligned} T\,\mathrm{E}\left[\left(\frac{X}{b_1(T)}Y\right)^{\delta} \mathbf{1}_{\left[\frac{X}{b_1(T)}<\varepsilon\right]}\right] &= \frac{T}{2}\,\mathrm{E}\left[\left(\frac{U_1}{b_1(T)}\right)^{\delta} \mathbf{1}_{\left[\frac{U_1}{b_1(T)}<\varepsilon\right]}\right] \mathrm{E}\left[V_1^{\delta}\right] \\ &\quad + \frac{T}{2}\,\mathrm{E}\left[\left(\frac{U_2}{b_1(T)}V_2\right)^{\delta} \mathbf{1}_{\left[\frac{U_2}{b_1(T)}<\varepsilon\right]}\right] \end{aligned} \tag{2.20}$$

So, if $\delta \geqslant \alpha_2$, then $\mathrm{E}\left(V_1^{\delta}\right) = \infty$ and hence condition (2.11) fails. Also a closer look at the proof of Lemma 2.2.4 will show

$$\lim_{\varepsilon \downarrow 0} \limsup_{T\to\infty} T\,\mathrm{E}\left[\left(\frac{X}{b_1(T)}\right)^{\delta} \mathbf{1}_{\left[\frac{U_1}{b_1(T)}<\varepsilon\right]}\right] = 0$$

iff $\delta > \alpha_1 > \alpha_2$. Thus, for no positive δ, the first term in the right side of (2.20) can go to zero. Since $\frac{b_X(T)}{b_1(T)} = 2^{-1/\alpha_1}$, condition (2.11) fails even if we scale by b_X.

We can also similarly check that X is asymptotically independent of Y with $b_Y = 2^{-1/\alpha_2} b_2$, and the limiting distribution $\nu_{\alpha_2} \times F_{U_1}$. However, the truncated moment condition (2.11) still fails for the pair (Y, X).

So we observe that the asymptotic independence condition (2.4) is insufficient to conclude something about the tail behavior of the product and the truncated moment condition (2.11) is indispensable.

2.4 The infinite source Poisson model

We use the concept of asymptotic independence to study a network traffic model with random transmission rate. Consider the M/G/∞ input model of incoming traffic to a communication network. Let $\{\Gamma_k, k \geq 1\}$ denote the points of a homogeneous Poisson process on $[0, \infty)$ with rate λ. Suppose at time Γ_k, a source starts a transmission, and continues to transmit for a period of length L_k, at a fixed rate R_k, both chosen at random. The total volume of traffic injected into the network between 0 and t is

$$X(t) = \sum_{k=1}^{\infty} ((t - \Gamma_k)_+ \wedge L_k) R_k, \ t \geq 0. \tag{2.21}$$

We assume that (L_k, R_k) are i.i.d. with joint distribution function F and let F_L and F_R be the marginal distributions of L_k and R_k respectively. We make the following assumptions on the distribution of (L_k, R_k):

$$F(\mathbb{R}_+^2) = 1, \text{ where } \mathbb{R}_+^2 = (\mathbf{0}, \boldsymbol{\infty}), \tag{2.22}$$

$$\overline{F}_L(x) = \in RV_{-\alpha_L} \text{ and } \overline{F}_R(x) \in RV_{-\alpha_R}, \ \alpha_L, \alpha_R \in (1, 2). \tag{2.23}$$

We denote the quantile functions of L_1 and R_1 by b_L and b_R respectively. Also, let b_P denote the quantile function of the product $L_1 R_1$, namely,

$$b_P(T) = \inf \left\{ z : \mathrm{P}[L_1 R_1 > z] \leqslant T^{-1} \right\}.$$

We make further assumptions on the joint distribution F to analyze the total traffic process. We have two sets of assumptions depending on which of the two variables, the transmission length and the transmission rate, has heavier tail.

Case I The transmission length has a heavier tail.

(IA) $\alpha_L < \alpha_R$.

(IB) R_1 is asymptotically independent of L_1, namely

$$T\,\mathrm{P}\left[\left(\frac{L_1}{b_L(T)}, R_1\right) \in \cdot\right] \xrightarrow{\mathrm{v}} \nu_{\alpha_L} \times G(\cdot) \text{ on } D := (0,\infty] \times [0,\infty],$$

where G is a probability measure with $G(\mathbb{R}_+) = 1$ with finite α_L-th moment.

(IC) (L_1, R_1) satisfy the truncated moment condition (2.11), namely,

$$\lim_{\varepsilon\downarrow 0}\limsup_{T\to\infty} T\,\mathrm{E}\left[\left(\frac{L_1}{b_L(T)}R_1\right)^{\delta}\mathbf{1}_{\left[\frac{L_1}{b_L(T)}<\varepsilon\right]}\right] = 0 \text{ for some } \delta > 0.$$

Case II The transmission rate has a heavier tail.

(IIA) $\alpha_R < \alpha_L$.

(IIB) L_1 is asymptotically independent of R_1, namely

$$T\,\mathrm{P}\left[\left(\frac{R_1}{b_R(T)}, L_1\right) \in \cdot\right] \xrightarrow{\mathrm{v}} \nu_{\alpha_R} \times G(\cdot) \text{ on } D := (0,\infty] \times [0,\infty],$$

where G is a probability measure with $G(\mathbb{R}_+) = 1$ with finite α_R-th moment.

(IIC) (R_1, L_1) satisfy the truncated moment condition (2.11), namely,

$$\lim_{\varepsilon\downarrow 0}\limsup_{T\to\infty} T\,\mathrm{E}\left[\left(\frac{R_1}{b_R(T)}L_1\right)^{\delta}\mathbf{1}_{\left[\frac{R_1}{b_R(T)}<\varepsilon\right]}\right] = 0 \text{ for some } \delta > 0.$$

We need to distinguish between above two cases since the object of our analysis, the total traffic process, $\{X(t) : t \geqslant 0\}$ is not symmetric in the transmission rate and the transmission length. In light of Theorem 2.2.1, we can rewrite the conditions (IB) and (IIB) in terms of multivariate regular variation on the cone D:

(IB′) $T\,\mathrm{P}\left[\dfrac{(L_1, L_1R_1)}{b_L(T)} \in \cdot\right] \stackrel{\mathrm{v}}{\to} \nu(\cdot)$ on D,
where ν is a homogeneous Radon measure of order $-\alpha_L$.

(IIB′) $T\,\mathrm{P}\left[\dfrac{(R_1, L_1R_1)}{b_R(T)} \in \cdot\right] \stackrel{\mathrm{v}}{\to} \nu(\cdot)$ on D,
where ν is a homogeneous Radon measure of order $-\alpha_R$.

Using Corollary 2.2.1 and conditions (IC) and (IIC), we can extend the multivariate regular variation in the conditions (IB′) and (IIB′) to the cone $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$.

2.5 Lévy Approximation

We now give a Lévy approximation to the cumulative input process when input rates are random. Observe from Theorem 2.3.1 that the product L_1R_1 has a tail of index $\alpha_P := \alpha_L \wedge \alpha_R$, and hence has a finite mean. Let us call it $\mu_P := \mathrm{E}(L_1R_1)$. Then we have the following asymptotic behavior of the process $X(t)$, defined as in (2.21), measured at a large scale.

Theorem 2.5.1. *Assume* (2.22)–(2.23) *and* (IA)–(IC) *or* (IIA)–(IIC)*. Define*

$$Y_T(t) = \frac{X(Tt) - \lambda T t \mu_P}{b_P(T)}.$$

Then

$$Y_T(\cdot) \stackrel{\mathrm{fidi}}{\to} Z_{\alpha_P}(\cdot),$$

where the above convergence is the usual weak convergence of finite dimensional distributions and Z_α *is a mean* 0*, skewness* 1*,* α*-stable Lévy motion with scale parameter* $(\lambda/C_\alpha)^{1/\alpha}$ *and*

$$C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}.$$

We shall prove the theorem in two parts. First we prove the one-dimensional convergence and then we prove the finite dimensional convergence for any number of dimensions.

For the analysis, it helps to consider the Poisson point process,

$$M = \sum_{k=1}^{\infty} \varepsilon_{(\Gamma_k, L_k, R_k)}$$

with mean measure $\lambda\, dt \times F$ on $(0,\infty)^3$.

The random variable $X(T)$ is a function of the random measure restricted to $\mathcal{R}(T) = \{(x,y,z) \in (0,\infty)^3 : x < T\}$. It helps to split $\mathcal{R}(T)$ into two disjoint sets

$$\mathcal{R}_1(T) = \{(x,y,z) \in (0,\infty)^3 : x + y \leqslant T\},$$
$$\mathcal{R}_2(T) = \{(x,y,z) \in (0,\infty)^3 : x < T < x + y\}.$$

The corresponding input processes are

$$X_1(T) = \sum_{k=1}^{\infty} R_k L_k \mathbf{1}_{[(\Gamma_k, L_k, R_k) \in \mathcal{R}_1(T)]}, \tag{2.24}$$

$$X_2(T) = \sum_{k=1}^{\infty} R_k (T - \Gamma_k) \mathbf{1}_{[(\Gamma_k, L_k, R_k) \in \mathcal{R}_2(T)]}. \tag{2.25}$$

with $X(T) = X_1(T) + X_2(T)$. Since $X_i(T)$, $i = 1, 2$ are functions of $M|_{\mathcal{R}_i(T)}$, $i = 1, 2$ respectively, and $\mathcal{R}_1(T) \cap \mathcal{R}_2(T) = \emptyset$, we have $X_1(T)$ and $X_2(T)$ are independent.

Now,

$$\mathrm{E}\left[M(\mathcal{R}_1(T))\right] = \int\limits_{x=0}^{T} \int\limits_{y\in(0,T-x]} \int\limits_{z=0}^{\infty} \lambda dx F(dy,dz) = \lambda \int\limits_{x=0}^{T} F_L(T-x)dx = \lambda \int\limits_{0}^{T} F_L(x)dx,$$

which we denote by $\lambda\widehat{F_L}(T)$, and, similarly, as $T\to\infty$,

$$\mathrm{E}\left[M\left(\mathcal{R}_2(T)\right)\right] = \lambda \int\limits_{x=0}^{T} \overline{F}_L(x)\,dx =: \lambda m_L(T) \to \lambda\mu_L,$$

where $\mu_L = \mathrm{E}(L_1) < \infty$ as $\alpha_L > 1$. Since $\mathrm{E}\left(M\left(\mathcal{R}_i(T)\right)\right) < \infty$, $i = 1, 2$, we have the representation

$$M|_{\mathcal{R}_1(T)} \stackrel{\mathrm{d}}{=} \sum_{k=1}^{P(T)} \varepsilon_{\left(\tau_k^{(T)}, V_k^{(T)}, S_k^{(T)}\right)},$$

where $P(T) \sim \mathrm{POI}(\lambda\widehat{F}_L(T))$; i. e., a Poisson random variable with parameter $\lambda\widehat{F}_L(T)$, independent of the i.i.d. random vectors

$$\left(\tau_k^{(T)}, V_k^{(T)}, S_k^{(T)}\right) \sim \left.\frac{dx\,F(dy,dz)}{\widehat{F_L}(T)}\right|_{\mathcal{R}_1(T)}, \tag{2.26}$$

where the above statement means the vector on the left has a distribution given on the right. Similarly

$$M|_{\mathcal{R}_2(T)} \stackrel{\mathrm{d}}{=} \sum_{k=1}^{P'(T)} \varepsilon_{\left(\tau_k'^{(T)}, V_k'^{(T)}, S_k'^{(T)}\right)},$$

where $P'(T) \sim \mathrm{POI}(\lambda m_L(T))$ independent of the i.i.d. random vectors

$$\left(\tau_k'^{(T)}, V_k'^{(T)}, S_k'^{(T)}\right) \sim \left.\frac{dx\,F(dy,dz)}{m_L(T)}\right|_{\mathcal{R}_2(T)}. \tag{2.27}$$

The key step in the entire analysis is to study the tail behavior of $V_1^{(T)}S_1^{(T)}$.

Lemma 2.5.1. *Under the assumptions of Theorem 2.5.1,*

$$T\,\mathrm{P}\left[\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)} > w\right] \to w^{-\alpha} \text{ as } T\to\infty,$$

where the convergence is uniform for $w\in[a,\infty)$, $\forall a > 0$.

Proof. We consider Case (I) assumptions only. The analysis in Case (II) is similar and details can be found in the technical report by Maulik et al. (2000).

Using (2.26) and the fact that $\frac{1}{T}\widehat{F}_L(T) = \frac{1}{T}\int_0^T F_L(u)\,du \sim F_L(T) \to 1$, we have,

$$T\,\mathrm{P}[V_1^{(T)}S_1^{(T)} > b_L(T)w] \sim \iint\limits_{\substack{y\in(0,T)\\ yz>b_L(T)w}} \int\limits_y^T du\,F(dy,dz) = \int\limits_0^T \iint\limits_{\substack{y\in(0,u]\\ yz>b_L(T)w}} F(dy,dz)\,du$$

$$= \frac{1}{T}\int\limits_0^T T\,\mathrm{P}\left[L_1 \leqslant u, \frac{L_1}{b_L(T)}R_1 > w\right]du \tag{2.28}$$

$$= \frac{b_L(T)}{T}\int\limits_0^{\frac{T}{b_L(T)}} T\,\mathrm{P}\left[\frac{(L_1, L_1R_1)}{b_L(T)} \in (0,u]\times(w,\infty)\right]du. \tag{2.29}$$

Now, $(0,u]\times(w,\infty)$ is bounded away from $\mathbf{0}$ and hence is relatively compact in $[\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\}$ and has boundary with ν-measure zero. Hence by the assumptions (IB) or (IC) and using Corollary 2.2.1, we have, as $T \to \infty$,

$$T\,\mathrm{P}\left[\frac{(L_1, L_1R_1)}{b_L(T)} \in (0,u]\times(w,\infty)\right] \to \nu\big((0,u]\times(w,\infty)\big) = (\nu_{\alpha_L}\times G)(A_{u,w}),$$

where $A_{u,w} = \{(y,z) : y \leqslant u, yz > w\}$ and

$$\lim_{u\to\infty}(\nu_{\alpha_L}\times G)(A_{u,w}) = w^{-\alpha_L}\int\limits_0^\infty u^{\alpha_L}\,G(du) =: c_w < \infty.$$

Also, the integrand on the right side of (2.29) is bounded above by $T\,\mathrm{P}[L_1R_1 > b_L(T)w]$, which, by Theorem 2.3.1, converges to c_w. Hence, from (2.29), we get,

$$T\,\mathrm{P}\left[\frac{V_1^{(T)}S_1^{(T)}}{b_L(T)} > w\right] \to w^{-\alpha_L}\int\limits_0^\infty u^{\alpha_L}\,G(du).$$

and, hence,

$$T\,\mathrm{P}\left[\frac{V_1^{(T)}S_1^{(T)}}{\tilde{b}(T)} > w\right] \to w^{-\alpha_P}, \tag{2.30}$$

where $\tilde{b}(T) := (\int_0^\infty u^{\alpha_P}\, G(du))^{1/\alpha_P} b_L(T) \sim b_P(T)$ by Theorem 2.3.1. The left side of (2.30) is monotone non-increasing and the right side is continuous in $(0,\infty)$. Hence, (cf. Resnick, 1987, pg. 1) pointwise convergence implies locally uniform convergence in $(0,\infty)$, and thus

$$T\,\mathrm{P}\left[\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)} > w\right] \to w^{-\alpha_P}. \tag{2.31}$$

Since the left side in (2.31) above is monotone non-increasing with a continuous pointwise limit on $(0,\infty)$ which has a finite limit at ∞, the convergence is uniform on $[a,\infty)$. □

To complete the proof of Theorem 2.5.1 for one-dimensional convergence, we need the following lemma studying the moment conditions of $V_1^{(T)}S_1^{(T)}/b_P(T)$, which follows easily using Lemma 2.5.1 and Karamata's theorem.

Lemma 2.5.2. *Under the assumptions of Theorem 2.5.1, we have the following results:*

$$\lim_{M\to\infty}\limsup_{T\to\infty} T\,\mathrm{E}\left(\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}\mathbf{1}_{\left[\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}>M\right]}\right) = 0, \tag{2.32}$$

$$\limsup_{T\to\infty} T\,\mathrm{Var}\left(\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}\mathbf{1}_{\left[\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}\leqslant M\right]}\right) \leqslant \frac{\alpha_P}{2-\alpha_P}M^{2-\alpha_P} \quad \forall M>0, \tag{2.33}$$

and hence,

$$\lim_{\varepsilon\downarrow 0}\limsup_{T\to\infty} T\,\mathrm{Var}\left(\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}\mathbf{1}_{\left[\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}\leqslant\varepsilon\right]}\right) = 0. \tag{2.34}$$

Further,

$$\limsup_{T\to\infty} b_P(T)\,\mathrm{E}\left(\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}\right) \leqslant \mathrm{E}[L_1 R_1] \tag{2.35}$$

and hence,

$$\lim_{T\to\infty} \mathrm{E}\left(\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}\right) = 0. \tag{2.36}$$

Now, we prove Theorem 2.5.1 for the process X_1 defined in (2.24), albeit with a different centering.

Theorem 2.5.2. *Under assumptions* (2.22)–(2.23) *and* (IA)–(IC) *or* (IIA)–(IIC), *we have*

$$\frac{X_1(T) - P(T)\,\mathrm{E}\left(V_1^{(T)}S_1^{(T)}\right)}{b_P(T)} \Rightarrow Z_{\alpha_P}(1), \tag{2.37}$$

where Z_{α_P} is as defined in Theorem 2.5.1.

Proof. As in Section 2 of Resnick and Samorodnitsky (2000), using (2.32), (2.34), (2.36), we get

$$S_T \Rightarrow S_{\alpha_P} \text{ in } D([0,\infty)),$$

where

$$S_T(t) := \sum_{k=1}^{\lfloor Tt \rfloor}\left[\frac{V_k^{(T)}S_k^{(T)}}{b_P(T)} - \mathrm{E}\left(\frac{V_k^{(T)}S_k^{(T)}}{b_P(T)}\right)\right]$$

and S_{α_P} is an α_P-Lévy motion with the skewness parameter 1, mean 0, and scaling parameter $C_{\alpha_P}^{-\frac{1}{\alpha_P}}$.

Since $P(T) \sim \mathrm{POI}(\lambda \widehat{F}_L(T))$ and $\lambda \widehat{F}_L(T) \sim \lambda T \to \infty$, we have $P(T)/T \Rightarrow \lambda$ in $[0,\infty)$. By independence of S_T and $P(T)/T$, we have

$$\left(S_T, \frac{P(T)}{T}\right) \Rightarrow (S_{\alpha_P}, \lambda) \text{ in } D([0,\infty)) \times [0,\infty).$$

Hence, by Whitt (1980),

$$S_T\left(\frac{P(T)}{T}\right) \Rightarrow S_{\alpha_P}(\lambda) \text{ in } \mathbb{R}.$$

Thus,

$$S_T\left(\frac{P(T)}{T}\right) = \frac{X_1(T)}{b_P(T)} - P(T)\,\mathrm{E}\left(\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}\right) \Rightarrow S_{\alpha_P}(\lambda) \text{ in } \mathbb{R}.$$

Note, $S_{\alpha_P}(\lambda)$ is α_P-stable random variable with skewness parameter 1, mean 0 and scaling parameter $(\lambda/C_{\alpha_P})^{1/\alpha_P}$ and hence has same distribution as $Z_{\alpha_P}(1)$, and the result is proved. $\square$

We prove X_2 is negligible in the following theorem.

Theorem 2.5.3. *If X_2 is defined as in (2.25), then*

$$\frac{X_2(T)}{b_P(T)} \xrightarrow{P} 0. \tag{2.38}$$

Proof. Again, we consider Case (I) only. The details of the proof of Case (II), which are similar, can be found in Maulik et al. (2000). It is enough to show $\frac{X_2(T)}{b_L(T)} \xrightarrow{P} 0$, since $\frac{b_P(T)}{b_L(T)}$ converges to a constant, which is positive and finite.

Fix $\varepsilon > 0$, $\eta > 0$. Observe that $P'(T) \sim \mathrm{POI}(\lambda m_L(T))$ and $m_L(T) \to \mu_L$. Thus, $P'(T) \Rightarrow P'$, where $P' \sim \mathrm{POI}(\lambda\mu_L)$. Choose M such that $\mathrm{P}[P' > M] < \frac{\varepsilon}{2}$. Then for all large enough T, we have $\mathrm{P}[P'(T) > M] < \frac{\varepsilon}{2}$ and

$$\begin{aligned}
\mathrm{P}\left[\frac{X_2(T)}{b_L(T)} > \eta\right] &\leqslant \mathrm{P}\left[\frac{X_2(T)}{b(T)} > \eta, P'(T) \leqslant M\right] + \mathrm{P}[P'(T) > M] \\
&\leqslant \mathrm{P}\left[\sum_{k=1}^{M}\left(T - \tau_k'^{(T)}\right)\frac{S_k'^{(T)}}{b_L(T)} > \eta\right] + \frac{\varepsilon}{2} \\
&\leqslant M\,\mathrm{P}\left[\left(T - \tau_1'^{(T)}\right)\frac{S_1'^{(T)}}{b_L(T)} > \frac{\eta}{M}\right] + \frac{\varepsilon}{2}.
\end{aligned}$$

Thus, it is enough to show that $\left(T - \tau_1'^{(T)}\right) S_1'^{(T)}/b(T)$ converges to 0 in probability. Now, by (2.27), we have,

$$\mathrm{P}\left[T - \tau_1'^{(T)} \leqslant s\right] = \mathrm{P}\left[\tau_1'^{(T)} \geqslant T - s\right] = \frac{\int\limits_{x=T-s}^{T} \overline{F}_L(T-x)\,dx}{m_L(T)} = \int\limits_0^s \frac{\overline{F}_L(x)}{m_L(T)}\,dx.$$

Thus, $T - \tau_1'^{(T)}$ has density $\overline{F}_L(\cdot)/m_L(T)$, supported on $(0, T)$, which converges pointwise to a density function $\overline{F}_L(\cdot)/\mu_L$, supported on $\mathbb{R}_+$. Hence, by Scheffé's theorem, $T - \tau_1'^{(T)}$ converges weakly to a positive random variable with density $\overline{F}_L(\cdot)/\mu_L$. So it is enough to show, by Slutsky's theorem, that $S_1'^{(T)}/b(T)$ is negligible in probability.

Fix $\eta > 0$. Now observe, by (2.27),

$$\begin{aligned}
\mathrm{P}\left[\frac{S_1'^{(T)}}{b_L(T)} > \eta\right] &= \frac{1}{m_L(T)} \int\limits_0^T \mathrm{P}\left[L_1 > T - x, \frac{R_1}{b_L(T)} > \eta\right] dx \\
&= \frac{1}{m_L(T)} \int\limits_0^T \mathrm{P}\left[L_1 > x, \frac{R_1}{b_L(T)} > \eta\right] dx \\
&\leqslant \frac{1}{m_L(T)} T\,\mathrm{P}\left[\frac{R_1}{b_L(T)} > \eta\right] = \frac{1}{m_L(T)} T\,\mathrm{P}\left[\frac{R_1}{b_R(T)} > \frac{b_L(T)}{b_R(T)}\eta\right] \to 0
\end{aligned}$$

since $\frac{b_L(T)}{b_R(T)} \in RV_{\frac{1}{\alpha_L} - \frac{1}{\alpha_R}}$ and $\alpha_L < \alpha_R$ imply $\frac{b_L(T)}{b_R(T)} \to \infty$ and $m_L(T) \to \mu_L < \infty$. Therefore, $S_1'^{(T)}/b_L(T) \xrightarrow{P} 0$. □

Combining (2.37) and (2.38) and using Slutsky's theorem, we get Theorem 2.5.1 for one-dimensional convergence with a random centering:

$$\frac{X(T)}{b_P(T)} - P(T)\,\mathrm{E}\left(\frac{V_1^{(T)} S_1^{(T)}}{b_P(T)}\right) \Rightarrow Z_{\alpha_P}(1), \tag{2.39}$$

We now obtain the correct centering. Observe that we need to have the centering $\frac{\lambda T\mu_P}{b_P(T)}$ in place of $P(T)\,\mathrm{E}\left[\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}\right]$ in (2.39) to get the required result. We shall show the difference of the above two expressions goes to 0 in probability. Now,

$$\begin{aligned} &P(T)\,\mathrm{E}\left(\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}\right) - \frac{\lambda T\mu_P}{b_P(T)} \\ =&\frac{P(T)-\lambda\widehat{F}_L(T)}{\sqrt{\lambda\widehat{F}_L(T)}}\sqrt{\lambda\widehat{F}_L(T)}\,\mathrm{E}\left(\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}\right) - \frac{\lambda}{b_P(T)}\left[T\mu_P - \widehat{F}_L(T)\,\mathrm{E}\left(V_1^{(T)}S_1^{(T)}\right)\right]. \end{aligned} \tag{2.40}$$

Now, as in (2.35),

$$\limsup_{T\to\infty}\sqrt{\lambda\widehat{F}_L(T)}\,\mathrm{E}\left(\frac{V_1^{(T)}S_1^{(T)}}{b_P(T)}\right) \leqslant \lim_{T\to\infty}\frac{\lambda\widehat{F}_L(T)}{b_P(T)}\,\mathrm{E}[L_1R_1] = \lim_{T\to\infty}\frac{\sqrt{T\lambda}}{b_P(T)}\,\mathrm{E}(L_1R_1),$$

which goes to 0, since $\frac{\sqrt{T}}{b_P(T)} \in RV_{\frac{1}{2}-\frac{1}{\alpha_P}}$ and $\alpha_P < 2$. Since $\widehat{F}_L(T) \sim T$, using the central limit theorem, we have $(P(T)-\lambda\widehat{F}_L(T))/\lambda\widehat{F}_L(T)^{1/2}$ is bounded in probability and we get the first term on the right side of (2.40) goes to 0 in probability. Thus, we only need to show the second term on the right side of (2.40), which is just a number, goes to zero. Observe, from (2.26),

$$\begin{aligned} &T\mu_P - \widehat{F}_L(T)\,\mathrm{E}(V_1^{(T)}S_1^{(T)}) && (2.41)\\ &=\int_{x=0}^{T}\iint_{(y,z)\in\mathbb{R}_+^2} yz\,F(dy,dz)\,dx - \int_{x=0}^{T}\int_{y\leqslant T-x}\int_{z\in(0,\infty)} yz\,F(dy,dz)\,dx \\ =&\int_{x=0}^{T}\int_{y>T-x}\int_{z\in(0,\infty)} yz\,F(dy,dz)\,dx = \int_{x=0}^{T}\int_{y>x}\int_{z\in(0,\infty)} yz\,F(dy,dz)\,dx && (2.42)\\ =&\int_{x=0}^{T}\left[\int_{y>x}\int_{z\in(0,1]} yz\,F(dy,dz) + \int_{y>x}\int_{z>1} yz\,F(dy,dz)\right]dx \end{aligned}$$

$$\leqslant \int_{x=0}^{T} \left[\int_{y>x} y\, F_L(dy) + \int_{u>x} u\, F_P(du) \right] dx, \tag{2.43}$$

where $L_1 R_1$ has distribution F_P. Then, by (2.42) and (2.43), we get,

$$\begin{aligned} 0 &\leqslant \frac{\lambda}{b_P(T)} \left[T\mu_P - \widehat{F}_L(T)\, \mathrm{E}\left(V_1^{(T)} S_1^{(T)} \right) \right] \\ &\leqslant \frac{\lambda}{b_P(T)} \int_{x=0}^{T} \int_{y>x} y\, F_L(dy)\, dx + \frac{\lambda}{b_P(T)} \int_{x=0}^{T} \int_{u>x} u\, F_P(du)\, dx. \end{aligned} \tag{2.44}$$

Now, using Karamata's theorem, we get,

$$\frac{\lambda}{b_P(T)} \int_{x=0}^{T} \int_{y>x} y\, F_L(dy)\, dx \sim \frac{\lambda}{(2-\alpha_L)(\alpha_L - 1)} \frac{T^2 \overline{F}_L(T)}{b_P(T)} \to 0,$$

since it is a regularly varying function of index $2 - \alpha_L - {\alpha_P}^{-1} \leqslant 2 - \alpha_P - {\alpha_P}^{-1} = -(\alpha_P - 1)^2/\alpha_P < 0$. Similarly,

$$\frac{\lambda}{b_P(T)} \int_{x=0}^{T} \int_{u>x} u\, F_P(du)\, dx \sim \frac{\lambda}{(2-\alpha_P)(\alpha_P - 1)} \frac{T^2 \overline{F_P}(T)}{b_P(T)} \in RV_{2-\alpha_P - \frac{1}{\alpha_P}},$$

and hence goes to zero. Thus, by (2.44),

$$\frac{\lambda}{b_P(T)} \left[T\mu_P - \widehat{F}_L(T)\, \mathrm{E}\left(V_1^{(T)} S_1^{(T)} \right) \right] \to 0. \tag{2.45}$$

Thus, on $\mathbb{R}$,

$$\frac{X(T) - \lambda T \mu_P}{b_P(T)} \Rightarrow Z_{\alpha_P}(1),$$

so that for all $t \geqslant 0$,

$$\frac{X(Tt) - \lambda T t \mu_P}{b_P(T)} = \frac{b_P(Tt)}{b_P(T)} \frac{X(Tt) - \lambda T t \mu_P}{b_P(Tt)} \Rightarrow t^{\frac{1}{\alpha_P}} Z_{\alpha_P}(1) \stackrel{\mathrm{d}}{=} Z_{\alpha_P}(t).$$

This is the required one-dimensional convergence:

$$\frac{X(Tt) - \lambda T t \mu_P}{b_P(T)} \Rightarrow Z_{\alpha_P}(t) \text{ in } \mathbb{R} \quad \forall t \geqslant 0. \tag{2.46}$$

Finally, we consider the finite dimensional convergence, which will complete the proof of Theorem 2.5.1. Let $0 < s < t$. Observe that the process $X_1(T\cdot)$ has independent increments. Also, let us define,

$$X_1(Tt) - X_1(Ts) = B_T(s,t) + C_T(s,t), \tag{2.47}$$

where

$$B_T(s,t) = \iiint\limits_{\substack{0\leqslant x\leqslant Ts\\ Ts<x+y\leqslant Tt}} yz\, M(dx,dy,dz,) \text{ and } C_T(s,t) = \iiint\limits_{\substack{Ts<x\leqslant Tt\\ Ts<x+y\leqslant Tt}} yz\, M(dx,dy,dz).$$

It is easy to check that $C_T(s,t) \overset{\mathrm{d}}{=} X_1(T(t-s))$. So, by (2.46), we get,

$$\frac{C_T(s,t) - \lambda T(t-s)\mu_P}{b_P(T)} \Rightarrow Z_{\alpha_P}(t-s) \overset{\mathrm{d}}{=} Z_{\alpha_P}(t) - Z_{\alpha_P}(s). \tag{2.48}$$

Also, by (2.42) and (2.45), we get,

$$\mathrm{E}\left(\frac{B_T(s,t)}{b_P(T)}\right) = \frac{\lambda}{b_P(T)} \iiint\limits_{\substack{0\leqslant x\leqslant Ts\\ Ts<x+y\leqslant Tt}} yz\, F(dy,dz)\, dx \leqslant \frac{\lambda}{b_P(T)} \iiint\limits_{\substack{0\leqslant x\leqslant Ts\\ x+y>Ts}} yz\, F(dy,dz)\, dx$$

$$= \frac{b_P(Ts)}{b_P(T)} \lambda \frac{Ts\mu_P - \widehat{F}_L(Ts)\,\mathrm{E}\left(V_1^{(Ts)} S_1^{(Ts)}\right)}{b_P(Ts)} \to s^{-\alpha_P} \cdot 0,$$

which implies

$$\frac{B_T(s,t)}{b_P(T)} \xrightarrow{P} 0. \tag{2.49}$$

Then, by (2.47) – (2.49),

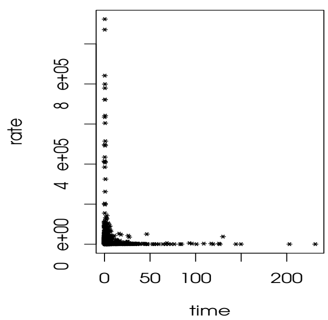
$$\frac{(X_1(Tt) - \lambda Tt\mu_P) - (X_1(Ts) - \lambda Ts\mu_P)}{b_P(T)} \Rightarrow Z_{\alpha_P}(t) - Z_{\alpha_P}(s) \text{ in } \mathbb{R}.$$

By the independent increment property of $Y_T^{(1)}$ and Z_{α_P}, coordinatewise convergence of increments of $Y_T^{(1)}$ and Z_{α_P} implies joint convergence of increments. Thus,

$Y_T^{(1)} \stackrel{\text{fidi}}{\to} Z_{\alpha_P}$. Also, from (2.38), we have, $(X_2(Tt_1), \cdots, X_2(Tt_k))/b_P(T) \stackrel{P}{\to} \mathbf{0}$, for all $0 \leqslant t_1 < \cdots < t_k$. Now, $Y_T(t) = \frac{X(Tt) - \lambda Tt\mu_P}{b_P(T)} = \frac{X_1(Tt) - \lambda Tt\mu_P}{b_P(T)} + \frac{X_2(Tt)}{b_P(T)}$ implies, $Y_T \stackrel{\text{fidi}}{\to} Z_{\alpha_P}$. □

2.6 Conclusion

Our result is dependent on the modelling decisions for the joint distribution of (L, R). If our model of asymptotic independence holds, then so does the classical asymptotic independence model. However, our estimate of the spectral distribution function, defined in (2.3), for the time and the rate of transmission (cf. de Haan and de Ronde, 1998, Sections 4 and 5), given in Figure 2.3, does not seem to be supported on $\{0, \frac{\pi}{2}\}$, and may suggest a lack of asymptotic independence between the two random variables. So our model, though successful in accommodating a random transmission rate, may not fit this particular data well. The possible failure, as pointed out by the referee, may be due to the assumption of asymptotic independence of the transmission length and the transmission rate. It is more realistic to consider to asymptotic independence between the transmission rate and the file size. This issue is addressed in a later work.



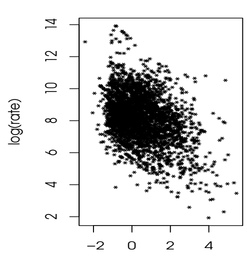
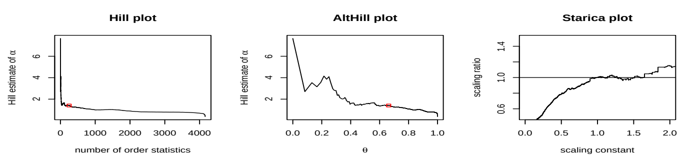


Figure 2.1: Plot of the time of transmission against the rate of the transmission of the BUburst data: *left*) in natural scale, *right*) in log-log scale


Hill plot
Hill estimate of α
number of order statistics
AltHill plot
θ
Starica plot
scaling ratio
scaling constant

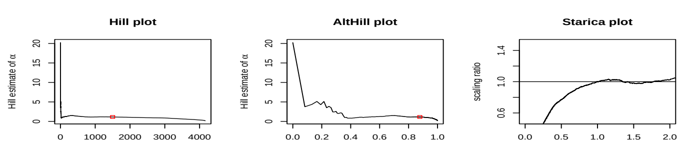

Hill plot
Hill estimate of α
number of order statistics
AltHill plot
θ
Starica plot
scaling ratio
scaling constant

Figure 2.3: Spectral measure estimates of time and rate of transmission

Chapter 3

Small and Large Time Scale Analysis of a Network Traffic Model

3.1 Introduction

In Chapter 2, the linear transmission schedule with randomly chosen slope has been considered. However, none of these efforts could capture the complete essence of a random transmission schedule process. Also, the above attempt modeled the transmission by choosing a transmission schedule and the length of transmission, which is not very realistic. In this chapter, we try to address these two issues among others. We model by considering a random, time-dependent transmission schedule A_k and choosing the file size J_k at the beginning of each transmission. The length of transmission L_k is obtained as a function of these two random quantities.

Both fBm and stable Lévy motion possess self-similarity, which is consistent with the macroscopic analysis of the network traffic data at a time scale of a few hundred milliseconds or larger. However, these models were posed without considering the complicated multifractal behavior of the WAN traffic observed at fine time scales below a few hundred milliseconds. Paxson and Floyd (1995) observed the limitations of the usual model in their study. Later Riedi and Lévy Véhel (1997) and Mannersalo and Norros (1997) analyzed different WAN traces to empirically observe the multifractal behavior of ATM WAN traces. These observations stimulated researchers to look for a model which could explain both the microscopic as well as the macroscopic behaviors. In Feldman et al. (1998), Feldmann et al. (1998), Gilbert et al. (1998, 1999), Kulkarni et al. (2001), Riedi and Willinger (2000), attempts were made to consider a conservative cascade model as the transmission schedule. We explain the fine time scale behavior by assuming individual transmission schedules exhibit multifractality. This results in multifractal behavior for the cumulative taffic process at the microscopic level and still gives a stable Lévy motion as the macroscopic approximation.

Thus the present model offers an explanation of both the micro and macroscopic behavior of the cumulative traffic process and suggests that the multifractal behavior of the cumulative traffic process results from similar behavior of the individual input processes. This suggests empirical studies should examine the behavior of individual, user level input processes.

This chapter is organized as follows: In Section 3.2, we review the basic concepts of the multifractal spectrum as required in our discussion. Section 3.3 is a quick review of the space $\mathbb{D}[0,\infty)$ endowed with Skorohod's (1956) M_1 topology. We state the model in detail and the main results at the micro and macro levels in Section 3.4, and discuss the conditions under which the main results hold. Section 3.5 considers the multifractal analysis, whereas Section 3.6 proves the approximation for large time scales.

3.2 Hölder Exponent and Multifractal Spectrum

We first recall the definition of the Hölder exponent of a function. In the literature, two types of Hölder exponents have been considered, namely, the one based on exponential growth rate and the one based on polynomial approximation. In this dissertation, we shall emphasize the first one.

Definition 3.2.1. The *Hölder exponent based on exponential growth rate* of the function x at t is defined as

$$h_x(t) := \liminf_{\varepsilon\downarrow 0} \frac{\log \sup_{u:|u-t|\leqslant\varepsilon} |x(u)-x(t)|}{\log \varepsilon}. \tag{3.1}$$

The Hölder exponent of the sum of two functions satisfies the following inequality.

Proposition 3.2.1. *For two functions x and y, we have*

$$h_{x+y}(t) \geqslant h_x(t) \wedge h_y(t). \tag{3.2}$$

Furthermore, equality holds if $h_x(t) \neq h_y(t)$.

Proof. By the triangle inequality, $|(x+y)(u)-(x+y)(t)| \leqslant |x(u)-x(t)|+|y(u)-y(t)|$ and hence

$$\sup_{u:|u-t|\leqslant\varepsilon} |(x+y)(u)-(x+y)(t)| \leqslant \sup_{u:|u-t|\leqslant\varepsilon} |x(u)-x(t)| + \sup_{u:|u-t|\leqslant\varepsilon} |y(u)-y(t)|.$$

Since $\log\varepsilon < 0$, we have

$$\begin{aligned} h_{x+y}(t) &= \liminf_{\varepsilon\downarrow 0} \frac{\log \sup_{u:|u-t|\leqslant\varepsilon} |(x+y)(u)-(x+y)(t)|}{\log\varepsilon} \\ &\geqslant \liminf_{\varepsilon\downarrow 0} \frac{\log\left[\sup_{u:|u-t|\leqslant\varepsilon} |x(u)-x(t)| + \sup_{u:|u-t|\leqslant\varepsilon} |y(u)-y(t)|\right]}{\log\varepsilon} \\ &= \liminf_{\varepsilon\downarrow 0} \frac{\log \sup_{u:|u-t|\leqslant\varepsilon} |x(u)-x(t)|}{\log\varepsilon} \bigwedge \liminf_{\varepsilon\downarrow 0} \frac{\log \sup_{u:|u-t|\leqslant\varepsilon} |y(u)-y(t)|}{\log\varepsilon} \qquad (3.3) \\ &= h_x(t) \wedge h_y(t). \qquad (3.4) \end{aligned}$$

Here is the explanation for the equality in (3.4): For any sequence, for which the limit for the quotient on the left side of (3.4) exists, choose a further subsequence such that limits for both the quotients on the right side of (3.4) also exist, and hence are greater than or equal to $h_x(t)$ and $h_y(t)$ respectively. Then the limit for the quotient on the left side will be equal to the minimum of the limits for x and y, and hence greater than or equal to $h_x(t) \wedge h_y(t)$. For the reverse inequality, assume without loss of generality that $h_x(t) \leqslant h_y(t)$. Fix $\varepsilon > 0$. Choose a sequence such that the limit for the quotient on the right side corresponding to x is less than $h_x(t) + \varepsilon$. Choose a further subsequence such that the limit for the quotient corresponding to y exists. Then the limit for the quotient on the left side along this subsequence is the minimum of the limits for the quotients corresponding to x and

y along it, and hence is smaller than $h_x(t)+\varepsilon$. Thus, the left side of (3.4) is smaller than $h_x(t)+\varepsilon$, for all $\varepsilon > 0$. Hence the left side of (3.4) is smaller than or equal to $h_x(t) = h_x(t) \wedge h_y(t)$.

Now assume $h_x(t) \neq h_y(t)$ and without loss of generality assume $h_x(t) < h_y(t)$. Now, observe $h_{-y}(t) = h_y(t)$. Since $x = (x+y) + (-y)$, we have, by previous discussion, $h_x(t) \geqslant h_{x+y}(t) \wedge h_{-y}(t) = h_{x+y}(t) \wedge h_y(t)$. But since, by assumption, $h_x(t) < h_y(t)$, we must have $h_x(t) \wedge h_y(t) = h_x(t) \geqslant h_{x+y}(t)$. Thus, $h_x(t) \wedge h_y(t) = h_{x+y}(t)$. □

Remark 3.2.1. A strict inequality may hold in (3.2) if $h_x(t) = h_y(t)$. For example, consider $x(t) = t^2 - t$ and $y(t) = t^2 + t$. Then $(x+y)(t) = 2t^2$ and $h_x(0) = h_y(0) = 1$, but $h_{x+y}(0) = 2$.

For non-decreasing functions, the definition of $h_x(t)$ in (3.1) simplifies to

$$\begin{aligned} h_x(t) &= \liminf_{\varepsilon \downarrow 0} \frac{\log[(x(t+\varepsilon) - x(t)) \wedge (x(t) - x(t-\varepsilon))]}{\log \varepsilon} \\ &= \liminf_{\varepsilon \downarrow 0} \frac{\log(x(t+\varepsilon) - x(t-\varepsilon))}{\log \varepsilon} \end{aligned} \tag{3.5}$$

The above definition (3.5) helps us to conclude an equality in (3.2) if x and y are non-decreasing.

Proposition 3.2.2. *If x and y are two non-decreasing functions, then $h_{x+y}(t) = h_x(t) \wedge h_y(t)$.*

Proof. In the case of non-decreasing functions, we use the representation (3.5) and then

$$h_{x+y}(t) = \liminf_{\varepsilon \downarrow 0} \frac{\log((x+y)(t+\varepsilon) - (x+y)(t-\varepsilon))}{\log \varepsilon}$$

$$= \liminf_{\varepsilon \downarrow 0} \frac{\log[(x(t+\varepsilon) - x(t-\varepsilon)) + (y(t+\varepsilon) - y(t-\varepsilon))]}{\log \varepsilon}$$
$$= \liminf_{\varepsilon \downarrow 0} \frac{\log(x(t+\varepsilon) - x(t-\varepsilon))}{\log \varepsilon} \bigwedge \liminf_{\varepsilon \downarrow 0} \frac{\log(y(t+\varepsilon) - y(t-\varepsilon))}{\log \varepsilon}$$
$$= h_x(t) \wedge h_y(t).$$

The penultimate equality holds due to an argument similar to that for (3.4). □

As mentioned earlier, there is another Hölder exponent based on polynomial approximation. We discuss this exponent in the following definitions.

Definition 3.2.2. We say *a function x belongs to the class C_h at t*, and write $x \in C_h(t)$, if there exists a polynomial P of degree at most h, and $\varepsilon > 0$, and $C > 0$, such that

$$|x(u) - P(u)| < C|u-t|^h, \text{ for all } u \text{ with } |u-t| < \varepsilon.$$

Definition 3.2.3. The *Hölder exponent based on polynomial approximation* of the function x at the point t is defined as

$$H_x(t) := \sup\{h \geqslant 0 : x \in C_h(t)\}.$$

In general, we may only conclude that the Hölder exponent based on polynomial approximation for the sum of two functions is greater than or equal to the minimum of that of each individual one. The equality may fail to hold even if the functions are increasing, as given in the following example.

Example 3.2.1. Let us consider the functions

$$x(t) = t + t^{\langle 1.5 \rangle} \quad \text{and} \quad y(t) = t - t^{\langle 1.5 \rangle}, \quad \text{for } t \in (-\tfrac{1}{3}, \tfrac{1}{3}),$$

where $t^{\langle u \rangle} = \operatorname{sgn}(t)|t|^u$. It is easy to check that $t^{\langle u \rangle}$ is differentiable for $u \geqslant 1$ and the derivative is $u|t|^{u-1}$. Thus we have $x'(t) = 1 + 1.5\sqrt{|t|} > 0$ and hence is increasing. Also $y'(t) = 1 - 1.5\sqrt{|t|} > 0$ for $t \in (-\frac{1}{3}, \frac{1}{3})$ and hence is increasing.

Observe $(x + y)(t) = 2t$ and clearly, $H_{x+y} \equiv \infty$. On the other hand,

$$|x(t) - t| = |t^{\langle 1.5 \rangle}| = |t|^{1.5}$$

and hence $H_x(0) \geqslant 1.5$. Now, let $0 < h < H_x(0)$ and P be the corresponding approximating polynomial. Clearly, $P(0) = x(0) = 0$ and if the coefficient of the linear term is 1, then $P(t) = t^2 Q(t)$, where Q is a polynomial, and hence $|f(t) - P(t)| = |t^{1.5} + t^2 Q(t)| = O(t^{1.5})$ as $t \to 0$. Thus $h \leqslant 1.5$. If the coefficient of the linear term is not 1, then $P(t) = t + tQ(t)$, for some polynomial Q, and hence $|f(t) - P(t)| = |t^{1.5} + tQ(t)| = O(t)$ as $t \to 0$. Thus $h \leqslant 1$. So we conclude $H_x(0) = 1.5$. Similarly, it can be checked $H_y(0) = 1.5$. Thus, we have $H_{x+y}(0) = \infty > 1.5 = H_x(0) \wedge H_y(0)$.

This lack of equality makes the Hölder exponent based on polynomial approximation more difficult to analyze and we shall not emphasize it. Still it is worth mentioning that the following relation holds between two types of Hölder exponents.

Proposition 3.2.3. *We have the following relation between two Hölder exponents:*

$$h_x(t) \leqslant H_x(t).$$

Further, if $h_x(t) \notin \mathbb{N}$, then $h_x(t) = H_x(t)$.

Proof. See Lemma 2.3 of Riedi (2001) and the discussion preceding it. □

We end the section by defining the multifractal spectrum of the Hölder exponent.

Definition 3.2.4. *The multifractal spectrum of the function x for the Hölder exponent based on exponential growth rate* is

$$d_x(a) = \dim(\{t > 0 : h_x(t) = a\}), \quad a \in [0, \infty),$$

where for a set Λ, $\dim(\Lambda)$ is the Hausdorff dimension (cf. Falconer, 1990, Chapter 2) of Λ.

3.3 The Space $\mathbb{D}$ and the M_1 Topology

In this section, we review the space $\mathbb{D}$ endowed with the M_1 topology. Throughout this dissertation, we shall consider the space $\mathbb{D}$ to be endowed with the M_1 topology, unless otherwise mentioned. A good reference for the topics considered in this section is the forthcoming book by Whitt (2002). We define $\mathbb{D}$ to be the set of all càdlàg functions on $[0, \infty)$. We denote $\mathbb{D}_T$ to be the set of all càdlàg functions on $[0, T]$. We first define the M_1 topology on $\mathbb{D}_T$.

To define the metric, we need to define the *completed graph* Γ_x of a function $x \in \mathbb{D}_T$, which is the set

$$\Gamma_x = \{(t, z) \in [0, T] \times \mathbb{R} : z = \alpha x(t-) + (1 - \alpha)x(t) \text{ for some } \alpha \in [0, 1]\}.$$

Γ_x is a connected set in $\mathbb{R}^2$ obtained by connecting $(t, x(t-))$ and $(t, x(t))$. We define the natural order on the points of Γ_x as follows: for $(t_1, z_1), (t_2, z_2) \in \Gamma_x$, we say $(t_1, z_1) \prec (t_2, z_2)$ if $t_1 < t_2$ or $t_1 = t_2$ and $|z_1 - x(t_1)| > |z_2 - x(t_2)|$, i.e., we rank the points in an increasing order as we traverse the completed graph from the left end $(0, x(0))$ to the right end $(T, x(T))$.

A parametric representation of the completed graph Γ_x is a continuous non-decreasing (according to the above-defined order) function $(r(\cdot), u(\cdot))$ mapping $[0,1]$ onto Γ_x. Let Π_x be the collection of all parametric representations of Γ_x. Then the metric giving the M_1 topology on $\mathbb{D}_T$ is given by

$$d_T(x_1, x_2) = \inf_{\substack{(r_j,u_j)\in\Pi_{x_j}\\ j=1,2}} \{\|r_1 - r_2\| \vee \|u_1 - u_2\|\},$$

where for a function $f : [0,1] \mapsto \mathbb{R}$ we define

$$\|f\| = \sup_{t\in[0,1]} |f(t)|.$$

Finally we define the M_1 metric on $\mathbb{D}$ as:

$$d(x_1, x_2) = \int_0^\infty e^{-t}[d_t(x_1, x_2) \wedge 1]\, dt.$$

We also denote the space of non-decreasing and unbounded càdlàg functions by $\mathbb{D}_\uparrow$ and endow it with the relative topology of M_1 topology on $\mathbb{D}$. We use the following results on the M_1 topology in the sequel.

Theorem 3.3.1. *The relative topology of* $\mathbb{D}_\uparrow$ *is the topology given by pointwise convergence on a dense subset of* $[0,\infty)$ *including* 0*.*

See Corollary 12.5.1 of Whitt (2002) for a proof.

Corollary 3.3.1. *If* $X_k, k \geqslant 0$ *are random processes taking values in the space* $\mathbb{D}_\uparrow$ *with the* M_1 *topology, then* $X_k \Rightarrow X_0$ *if* $X_k \stackrel{\text{fidi}}{\to} X_0$*, where the above convergence is in the sense of weak convergence of finite dimensional distributions on* $[0,\infty)$*.*

Theorem 3.3.2. *The evaluation map* $\pi : [0,\infty) \times \mathbb{D} \to \mathbb{R}$ *defined by* $\pi(t,x) = x(t)$ *is continuous at* (t,x) *iff* x *is continuous at* t*.*

See Lemma 12.5.1 of Whitt (2002) for a proof.

Corollary 3.3.2. *The projection map* $\pi_{t_1,\ldots,t_k}$ *defined by* $\pi_{t_1,\ldots,t_k}(x) = (x(t_1),\ldots,x(t_k))$ *is continuous at* x *if* x *is continuous at* $(t_1,\ldots,t_k)$*.*

Theorem 3.3.3. *The right continuous inverse map* Φ *defined by* $\Phi(x) = x^{\rightarrow}$*, where* $x^{\rightarrow}$ *is as defined in* (1.2)*, is continuous on* $\mathbb{D}_{\uparrow}$*.*

The proof follows trivially from the theorem in Section 2 of Whitt (1971).

As we have seen in the definition (3.7) of L_k, we shall have opportunity to use the left continuous right limit, i.e., càglàd functions as well. Let us denote the space of càglàd functions on $[0,\infty)$ by $\tilde{\mathbb{D}}$. We endow this space with M_1 topology as well, which is defined similarly through the completed graph and increasing parametrization. Notice that for a function x, the left continuous inverse $x^{\leftarrow}$, defined by (1.1) and the right continuous inverse $x^{\rightarrow}$ differ only by the value at the jump points of x. Thus they have the same completed graphs. Hence, for a sequence of functions $\{x_n\}_{n\geqslant 0}$, we have $x_n^{\leftarrow} \to x^{\leftarrow}$ in the M_1 topology iff $x_n^{\rightarrow} \to x^{\rightarrow}$ in the M_1 topology. So, we have the following corollary to Theorem 3.3.3, where $\tilde{\mathbb{D}}_{\uparrow}$ is the space of unbounded increasing càglàd functions:

Corollary 3.3.3. *The left continuous inverse map* $\tilde{\Phi}$ *defined by* $\tilde{\Phi}(x) = x^{\leftarrow}$ *is continuous on* $\mathbb{D}_{\uparrow}$*.*

The following result extends Lamperti's theorem (cf. Theorem 2 of Lamperti, 1962, Theorem 2 and Durrett and Resnick, 1977, Section 2) where the convergence is extended to happen in $\mathbb{D}$ with M_1 toplogy.

Theorem 3.3.4. *Suppose Z is a process taking values in $\mathbb{D}$ endowed with the M_1 topology. If for some function σ and some process ζ with values in $\mathbb{D}$, which is proper, i.e., $\zeta(t)$ has non-degenerate distribution for all $t > 0$, we have*

$$\frac{Z(T\cdot)}{\sigma(T)} \Rightarrow \zeta(\cdot),$$

then there exists $H > 0$, such that $\sigma \in RV_H$ and ζ is H-self-similar (H-ss). Also, ζ is continuous in probability.

The proof of Theorem 3.3.4 is along the same lines of Durrett and Resnick (1977), since the proofs in section 2 of Durrett and Resnick (1977) depends only on finite dimensional convergence on a dense set.

Remark 3.3.1. Note that Corollary 3.3.2 implies finite dimensional convergence on a dense set of points $\mathcal{C}_\zeta$, which is the set of points where ζ is continuous in probability, unlike in Lamperti's theorem, where the finite dimensional convergence takes place on the entire half line $[0, \infty)$. However, Theorem 3.3.4 allows us to conclude the convergence in finite dimensional distributions, since ζ is continuous in probability.

3.4 Model Specification

In this section, we state the assumptions of the model. First we recall the basic assumptions stated in Section 1.

1. We denote the time when k-th transmission begins by Γ_k. $\{\Gamma_k\}$ is a sequence strictly increasing to ∞.

2. The size of the file transmitted is J_k and we assume $J_k > 0$.

3. The transmission schedule is denoted by $A_k(\cdot)$, where $A_k(t)$ denotes the amount of data transmitted in time t after the kth transmission has begun. It is a non-decreasing càdlàg function starting at 0 and increasing to ∞, which vanishes on the negative real axis.

Then the traffic process becomes

$$X(t) = \sum_{k=1}^{\infty} A_k(t - \Gamma_k) \wedge J_k. \tag{3.6}$$

In this case, the transmission length of k-th source is

$$L_k = \inf\{t : A_k(t) \geqslant J_k\} = A_k^{\leftarrow}(J_k). \tag{3.7}$$

3.4.1 Small time scale behavior

To study the behavior of the cumulative traffic process $X(\cdot)$ for small time scales, we need to make the following further minimal assumptions on the transmission schedule $\{A_k\}$:

4. We assume $\{A_k\}$ are identically distributed and have stationary increments.

5. The multifractal spectrum of $A_k(\cdot)$ is not degenerated to a single point, which ensures that we consider processes with paths that show real multifractal behavior.

6. The multifractal spectrum of $A_k(\cdot)$ restricted to any (non-random) interval is non-random.

Remark 3.4.1. If A_k is, for example, an increasing Lévy process, then, restricted to any interval, it has a non-random multifractal spectrum for the Hölder exponent based on exponential growth rate. (Jaffard (1999) shows that the multifractal spectrum of a Lévy process restricted to $[0, 1]$ for the Hölder exponent based on polynomial approximation is non-random. The same proof works for any interval. By Proposition 3.2.3, an upper bound for the Hölder exponent H is also an upper bound for the Hölder exponent h. Also the lower bounds for H in Proposition 2 of Jaffard (1999) works for h since the approximating polynomial for those lower bounds are constant.)

The following theorem summarizes the small time scale behavior of the cumulative traffic process.

Theorem 3.4.1. *If the assumptions* $(1) - (6)$ *hold, then with probability 1,* $d_X = d_{A_1}$.

So the aggregate traffic process $X(\cdot)$ inherits the multifractal structure of the individual transmission schedules. This implies that a possible explanation for observed multifractality in aggregate traffic is intermittancy of individual transmissions, presumably caused by blocking and congestion. See Mannersalo et al. (1999), Resnick and Samorodnitsky (2001).

3.4.2 Large time scale behavior

For multifractal analysis, we study the process path by path. However, for large time scales, we need to make additional distributional assumptions, which we summarize as follows:

7. We assume $\{\Gamma_k\}$ forms a homogeneous Poisson process with intensity parameter λ.

8. We also assume $\{(A_k, J_k) : k \geqslant 1\}$ are i.i.d. and independent of $\{\Gamma_k\}$.

   Remark 3.4.2. Then L_k are i.i.d. Let F_L be the marginal distribution of L_1. We have

$$F_L(x) = \mathrm{P}[L_1 \leqslant x] = \mathrm{P}[A_1(x) \geqslant J_1] \text{ by right continuity of paths of } A_1 \tag{3.8}$$

   and

$$\overline{F}_L(x) = \mathrm{P}[L_1 > x] = \mathrm{P}[A_1(x) < J_1] \tag{3.9}$$

9. Let us define

$$\tilde{A}_1^{(T)}(\cdot) = A_1(T\cdot).$$

   We assume there exists a regularly varying function $\tilde{\sigma}$ of index H and a proper random process χ with stationary increments, taking values in $\mathbb{D}$, such that for each fixed $\varepsilon > 0$,

$$\frac{1}{\overline{F}_J(\tilde{\sigma}(T))} \mathrm{P}\left[\frac{J_1}{\tilde{\sigma}(T)} > \varepsilon, \frac{\tilde{A}_1^{(T)}}{\tilde{\sigma}(T)} \in \cdot\right] \xrightarrow{\mathrm{w}} \varepsilon^{-\alpha_J} \mathrm{P}[\chi \in \cdot] \tag{3.10}$$

on $\mathbb{D}$, where the above convergence is in the sense of weak convergence of finite measures.

Remark 3.4.3. Observe that by (3), we have, for $t > 0$, $\tilde{A}_1^{(T)}(t)$ goes to ∞ with probability 1, as $T \to \infty$. Thus, we must have $\tilde{\sigma}(T) \to \infty$, as otherwise $\tilde{A}_1^{(T)}(\cdot)/\tilde{\sigma}(T)$ goes to a function identically equal to ∞ with probability 1, which contradicts (3.10). Hence, we also have $H \geqslant 0$.

Remark 3.4.4. Let Λ be a Borel subset of $\mathbb{D}$, such that $\mathrm{P}[\chi \in \partial\Lambda] = 0$, where $\partial\Lambda$ is the boundary of Λ. Then, the convergence

$$\frac{1}{\overline{F}_J(\tilde{\sigma}(T))} \mathrm{P}\left[\frac{J_1}{\tilde{\sigma}(T)} > \varepsilon, \frac{\tilde{A}_1^{(T)}}{\tilde{\sigma}(T)} \in \Lambda\right] \xrightarrow{\mathrm{w}} \varepsilon^{-\alpha_J} \mathrm{P}[\chi \in \Lambda]$$

is locally uniform in $\varepsilon \in (0, \infty]$, since the converging functions are monotone in ε and the limit is continuous in ε with a finite limit at ∞ (cf. Resnick, 1987, Section 0.1).

Remark 3.4.5. Note that, in assumption (4), we have only assumed that the marginal distribution of A_1 has stationary increments, which is not enough to conclude that χ has stationary increments. So we include that as part of the assumption (9).

10. For all $\gamma > 0$, assume

$$\lim_{\varepsilon \downarrow 0} \limsup_{T \to \infty} \frac{1}{\overline{F}_J(\tilde{\sigma}(T))} \mathrm{P}\left[\frac{J_1}{\tilde{\sigma}(T)} \leqslant \varepsilon, \frac{L_1}{T} > \gamma\right] = 0. \tag{3.11}$$

Recall that L_1 is defined in (3.7).

11. We finally assume that

$$\mathrm{E}\left[\chi(1)^{-\alpha_J}\right] < \infty. \tag{3.12}$$

Remark 3.4.6. Our assumptions are not restrictive but do require $A_1(0) = 0$. This might be an impediment to the modeling of the download of cached files where one might prefer to allow $P[A_1(0) > 0] > 0$.

The following theorem summarizes the large time scale behavior of the input process.

Theorem 3.4.2. *Suppose the assumptions* (1) – (3) *and* (7) – (11) *hold and define*

$$Y_T(t) = \frac{X(Tt) - \lambda Tt\,\mathrm{E}(J_1)}{b_J(T)}.$$

Then we have

$$Y_T \stackrel{\text{fidi}}{\to} Z_{\alpha_J},$$

where Z_α is mean 0, skewness 1, α-stable Lévy motion with scale parameter $\left(\frac{\lambda}{C_\alpha}\right)^{\frac{1}{\alpha}}$, *and*

$$C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}. \tag{3.13}$$

We now amplify the implications of the assumptions (7) – (11).

Proposition 3.4.1. *If* (3.10) *holds, then χ is H-self-similar.*

Proof. First observe that from (3.10), by considering the set $\mathbb{D}$, we have

$$\lim_{T\to\infty} \frac{\overline{F}_J(\tilde{\sigma}(T)\varepsilon)}{\overline{F}_J(\tilde{\sigma}(T))} = \varepsilon^{-\alpha_J} \tag{3.14}$$

and by Remark 3.4.4 the convergence is locally uniform in $\varepsilon \in (0, \infty]$. Thus we further have

$$\frac{\overline{F}_J(\tilde{\sigma}(Ts))}{\overline{F}_J(\tilde{\sigma}(T))} = \frac{\overline{F}_J\left(\tilde{\sigma}(T)\frac{\tilde{\sigma}(Ts)}{\tilde{\sigma}(T)}\right)}{\overline{F}_J(\tilde{\sigma}(T))} \to s^{-H\alpha_J}, \tag{3.15}$$

since $\tilde{\sigma} \in RV_H$.

Using (3.10) and Corollary 3.3.2, we have

$$\frac{1}{\overline{F}_J(\tilde{\sigma}(T))} \mathrm{P}\left[\frac{J_1}{\tilde{\sigma}(T)} > \varepsilon, \frac{A_1(Tu)}{\tilde{\sigma}(T)} \in \cdot\right] \xrightarrow{\mathrm{w}} \varepsilon^{-\alpha_J} \mathrm{P}[\chi(u) \in \cdot] \tag{3.16}$$

for all u such that χ is continuous in probability at u. By Remark 3.4.4, for every fixed Borel subset Λ of $\mathbb{R}$ with $\mathrm{P}[\chi(u) \in \partial\Lambda] = 0$, the convergence in (3.16)

$$\frac{1}{\overline{F}_J(\tilde{\sigma}(T))} \mathrm{P}\left[\frac{J_1}{\tilde{\sigma}(T)} > \varepsilon, \frac{A_1(Tu)}{\tilde{\sigma}(T)} \in \Lambda\right] \xrightarrow{\mathrm{w}} \varepsilon^{-\alpha_J} \mathrm{P}[\chi(u) \in \Lambda]$$

is locally convergent in $\varepsilon \in (0, \infty]$. Let $s, t \geqslant 0$ be such that χ is continuous in probability at t and st. Then, from (3.15), (3.16) and local uniform convergence, we have

$$\frac{1}{\overline{F}_J(\tilde{\sigma}(Ts))} \mathrm{P}\left[\frac{J_1}{\tilde{\sigma}(Ts)} > 1, \frac{A_1(Tst)}{\tilde{\sigma}(Ts)} \in \cdot\right] \xrightarrow{\mathrm{w}} \mathrm{P}[\chi(t) \in \cdot] \tag{3.17}$$

and

$$\begin{aligned} &\frac{1}{\overline{F}_J(\tilde{\sigma}(Ts))} \mathrm{P}\left[\frac{J_1}{\tilde{\sigma}(Ts)} > 1, \frac{A_1(Tst)}{\tilde{\sigma}(T)} \in \cdot\right] && (3.18)\\ =&\frac{\overline{F}_J(\tilde{\sigma}(T))}{\overline{F}_J(\tilde{\sigma}(Ts))} \frac{1}{\overline{F}_J(\tilde{\sigma}(T))} \mathrm{P}\left[\frac{J_1}{\tilde{\sigma}(T)} > \frac{\tilde{\sigma}(Ts)}{\tilde{\sigma}(T)}, \frac{A_1(Tst)}{\tilde{\sigma}(T)} \in \cdot\right] \\ \xrightarrow{\mathrm{w}}& s^{H\alpha_J} s^{-H\alpha_J} \mathrm{P}[\chi(st) \in \cdot] = \mathrm{P}[\chi(st) \in \cdot], && (3.19) \end{aligned}$$

where we also use the fact that $\tilde{\sigma} \in RV_H$. Now define the distribution functions

$$G_T(x) = \frac{1}{\overline{F}_J(\tilde{\sigma}(Ts))} \mathrm{P}\left[\frac{J_1}{\tilde{\sigma}(Ts)} > 1, A_1(Tst) \leqslant x\right].$$

Then from (3.17) and (3.19), we get

$$G_T(\tilde{\sigma}(Ts)\cdot) \xrightarrow{\mathrm{w}} \mathrm{P}[\chi(t) \leqslant \cdot]$$

and

$$G_T(\tilde{\sigma}(T)\cdot) \xrightarrow{\text{w}} \mathrm{P}[\chi(st) \leqslant \cdot]$$

Then by the Convergence of Types Theorem (cf. Proposition 0.3 of Resnick (1987)), we have

$$\frac{\tilde{\sigma}(Ts)}{\tilde{\sigma}(T)} \to C$$

and

$$\mathrm{P}[\chi(t) \leqslant x] = \mathrm{P}[\chi(st) \leqslant Cx].$$

However, $\tilde{\sigma}$ being a regularly varying function of index H, we must have $C = s^H$. Thus, we have $\chi(st) \stackrel{\text{d}}{=} s^H\chi(t)$, for all $s, t \geqslant 0$, such that χ is continuous in probability at st and t. Since the points, where χ is continuous in probability, are dense in $[0, \infty)$ (cf. pg. 138 of Billingsley (1999)) and χ has càdlàg paths, we can conclude that $\chi(st) \stackrel{\text{d}}{=} s^H\chi(t)$, for all $s, t \geqslant 0$. Similarly, using Corollary 3.3.2 and the multivariate analog of the Convergence of Types Theorem (cf. pg. 28 of Geffroy (1959)), we have that $(\chi(st_1), \ldots, \chi(st_k)) \stackrel{\text{d}}{=} s^H(\chi(t_1), \ldots, \chi(t_k))$ for $t_1, \ldots, t_k \in [0, \infty)$. So we conclude that χ is H-ss. □

Remark 3.4.7. From the assumption (9), we know that χ has stationary increments. Thus χ is an H-self-affine (H-sa) process, i.e, an H-ss process with stationary increments. Since χ is càdlàg (and hence has a measurable version, cf. Doob, 1953, Theorem 2.6) and non-degenerate, using Lemma 1.2 and Theorem 1.3 of Vervaat (1985) we also have $H > 0$.

Further observe that $\mathbb{D}_{\uparrow}$ is a closed subset of $\mathbb{D}$ with M_1 topology. Since by assumption A_1 has almost surely non-decreasing paths, the finite measures on the left side of (3.10) have support $\mathbb{D}_\uparrow$. Thus from the convergence in (3.10) with $\varepsilon = 1$, we conclude that the distribution of χ is supported on $\mathbb{D}_\uparrow$ and so χ has almost surely non-decreasing paths.

Since χ is proper, we have $\mathrm{P}[\chi \equiv 0] = 0$, and since χ is a non-decreasing, H-sa process, we have, from Theorem 2.1 of Vervaat (1994), that $H \geqslant 1$. Also, in case $H = 1$, we have $\chi(t) \equiv t\chi(1)$ almost surely, which is the case of random but time-invariant transmission rate. Such case has been considered in Chapter 2, though under a slightly different set of hypotheses. Levy and Taqqu (2000), Pipiras and Taqqu (2000), Pipiras et al. (2000) also considered a similar case for superposition of on-off processes in context of renewal-reward processes. However, since in the case $H = 1$, the paths are almost everywhere linear and hence non-fractal, they are excluded from the current discussion and we assume $H > 1$. For the case $H > 1$, we have, from Theorem 3.1 of Vervaat (1994), that $\mathrm{E}\left[\chi(1)^p\right] = \infty$ for $p \geqslant \frac{1}{H}$, i.e., $\chi(1)$ has infinite mean.

Remark 3.4.8. Since we know from Remark 3.4.7 that $H > 0$, we can replace $\tilde{\sigma}$ in (3.10) with a strictly increasing and continuous regularly varying function $\sigma \sim \tilde{\sigma}$ (cf. Resnick, 1987, Proposition 0.8(vii)).

In the following lemma, we study the tail behavior of J_1.

Lemma 3.4.1. *Under the assumption* (9), *J_1 has a tail of index of α_J.*

Proof. We have already seen that the limit in (3.15) converges locally uniformly in $\varepsilon \in (0, \infty]$. So, we can have, using the fact from Remark 3.4.8 that $\sigma \sim \tilde{\sigma}$,

$$\lim_{T \to \infty} \frac{\overline{F}_J(\sigma(T)\varepsilon)}{\overline{F}_J(\tilde{\sigma}(T))} = \varepsilon^{-\alpha_J}.$$

Then evaluating at $\varepsilon = 1$, we have

$$\overline{F}_J \circ \sigma \sim \overline{F}_J \circ \tilde{\sigma} \tag{3.20}$$

and hence we have

$$\lim_{T\to\infty} \frac{\overline{F}_J(\sigma(T)\varepsilon)}{\overline{F}_J(\sigma(T))} = \varepsilon^{-\alpha_J}.$$

Now, from Remarks 3.4.3 and 3.4.8, we know that $\sigma(T)$ increases to ∞ continuously. Hence we conclude that $\overline{F}_J \in RV_{-\alpha_J}$, i.e., J_1 has a tail of index α_J. □

Remark 3.4.9. Since J_1 has a tail of index α_J, following the discussion in Section 1.2, we may choose a continuous, strictly increasing function b_J, which is regularly varying of index α_J^{-1}, such that for all $w > 0$, we have

$$\lim_{T\to\infty} T\,\mathrm{P}[J_1 > b_J(T)w] = w^{-\alpha_J}. \tag{3.21}$$

Remark 3.4.10. Using (3.20), we can replace $\overline{F}_J(\tilde{\sigma}(T))$ in (3.10) with $\overline{F}_J(\sigma(T))$. Using Remark 3.4.4 we can also replace $\frac{J_1}{\tilde{\sigma}(T)}$ by $\frac{J_1}{\sigma(T)}$, to obtain

$$\frac{1}{\overline{F}_J(\sigma(T))}\,\mathrm{P}\left[\frac{J_1}{\sigma(T)} > \varepsilon, \frac{\tilde{A}_1^{(T)}}{\tilde{\sigma}(T)} \in \cdot\right] \xrightarrow{\mathrm{w}} \varepsilon^{-\alpha_J}\,\mathrm{P}[\chi \in \cdot] \tag{3.22}$$

on $\mathbb{D}$. For $\mathbf{t} = (t_1, \ldots, t_k)$ and $\mathbf{x} = (x_1, \ldots, x_k)$, let us define

$$G_{\mathbf{t}}^{(T)}(\mathbf{x}) = \frac{1}{\overline{F}_J(\sigma(T))}\,\mathrm{P}\left[\frac{J_1}{\sigma(T)} > \varepsilon, \frac{\tilde{A}_1^{(T)}(t_1)}{\tilde{\sigma}(T)} \leqslant x_1, \ldots, \frac{\tilde{A}_1^{(T)}(t_k)}{\tilde{\sigma}(T)} \leqslant x_k\right]$$

and

$$G_{\mathbf{t}}^{(0)}(\mathbf{x}) = \varepsilon^{-\alpha_J}\,\mathrm{P}[\chi(t_1) \leqslant x_1, \ldots, \chi(t_k) \leqslant x_k].$$

Since χ is H-ss, it is continuous in probability, and so the convergence in (3.22) implies, by Corollary 3.3.2, that

$$\lim_{T\to\infty} G_{\mathbf{t}}^{(T)}(\mathbf{x}) = G_{\mathbf{t}}^{(0)}(\mathbf{x})$$

for all $\mathbf{t}$ and continuity points $\mathbf{x}$ of $G_{\mathbf{t}}^{(0)}(\cdot)$. It is also easily checked that for all continuity points $\mathbf{x}$ of $G_{\mathbf{t}}^{(0)}(\cdot)$, if $(x_{T,1},\ldots,x_{T,k}) := \mathbf{x}_T \to \mathbf{x}$, then

$$\lim_{T\to\infty} G_{\mathbf{t}}^{(T)}(\mathbf{x}_T) = G_{\mathbf{t}}^{(0)}(\mathbf{x}).$$

Thus, we get, as $T\to\infty$,

$$\frac{1}{\overline{F}_J(\sigma(T))}\,\mathrm{P}\left[\frac{J_1}{\sigma(T)} > \varepsilon, \frac{\tilde{A}_1^{(T)}(t_1)}{\sigma(T)} \leqslant x_1, \ldots, \frac{\tilde{A}_1^{(T)}(t_k)}{\sigma(T)} \leqslant x_k\right] = G_{\mathbf{t}}^{(T)}\left(\frac{\tilde{\sigma}(T)}{\sigma(T)}\mathbf{x}\right),$$

which converges to $G_{\mathbf{t}}^{(0)}(\mathbf{x})$ for all $\mathbf{t}$ and for all continuity points $\mathbf{x}$ of $G_{\mathbf{t}}^{(0)}(\cdot)$. Then, by Corollary 3.3.1, we conclude that

$$\frac{1}{\overline{F}_J(\sigma(T))}\,\mathrm{P}\left[\frac{J_1}{\sigma(T)} > \varepsilon, \frac{\tilde{A}_1^{(T)}}{\sigma(T)} \in \cdot\right] \xrightarrow{\mathrm{w}} \varepsilon^{-\alpha_J}\,\mathrm{P}[\chi \in \cdot] \tag{3.23}$$

on $\mathbb{D}$.

Remark 3.4.11. Using (3.20), we can replace $\overline{F}_J(\tilde{\sigma}(T))$ in (3.11) with $\overline{F}_J(\sigma(T))$ to get

$$\lim_{\varepsilon\downarrow 0}\limsup_{T\to\infty} \frac{1}{\overline{F}_J(\sigma(T))}\,\mathrm{P}\left[\frac{J_1}{\tilde{\sigma}(T)} \leqslant \varepsilon, \frac{L_1}{T} > \gamma\right] = 0.$$

Also, since $\sigma \sim \tilde{\sigma}$, we have,

$$\begin{aligned}&\limsup_{T\to\infty} \frac{1}{\overline{F}_J(\sigma(T))}\,\mathrm{P}\left[\frac{J_1}{\sigma(T)} \leqslant \varepsilon, \frac{L_1}{T} > \gamma\right] \\ =&\limsup_{T\to\infty} \frac{1}{\overline{F}_J(\sigma(T))}\,\mathrm{P}\left[\frac{J_1}{\tilde{\sigma}(T)} \leqslant \frac{\sigma(T)}{\sigma(\tilde{T})}\varepsilon, \frac{L_1}{T} > \gamma\right]\end{aligned}$$

$$\leqslant \limsup_{T\to\infty} \frac{1}{\overline{F}_J(\sigma(T))} \mathrm{P}\left[\frac{J_1}{\sigma(T)} \leqslant 2\varepsilon, \frac{L_1}{T} > \gamma\right].$$

Hence, we get, for all $\gamma > 0$,

$$\lim_{\varepsilon\downarrow 0} \limsup_{T\to\infty} \frac{1}{\overline{F}_J(\sigma(T))} \mathrm{P}\left[\frac{J_1}{\sigma(T)} \leqslant \varepsilon, \frac{L_1}{T} > \gamma\right] = 0. \tag{3.24}$$

Recall that σ is a regularly varying function of index H. We know that b_J is also a regularly varying function of index α_J^{-1}. So $\sigma^{\leftarrow} \circ b_J$ is a regularly varying function of index $(H\alpha_J)^{-1}$ continuously increasing to ∞.

Then taking the limits in (3.23) and (3.24) along $\sigma^{\leftarrow}(b_J(T))$, we get the following equivalent formulations of the assumptions (9) and (10):

($9'$) Define

$$A_1^{(T)}(\cdot) = A_1(\sigma^{\leftarrow}(b_J(T))\cdot).$$

There exists a regularly varying function σ of index H and a $\mathbb{D}$-valued random process χ with stationary increments which is also proper, such that for each fixed $\varepsilon > 0$,

$$T\,\mathrm{P}\left[\frac{J_1}{b_J(T)} > \varepsilon, \frac{A_1^{(T)}}{b_J(T)} \in \cdot\right] \xrightarrow{\mathrm{w}} \varepsilon^{-\alpha_J}\,\mathrm{P}[\chi \in \cdot] \tag{3.25}$$

on $\mathbb{D}$.

($10'$) For all $\gamma > 0$, assume

$$\lim_{\varepsilon\downarrow 0} \limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{J_1}{b_J(T)} \leqslant \varepsilon, \frac{L_1}{\sigma^{\leftarrow}(b_J(T))} > \gamma\right] = 0. \tag{3.26}$$

Henceforth we shall use the assumptions (9) and (10) interchangeably with the assumptions ($9'$) and ($10'$).

The following lemma shows that the assumption ($9'$) is not vacuous.

Lemma 3.4.2. *If J_1 and A_1 are independent, then* (3.25) *holds iff*

(i) *J_1 has tail of index α_J and*

(ii) *A_1 belongs to the domain of a random càdlàg process χ, which is proper, i.e., there exists a function σ such that*

$$\frac{A_1(T\cdot)}{\sigma(T)} \stackrel{\text{fidi}}{\to} \chi(\cdot).$$

Remark 3.4.12. If A_1 belongs to the domain of a càdlàg process, then, by Theorem 2 of Lamperti (1962), we must necessarily have $\sigma \in RV_H$ and χ is H-ss.

Proof of Lemma 3.4.2. Observe that (3.25) reads for $\varepsilon = 1$:

$$T\,\mathrm{P}\left[\frac{J_1}{b_J(T)} > 1, \frac{A_1^{(T)}}{b_J(T)} \in \cdot\right] = T\,\mathrm{P}\left[\frac{J_1}{b_J(T)} > 1\right]\mathrm{P}\left[\frac{A_1^{(T)}}{b_J(T)} \in \cdot\right] \stackrel{\mathrm{w}}{\to} \mathrm{P}[\chi \in \cdot] \quad (3.27)$$

on $\mathbb{D}$. We have seen in Remark 3.4.9 that, if (3.25) and hence (3.10) holds, then (i) holds and, hence in particular, $T\,\mathrm{P}\left[\frac{J_1}{b_J(T)} > 1\right] \to 1$. Thus, if J_1 and A_1 are indeed independent, from (3.27), we have

$$\frac{A_1(\sigma^{\leftarrow}(b_J(T))\cdot)}{b_J(T)} \Rightarrow \chi$$

on $\mathbb{D}$. Taking the above limit along $(\overline{F}_J(\sigma(T)))^{-1}$ instead of T and using finite dimensional convergence and locally uniform convergence arguments as in Remark 3.4.10, we get

$$\frac{A_1(T\cdot)}{\sigma(T)} \Rightarrow \chi(\cdot)$$

on $\mathbb{D}$. Also, since χ is H-ss (from Proposition 3.4.1),χ has no fixed point of discontinuity. Since the projection map $\pi_{t_1 t_2 \dots t_k}(x) = (x(t_1), x(t_2), \dots, x(t_k))$ is continuous at continuity points $t_1, t_2, \ldots, t_k$ of χ, by the Continuous Mapping Theorem, we have the required finite dimensional convergence. This completes the proof of "only if" part.

Conversely, assume (i) and (ii) hold. From (ii), we can conclude that

$$\frac{A_1(\sigma^{\leftarrow}(b_J(T)\cdot))}{b_J(T)} \stackrel{\text{fidi}}{\rightarrow} \chi$$

and then using Corollary 3.3.1, we have

$$\frac{A_1^{(T)}}{b_J(T)} \Rightarrow \chi.$$

Also, since J_1 has tail of index α_J, (3.21) holds and then using independence of J_1 and A_1, we conclude that (3.25) holds. This completes the proof of "if" part. $\square$

The following proposition gives a set of sufficient conditions for conditions (3.25), (3.26) and (3.12) to hold.

Proposition 3.4.2. *Assume that*

(i) J_1 *and* A_1 *are independent.*

(ii) J_1 *has a tail of index* α_J*.*

(iii) A_1 *is itself a proper* H*-ss process.*

(iv) $\mathrm{E}[A_1(1)^{-\rho}] < \infty$ *for some* $\rho > \alpha_J$*.*

Then (3.25), (3.26) *and* (3.12) *hold.*

Proof. It is trivial to check that a H-ss process A_1 is in the domain of attraction of itself corresponding to $\sigma(T) = T^H$. Thus Lemma 3.4.2 shows that, under independence of J_1 and A_1, (3.25) holds for the above $\sigma(T)$. Also (3.12), which states that $A_1(1)^{-1}$ has α_J-th moment finite, holds since we have assumed the existence of an even higher moment in (iv). So we need to check (3.26) only. Note that $\sigma^{\leftarrow}(T) = T^{\frac{1}{H}}$. Also, by self-similarity, $\mathrm{E}[A_1(\gamma)^{-\rho}] < \infty$.

Observe that,

$$
\begin{aligned}
&T\,\mathrm{P}\left[\frac{J_1}{b_J(T)} \leqslant \varepsilon, \frac{A_1^{\leftarrow}(J_1)}{\sigma^{\leftarrow}(b_J(T))} > \gamma\right] \\
=&T\,\mathrm{P}\left[\frac{A_1\left(b_J(T)^{\frac{1}{H}}\gamma\right)}{b_J(T)} < \frac{J_1}{b_J(T)} \leqslant \varepsilon\right] \\
=&T\,\mathrm{P}\left[\frac{A_1(\gamma)}{b_J(T)} < \frac{J_1}{b_J(T)} \leqslant \varepsilon\right] \quad \text{by self-similarity} \\
=&T\int_0^{\varepsilon} \mathrm{P}[A_1(\gamma) < s]\, F_J(b_J(T)ds) \\
=&T\int_0^{\varepsilon} \mathrm{P}\left[A_1(\gamma)^{-1} > s^{-1}\right]\, F_J(b_J(T)ds) \\
\leqslant&\,\mathrm{E}[A_1(\gamma)^{-\rho}]\, T\int_0^{\varepsilon} s^{\rho}\, F_J(b_J(T)ds) \\
=&\,\mathrm{E}[A_1(\gamma)^{-\rho}]\frac{T}{b_J(T)^{\rho}}\int_0^{b_J(T)\varepsilon} s^{\rho}\, F_J(ds) \\
\sim&\,\mathrm{E}[A_1(\gamma)^{-\rho}]\frac{T}{b_J(T)^{\rho}}\frac{\alpha_J}{\rho-\alpha_J}(b_J(T)\varepsilon)^{\rho}\overline{F}_J(b_J(T)\varepsilon) \qquad (3.28) \\
\to&\,\mathrm{E}[A_1(\gamma)^{-\rho}]\frac{\alpha_J}{\rho-\alpha_J}\varepsilon^{\rho-\alpha_J},
\end{aligned}
$$

where the limit in (3.28) holds by Lemma on pages 578-579 of Feller (1971) as

$\rho > \alpha_J$. Also, $\rho > \alpha_J$ again implies (3.26) holds. □

Remark 3.4.13. If we assume that A_1 is a H-ss process with stationary, independent increments, then $A_1(1)$ is a positive stable random variable of index $\frac{1}{H}$ and hence has a density which decays exponentially near 0 (cf. Zolotarev, 1986, Theorem 2.5.2) and so has all negative moments finite. Thus it satisfies the conditions of Proposition 3.4.2. Also Remark 3.4.1 shows that a Lévy process satisfies the conditions for the multifractal analysis. Thus a $\frac{1}{H}$-stable Lévy process satisfies the requirements of the transmission schedule.

3.5 Multifractal Analysis

In this section, we prove Theorem 3.4.1. We shall only use the assumptions made on the paths of the transmission schedule.

By the stationary increment property, the transmission schedule A_1 has same multifractal spectrum d restricted to any interval of length l, since

$$\{A_1(t) : 0 \leqslant t < l\} \stackrel{\mathrm{d}}{=} \{A_1(t) - A_1(a) : a \leqslant t < a + l\}$$

and the multifractal spectrum d of $\{A_1(t) : a \leqslant t < a + l\}$ is the same as that of $\{A_1(t) - A_1(a) : a \leqslant t < a + l\}$. Thus for any l, the multifractal spectra d of A_1 restricted to the intervals $[il, (i+1)l)$ for $i \geqslant 0$ are non-random and the same. So for all $l > 0$, there is a probability 1 set, on which, for all $i \geqslant 0$, the multifractal spectra d of A_1 restricted to the intervals $[il, (i+1)l)$ are the same and independent of ω. Also, these spectra are the same as d_{A_1} a.e., since d_{A_1} is the supremum of the spectra d of A_1 restricted to countable partitioning intervals. Thus, combining all these facts, we may conclude the following lemma:

Lemma 3.5.1. *If A_1 is a transmission schedule satisfying the conditions of the model, then there is a probability 1 set, on which for all $i, n \in \mathbb{N}$, the multifractal spectrum of A_1 based on exponential growth rate restricted to the intervals $[\frac{i-1}{n}, \frac{i}{n})$ is the same as that of d_{A_1} and independent of ω.*

In the following lemma, we calculate the Hölder exponent of the input process, h_X, bearing in mind Proposition 3.2.2.

Lemma 3.5.2. *If t is not the time of birth or death of a session, then $h_X(t) = \bigwedge h_{A_k}(t - \Gamma_k)$, where the minimum is taken over the indices corresponding to which sessions are active. (We use the convention that the minimum over an empty set is ∞.)*

Proof. First observe that, since X and the A_k's are non-decreasing, we can use the definition in (3.5). Also observe that, till time t, only a finite number of sessions have started, since $\Gamma_k \to \infty$. Thus, at time t, only a finite number of sessions, say n, are active. Since t is not a birth or death time of a session, there exists $\delta > 0$ such that, only these n sessions transmit in the interval $(t - \delta, t + \delta)$. These transmitting sessions contribute terms of the form $A_k(u - \Gamma_k) \wedge J_k = A_k(u - \Gamma_k)$, $u \in (t - \delta, t + \delta)$ to the sum $X(\cdot)$. The non-transmitting sessions contribute terms of the form $A_k(u - \Gamma_k) \wedge J_k$, which are constant at 0 or J_k for $u \in (t - \delta, t + \delta)$ and in either case, the Hölder exponent will be ∞ at t. If there are no active sessions, then X itself is constant and $h_X(t) = \infty$, the same as the minimum of the empty set. Otherwise, only the active sessions contribute to the calculation of the liminf in (3.5) and if $i_1, \ldots, i_n$ are the indices of the active sessions, we have

$$\begin{aligned} h_X(t) &= \liminf_{\varepsilon \downarrow 0} \frac{\log \sum_{k=1}^n (A_{i_k}(t - \Gamma_{i_k} + \varepsilon) - A_{i_k}(t - \Gamma_{i_k} - \varepsilon))}{\log \varepsilon} \\ &= \bigwedge_{k=1}^n \liminf_{\varepsilon \downarrow 0} \frac{\log(A_{i_k}(t - \Gamma_{i_k} + \varepsilon) - A_{i_k}(t - \Gamma_{i_k} - \varepsilon))}{\log \varepsilon} \\ &= \bigwedge_{k=1}^n h_{A_{i_k}}(t - \Gamma_{i_k}). \end{aligned}$$

The penultimate equality holds due to an argument similar to that for (3.4). □

Now we are ready to prove Theorem 3.4.1.

Proof of Theorem 3.4.1. Define for $a < \infty$,

$$E_a^X = \{t > 0 : h_X(t) = a\}.$$

Let E_a^* be the set of points of E_a^X except birth and death points of the sessions. Now observe that E_a^* differs from E_a only by countably many points and thus they have same Hausdorff dimension.

Now, suppose $t \in E_a^*$. Then by Lemma 3.5.2, we have

$$h_X(t) = \bigwedge h_{A_k}(t - \Gamma_k) = a, \tag{3.29}$$

where the minimum is taken over the indices corresponding to the active sessions. If there are no active sessions, then X is constant in a neighborhood of t and hence has Hölder exponent ∞ at t, contradicting the assumption that $a < \infty$. Thus, there are a positive and finite number of sessions running at time t and there exists some

k such that $h_{A_k}(t-\Gamma_k)=a$, that is, $t \in E_a^{A_k}+\Gamma_k$. So, $E_a^* \subset \cup_{k=1}^{\infty}(E_a^{A_k}+\Gamma_k)$ and hence

$$d_X(a) \leqslant \sup_k \dim(E_a^{A_k}+\Gamma_k) = \sup_k d_{A_k}(a),$$

where the last equality holds since Hausdorff dimension is translation-invariant. However, since the A_k's are identically distributed and $d_{A_k}(a)$ is non-random, we have $d_{A_k}(a)$ non-random and constant (independent of both ω and k). Thus,

$$d_X(a) \leqslant d_{A_1}(a) \text{ a.e.}$$

To prove the other inequality, consider the interval $I=[\Gamma_1,(\Gamma_1+\inf\{t : A_1(t) \geqslant J_1\}) \wedge \Gamma_2)$. Since Γ_k's are strictly increasing, $A_1(0+)=0<J_1$ a.e., we have that the interval I is almost surely non-empty. Also, only the first session is active on the interval I. Thus, $X(t)=A_1(t-\Gamma_1)$ and $h_X(t)=h_{A_1}(t-\Gamma_1)$ on I. Then,

$$E_a^X \supset (E_a^{A_1}+\Gamma_1)\cap I = (E_a^{A_1}\cap I')+\Gamma_1,$$

where $I'=[0,\inf\{t : A_1(t) \geqslant J_1\} \wedge (\Gamma_2-\Gamma_1))$. Thus,

$$\dim(E_a^X) \geqslant \dim((E_a^{A_1}\cap I')+\Gamma_1) = \dim(E_a^{A_1}\cap I'),$$

by the translation-invariance of Hausdorff dimension. We already know $\inf\{t : A_1(t) \geqslant J_1\} \wedge (\Gamma_2-\Gamma_1) > 0$ a.e. Then choose a probability 1 set on which the conclusions of Lemma 3.5.1 are satisfied and $\inf\{t : A_1(t) \geqslant J_1\} \wedge (\Gamma_2-\Gamma_1) > 0$ holds. For any ω in this set, choose $n \in \mathbb{N}$, such that,

$$\inf\{t : A_1(t) \geqslant J_1\} \wedge (\Gamma_2-\Gamma_1) > \tfrac{1}{n}.$$

Then we have, for that ω,

$$d_X(a) \geqslant \dim(E_a^{A_1} \cap I') \geqslant \dim(E_a^{A_1} \cap [0, \tfrac{1}{n})) = d_{A_1}(a),$$

where the last equality holds by Lemma 3.5.1. Thus we have with probability 1, $d_X \equiv d_{A_1}$. □

3.6 Large Time Scale Behavior

3.6.1 Poisson process representation

We consider the following Poisson point process to facilitate the analysis:

$$M = \sum_{k=1}^{\infty} \varepsilon_{(\Gamma_k, A_k, J_k)},$$

which has mean measure $\lambda d\gamma \times \mathrm{P}[A_1 \in da, J_1 \in dj]$ on $(0,\infty) \times \mathbb{D}_{\uparrow} \times (0,\infty)$. The random variable $X(T)$ is a function of M restricted to $\mathcal{R}(T) = \{(\gamma, a, j) \in (0,\infty) \times \mathbb{D}_{\uparrow} \times (0,\infty) : \gamma < T\}$ and more precisely,

$$X(T) = \sum_{k=1}^{\infty} [J_k \wedge A_k(T - \Gamma_k)] \, \mathbf{1}_{\mathcal{R}(T)}(\Gamma_k, A_k, J_k).$$

It helps to split $\mathcal{R}(T)$ in two disjoint sets

$$\mathcal{R}_{(1)}(T) = \{(\gamma, a, j) \in (0,\infty) \times \mathbb{D}_{\uparrow} \times (0,\infty) : \gamma < T, j \leqslant a(T-\gamma)\}$$

and

$$\mathcal{R}_{(2)}(T) = \{(\gamma, a, j) \in (0,\infty) \times \mathbb{D}_{\uparrow} \times (0,\infty) : \gamma < T, j > a(T-\gamma)\}.$$

$\mathcal{R}_1(T)$ and $\mathcal{R}_1(T)$ correspond to the regions where transmission has ended or is continuing respectively, by time T. Correspondingly, the input process X breaks into two sums:

$$X_1(T) = \sum_{k=1}^{\infty} J_k \mathbf{1}_{\mathcal{R}_1(T)}(\Gamma_k, A_k, J_k) \tag{3.30}$$

and

$$X_2(T) = \sum_{k=1}^{\infty} A_k(T - \Gamma_k) \mathbf{1}_{\mathcal{R}_2(T)}(\Gamma_k, A_k, J_k). \tag{3.31}$$

Since $X_i(T)$, $i = 1, 2$ are functions of $M|_{\mathcal{R}_i(T)}$, $i = 1, 2$ respectively with $\mathcal{R}_1(T) \cap \mathcal{R}_2(T) = \emptyset$, we have $X_1(T)$ and $X_2(T)$ are independent.

Next we analyze the part restricted to $\mathcal{R}_1(T)$, where the transmission has ended. This part will contribute towards the limiting behavior, the other part due to $\mathcal{R}_2(T)$ will be probabilistically negligible. Observe,

$$\begin{aligned}
\mathrm{E}\left[M\left(\mathcal{R}_1(T)\right)\right] &= \lambda \int_{\gamma=0}^{T} \mathrm{P}[J_1 \leqslant A_1(T-\gamma)]\, d\gamma \\
&= \lambda \int_{\gamma=0}^{T} \mathrm{P}[J_1 \leqslant A_1(\gamma)]\, d\gamma \\
&= \lambda \int_{\gamma=0}^{T} F_L(\gamma)\, d\gamma \qquad\qquad \text{by (3.8)} \\
&=: \lambda \widehat{F}_L(T).
\end{aligned}$$

Thus $\mathrm{E}\left[M\left(\mathcal{R}_1(T)\right)\right]$ is finite for all T and $\mathrm{E}\left[M\left(\mathcal{R}_1(T)\right)\right] \sim \lambda T$, as $\widehat{F}_L(T) \sim T$ and therefore $M|_{\mathcal{R}_1(T)}$ has the following representation:

$$M|_{\mathcal{R}_1(T)} \overset{\mathrm{d}}{=} \sum_{k=1}^{P(T)} \varepsilon_{\left(\tau_k^{(T)}, S_k^{(T)}, W_k^{(T)}\right)}, \tag{3.32}$$

where $P(T)$ is a Poisson random variable with mean $\lambda \widehat{F}_L(T)$ independent of i.i.d. random vectors

$$\left(\tau_k^{(T)}, S_k^{(T)}, W_k^{(T)}\right) \sim \frac{1}{\widehat{F}_L(T)} d\gamma \, \mathrm{P}[A \in da, J \in dj]\Big|_{\mathcal{R}_1(T)}.$$

Then, we have

$$X_1(T) \stackrel{\mathrm{d}}{=} \sum_{k=1}^{P(T)} W_k^{(T)}. \tag{3.33}$$

3.6.2 Tail behavior and moment conditions

Now we study the tail behavior of W_1^T.

Proposition 3.6.1. *For the above representation* (3.32)*, we have*

$$\lim_{T\to\infty} T\,\mathrm{P}\left[W_1^{(T)} > b_J(T)w\right] = w^{-\alpha_J}. \tag{3.34}$$

Proof. First observe that,

$$\begin{aligned} T\,\mathrm{P}[W_1^{(T)} > b_J(T)w] &= \frac{T}{\widehat{F}_L(T)} \int_{\gamma=0}^{T} \mathrm{P}[b_J(T)w < J_1 \leqslant A_1(T-\gamma)]\, d\gamma \\ &= \frac{T}{\widehat{F}_L(T)} \int_{\gamma=0}^{T} \mathrm{P}[b_J(T)w < J_1 \leqslant A_1(\gamma)]\, d\gamma & (3.35) \\ &\sim \int_{\gamma=0}^{T} \mathrm{P}[b_J(T)w < J_1 \leqslant A_1(\gamma)]\, d\gamma. & (3.36) \\ &\leqslant T\,\mathrm{P}[J_1 > b_J(T)w] \to w^{-\alpha_J}. & \text{by (3.21)} \end{aligned}$$

Thus, we have

$$\limsup_{T\to\infty} T\,\mathrm{P}\left[W_1^{(T)} > b_J(T)w\right] = w^{-\alpha_J}. \tag{3.37}$$

Till now we have not used any assumption about the joint distribution of (J_1, A_1). For the lower bound, we need to use (3.25) of the assumption (9′). Then, we have, from (3.36)

$$
\begin{aligned}
T\,\mathrm{P}[W_1^{(T)} > b_J(T)w] &\sim \int_{\gamma=0}^{T} \mathrm{P}[b_J(T)w < J_1 \leqslant A_1(\gamma)]\,d\gamma \\
&= \frac{\sigma^{\leftarrow}(b_J(T))}{T} \int_{\gamma=0}^{\frac{T}{\sigma^{\leftarrow}(b_J(T))}} T\,\mathrm{P}[b_J(T)w < J_1 \leqslant A_1(\sigma^{\leftarrow}(b_J(T))\gamma)]\,d\gamma \\
&\geqslant \frac{\sigma^{\leftarrow}(b_J(T))}{T} \int_{\gamma=N}^{\frac{T}{\sigma^{\leftarrow}(b_J(T))}} T\,\mathrm{P}[b_J(T)w < J_1 \leqslant A_1(\sigma^{\leftarrow}(b_J(T))N))]\,d\gamma
\end{aligned} \tag{3.38}
$$

$$
\begin{aligned}
&= \left(1 - N\frac{\sigma^{\leftarrow}(b_J(T))}{T}\right) T\,\mathrm{P}\left[w < \frac{J_1}{b_J(T)} \leqslant \frac{A_1(\sigma^{\leftarrow}(b_J(T))N)}{b_J(T)}\right] \\
&\sim T\,\mathrm{P}\left[w < \frac{J_1}{b_J(T)} \leqslant \frac{A_1(\sigma^{\leftarrow}(b_J(T))N)}{b_J(T)}\right] && (3.39)\\
&\geqslant T\,\mathrm{P}\left[w < \frac{J_1}{b_J(T)} \leqslant K, \frac{A_1(\sigma^{\leftarrow}(b_J(T))N)}{b_J(T)} \geqslant K\right] && (3.40)\\
&\to (w^{-\alpha_J} - K^{-\alpha_J})\,\mathrm{P}[\chi(N) \geqslant K] && (3.41)\\
&= (w^{-\alpha_J} - K^{-\alpha_J})\,\mathrm{P}[\chi(1) \geqslant KN^{-H}]. && (3.42)
\end{aligned}
$$

The inequality in (3.38) holds for any N and all large enough T. The asymptotic equivalence in (3.39) holds since $\frac{\sigma^{\leftarrow}(b_J(T))}{T} \to 0$ as the function is regularly varying of index $\frac{1}{H\alpha_J} - 1 < 0$. The inequality holds for all $K > w$. The convergence in (3.41) holds for all continuity points K of the distribution of $\chi(N)$ due to (3.25). Finally, (3.42) holds due to the H-self-similarity of χ. Thus, for all N and for all $K > w$, which are continuity points of the distribution of $\chi(N)$ for all $N \in \mathbb{N}$, we have

$$\liminf_{T\to\infty} T\,\mathrm{P}[W_1^{(T)} > Mb_J(T)] \geqslant (w^{-\alpha_J} - K^{-\alpha_J})\,\mathrm{P}[\chi(1) \geqslant KN^{-H}].$$

Since χ is proper and hence $\mathrm{P}[\chi \equiv 0] = 0$, we have, by Theorem 2.4(a) of Vervaat (1985), $\mathrm{P}[\chi(1) > 0] = 1$. Then first letting $N \to \infty$ through natural numbers and then letting $K \to \infty$ through the continuity points of the distribution of $\chi(N)$, for all $N \in \mathbb{N}$, we get

$$\liminf_{T\to\infty} T\,\mathrm{P}[W_1^{(T)} > Mb_J(T)] \geqslant w^{-\alpha_J}.$$

Combining this with (3.37), we have,

$$\lim_{T\to\infty} T\,\mathrm{P}[W_1^{(T)} > Mb_J(T)] = w^{-\alpha_J}.$$

□

Next we need to check a few moment conditions, which are summarized in the following lemmas.

Lemma 3.6.1. *For the above representation* (3.32)*, we have*

$$\lim_{K\to\infty} \limsup_{T\to\infty} T\,\mathrm{E}\left[\frac{W_1^{(T)}}{b_J(T)} \mathbf{1}_{\left[\frac{W_1^{(T)}}{b_J(T)} > K\right]}\right] = 0. \tag{3.43}$$

Proof. We have

$$T\,\mathrm{E}\left[\frac{W_1^{(T)}}{b_J(T)} \mathbf{1}_{\left[\frac{W_1^{(T)}}{b_J(T)} > K\right]}\right] = KT\,\mathrm{P}\left[W_1^{(T)} > Kb_J(T)\right] + \int_{w=K}^{\infty} T\,\mathrm{P}\left[W_1^{(T)} > b_J(T)w\right] dw \tag{3.44}$$

First we consider the second term on the right side of (3.44). Note that, by (3.35)

$$\begin{aligned}\int\limits_{w=K}^{\infty} T\,\mathrm{P}[W_1^{(T)} > b_J(T)w]\,dw &= \frac{T}{\widehat{F}_L(T)}\int\limits_{w=K}^{\infty}\int\limits_{\gamma=0}^{T}\mathrm{P}[b_J(T)w < J_1 \leqslant A_1(\gamma)]\,d\gamma\,dw\\ &\leqslant \frac{T^2}{\widehat{F}_L(T)}\int\limits_{w=K}^{\infty}\mathrm{P}[J_1 > b_J(T)w]\,dw\\ &\sim 1\cdot\frac{T}{b_J(T)}\frac{Kb_J(T)\,\mathrm{P}[J_1 > Kb_J(T)]}{\alpha_J - 1} \quad \text{by Karamata's theorem}\\ &\sim \frac{1}{\alpha_J - 1}K^{1-\alpha_J}. \end{aligned} \tag{3.45}$$

Also, for the first term of the right side of (3.44), we have, from (3.34),

$$\lim_{T\to\infty} KT\,\mathrm{P}\left[W_1^{(T)} > Kb_J(T)\right] = K^{1-\alpha_J}. \tag{3.46}$$

Thus, adding (3.46) and (3.45), and using (3.44), we have,

$$\limsup_{T\to\infty} T\,\mathrm{E}\left[\frac{W_1^{(T)}}{b_J(T)}\mathbf{1}_{\left[\frac{W_1^{(T)}}{b_J(T)} > K\right]}\right] \leqslant \frac{\alpha_J}{\alpha_J - 1}K^{1-\alpha_J}$$

and, since $\alpha_J > 1$, we have (3.43). □

Lemma 3.6.2. *For the above representation* (3.32), *we have for all* $K > 0$,

$$\limsup_{T\to\infty} T\,\mathrm{Var}\left[\frac{W_1^{(T)}}{b_J(T)}\mathbf{1}_{\left[\frac{W_1^{(T)}}{b_J(T)} \leqslant K\right]}\right] < \infty \tag{3.47}$$

and

$$\lim_{\varepsilon\to 0}\limsup_{T\to\infty} T\,\mathrm{Var}\left[\frac{W_1^{(T)}}{b_J(T)}\mathbf{1}_{\left[\frac{W_1^{(T)}}{b_J(T)} \leqslant \varepsilon\right]}\right] = 0 \tag{3.48}$$

Proof. Using (3.35) and Karamata's theorem, we have

$$\begin{aligned}
T\operatorname{Var}\left[\frac{W_1^{(T)}}{b_J(T)}\mathbf{1}_{\left[\frac{W_1^{(T)}}{b_J(T)}\leqslant K\right]}\right] &\leqslant T\operatorname{E}\left[\left(\frac{W_1^{(T)}}{b_J(T)}\right)^2\mathbf{1}_{\left[\frac{W_1^{(T)}}{b_J(T)}\leqslant K\right]}\right]\\
&= \int\limits_{w=0}^{K} 2wT\operatorname{P}\left[w<\frac{W_1^{(T)}}{b_J(T)}\leqslant K\right]dw\\
&= \frac{T}{\widehat{F}_L(T)}\int\limits_{w=0}^{K}\int\limits_{\gamma=0}^{T} 2w\operatorname{P}\left[J_1\leqslant A_1(\gamma), w<\frac{J_1}{b_J(T)}\leqslant K\right]d\gamma\,dw\\
&\leqslant \frac{T}{\widehat{F}_L(T)}\int\limits_{w=0}^{K} 2wT\operatorname{P}\left[w<\frac{J_1}{b_J(T)}\leqslant K\right]dw\\
&\sim 2T\int\limits_{w=0}^{K} wT\operatorname{P}\left[J_1>wb_J(T)\right]dw - TK^2\operatorname{P}\left[J_1>b_J(T)K\right]\\
&\sim \frac{2T}{(b_J(T))^2}\int\limits_{w=0}^{b_J(T)K} wT\operatorname{P}\left[J_1>w\right]dw - K^{2-\alpha_J}\\
&\sim \frac{2TK^2\operatorname{P}[J_1>b_J(T)K]}{2-\alpha_J} - K^{2-\alpha_J}\\
&\sim \frac{2}{2-\alpha_J}K^{2-\alpha_J} - K^{2-\alpha_J} = \frac{\alpha_J}{2-\alpha_J}K^{2-\alpha_J}<\infty
\end{aligned}$$

and hence we have (3.47). Also, since $\alpha_J<2$, we also have (3.48). □

Lemma 3.6.3. *For the above representation* (3.32), *we have*

$$\lim_{T\to\infty}\operatorname{E}\left[\frac{W_1^{(T)}}{b_J(T)}\right]=0. \tag{3.49}$$

Proof. Using (3.35), we have

$$\operatorname{E}\left[\frac{W_1^{(T)}}{b_J(T)}\right] = \int\limits_{w=0}^{\infty}\operatorname{P}\left[W_1^{(T)}>b_J(T)w\right]dw$$

$$= \frac{1}{\widehat{F}_L(T)} \int\limits_{w=0}^{\infty} \int\limits_{\gamma=0}^{T} \mathrm{P}[b_J(T)w < J_1 \leqslant A_1(\gamma)]\, d\gamma\, dw$$

$$\leqslant \frac{T}{\widehat{F}_L(T)} \int\limits_{w=0}^{\infty} \mathrm{P}[b_J(T)w < J_1]\, dw \sim \frac{\mathrm{E}[J_1]}{b_J(T)} \to 0.$$

□

3.6.3 One-dimensional convergence

Now we are ready to prove Theorem 3.4.2 for the process X_1 in the one-dimensional case, although with a different centering.

Lemma 3.6.4. *Under assumptions and notations used in Theorem 3.4.2, we have*

$$\frac{\sum\limits_{k=1}^{P(T)} W_k^{(T)} - P(T)\,\mathrm{E}\left[W_1^{(T)}\right]}{b_J(T)} \Rightarrow Z_{\alpha_J}(1),$$

where $P(T)$ and $W_k^{(T)}$ are defined in (3.32)*.*

Proof. Define

$$R_T(t) := \sum_{k=1}^{\lfloor Tt \rfloor} \left\{ \frac{W_k^{(T)}}{b_J(T)} - \mathrm{E}\left[\frac{W_1^{(T)}}{b_J(T)}\right] \right\}.$$

Then, using (3.34), (3.43) (3.47), (3.48) and (3.49), we have, as in Section 2 of Resnick and Samorodnitsky (2000), that $R_T \Rightarrow \Xi_{\alpha_J}(1)$ in $\mathbb{D}$ endowed with Skorohod's J_1 topology, where Ξ_α is α-stable Lévy motion with mean 0, skewness parameter 1 and scale parameter $C_\alpha^{-\frac{1}{\alpha}}$, where C_α is defined by (3.13).

Since $P(T)$ has a Poisson distribution with mean $\lambda \widehat{F}_L(T)$ and $\widehat{F}_L(T) \sim T \to \infty$, we have

$$\frac{P(T)}{T} \xrightarrow{\mathrm{P}} \lambda.$$

By independence of R_T and $P(T)$, we have

$$\left(R_T, \frac{P(T)}{T}\right) \Rightarrow (\Xi_{\alpha_J}, \lambda) \text{ in } \mathbb{D} \times [0, \infty). \tag{3.50}$$

Now, we know from Theorem 3.3.2 that the function $\pi : \mathbb{D} \times [0, \infty) \to [0, \infty)$ defined by

$$\pi(x, t) = x(t) \tag{3.51}$$

is continuous at (x, t) iff x is continuous at t. Also, since we know from Theorem 3.3.4 Ξ_{α_J} has no fixed point of discontinuity, we have that Ξ_{α_J} is continuous at λ with probability 1. Thus, using the Continuous Mapping Theorem on the convergence in (3.50), we get

$$\frac{\sum_{k=1}^{P(T)} W_k^{(T)} - P(T)\,\mathrm{E}\left[W_1^{(T)}\right]}{b_J(T)} = R_T\left(\frac{P(T)}{T}\right) \Rightarrow \Xi_{\alpha_J}(\lambda) = Z_{\alpha_J}(1) \text{ in } \mathbb{R}.$$

□

Next we change the centering to the one suggested in Theorem 3.4.2. We use the assumptions (9), (10) and (11) to study the tail behavior of L_1, which we then use to change the centering.

Proposition 3.6.2. *Under the assumptions* $(9'), (10')$ *and* (11)*, stated in* (3.25)*,* (3.26) *and* (3.12)*,* L_1 *has a tail of index* $H\alpha_J$*.*

Proof. We consider the function $\Psi : (0, \infty) \times \mathbb{D} \to (0, \infty) \times \tilde{\mathbb{D}}$ defined by $\Psi(t, x) = (t, x^{\leftarrow})$. By Corollary 3.3.3, Ψ is continuous at $(t, x) \in (0, \infty) \times \mathbb{D}_{\uparrow}$. Also the function π, defined by (3.51), is continuous at (t, x), if x is continuous at t. Now χ is supported on $\mathbb{D}_{\uparrow}$ and does not have any fixed point of discontinuity.

Thus the map $\pi \circ \Psi(t,x) = x^{\leftarrow}(t)$ is continuous with probability 1. Hence by the Continuous Mapping Theorem, we have, for all $\varepsilon > 0$,

$$T\,\mathrm{P}\left[\frac{A_1^{\leftarrow}\left(J_1\mathbf{1}_{\left[\frac{J_1}{b_J(T)}>\varepsilon\right]}\right)}{\sigma^{\leftarrow}(b_J(T))} \in \cdot\right] \xrightarrow{\mathrm{w}} \iint\limits_{\{(t,x):x^{\leftarrow}(t)\in\cdot\}} \nu^{\varepsilon}_{\alpha_J}(dt)\,\mathrm{P}[\chi \in dx].$$

Hence for all $\varepsilon > 0$ and $\gamma > 0$ such that $\{\gamma\}$ has zero limit measure, we have

$$T\,\mathrm{P}\left[\frac{A_1^{\leftarrow}\left(J_1\mathbf{1}_{\left[\frac{J_1}{b_J(T)}>\varepsilon\right]}\right)}{\sigma^{\leftarrow}(b_J(T))} > \gamma\right] \to \iint\limits_{\substack{t>\varepsilon\\ x^{\leftarrow}(t)>\gamma}} \nu^{\varepsilon}_{\alpha_J}(dt)\,\mathrm{P}[\chi \in dx]$$

$$\text{or, } T\,\mathrm{P}\left[\frac{A_1^{\leftarrow}(J_1)}{\sigma^{\leftarrow}(b_J(T))} > \gamma, \frac{J_1}{b_J(T)} > \varepsilon\right] \to \int (\varepsilon \vee x(\gamma))^{-\alpha_J}\,\mathrm{P}[\chi \in dx]$$

$$\text{or, } T\,\mathrm{P}\left[\frac{L_1}{\sigma^{\leftarrow}(b_J(T))} > \gamma, \frac{J_1}{b_J(T)} > \varepsilon\right] \to \mathrm{E}\left[(\varepsilon \vee \chi(\gamma))^{-\alpha_J}\right],$$

Finally, letting $\varepsilon \downarrow 0$, using the Monotone Convergence Theorem, we have

$$\lim_{\varepsilon\downarrow 0}\lim_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{\sigma^{\leftarrow}(b_J(T))} > \gamma, \frac{J_1}{b_J(T)} > \varepsilon\right] = \mathrm{E}\left[\chi(\gamma)^{-\alpha_J}\right] = \gamma^{-H\alpha_J}\,\mathrm{E}\left[\chi(1)^{-\alpha_J}\right],$$

which is finite due to (3.12). Also from (3.26), we have that

$$\lim_{\varepsilon\downarrow 0}\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{\sigma^{\leftarrow}(b_J(T))} > \gamma, \frac{J_1}{b_J(T)} \leqslant \varepsilon\right] = 0.$$

Thus, given $\delta > 0$, we can choose $\varepsilon > 0$, such that the following hold:

$$\gamma^{-H\alpha_J}\,\mathrm{E}\left[\chi(1)^{-\alpha_J}\right] - \delta < \lim_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{\sigma^{\leftarrow}(b_J(T))} > \gamma, \quad \frac{J_1}{b_J(T)} > \varepsilon\right] \leqslant \gamma^{-H\alpha_J}\,\mathrm{E}\left[\chi(1)^{-\alpha_J}\right],$$

and

$$0 \leqslant \limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{\sigma^{\leftarrow}(b_J(T))} > \gamma, \frac{J_1}{b_J(T)} \leqslant \varepsilon\right] < \frac{\delta}{2}.$$

Thus, there exists T_0 such that for all $T > T_0$, we have

$$\gamma^{-H\alpha_J} \operatorname{E}\left[\chi(1)^{-\alpha_J}\right] - \delta < T \operatorname{P}\left[\frac{L_1}{\sigma^{\leftarrow}(b_J(T))} > \gamma, \frac{J_1}{b_J(T)} > \varepsilon\right] < \gamma^{-H\alpha_J} \operatorname{E}\left[\chi(1)^{-\alpha_J}\right] + \frac{\delta}{2},$$

and

$$0 \leqslant T \operatorname{P}\left[\frac{L_1}{\sigma^{\leftarrow}(b_J(T))} > \gamma, \frac{J_1}{b_J(T)} \leqslant \varepsilon\right] < \frac{\delta}{2}.$$

Thus adding, we can conclude that for all $T > T_0$,

$$\gamma^{-H\alpha_J} \operatorname{E}\left[\chi(1)^{-\alpha_J}\right] - \delta < T \operatorname{P}\left[\frac{L_1}{\sigma^{\leftarrow}(b_J(T))} > \gamma\right] < \gamma^{-H\alpha_J} \operatorname{E}\left[\chi(1)^{-\alpha_J}\right] + \delta.$$

Hence, we have

$$\lim_{T\to\infty} T \operatorname{P}\left[\frac{L_1}{\sigma^{\leftarrow}(b_J(T))} > \gamma\right] = \gamma^{-H\alpha_J} \operatorname{E}\left[\chi(1)^{-\alpha_J}\right],$$

which in turn implies that L_1 has a tail of index $H\alpha_J$. □

Now, we use Proposition 3.6.2 to obtain the required centering.

Lemma 3.6.5. *Under assumptions and notations as used in Theorem 3.4.2, we have*

$$\frac{X_1(T) - \lambda T \mu_J}{b_J(T)} \Rightarrow Z_{\alpha_J}(1),$$

where $\mu_J = \operatorname{E}(J_1)$.

Proof. Due to (3.33) and Lemma 3.6.4, it is enough to prove that

$$P(T) \operatorname{E}\left[\frac{W_1^{(T)}}{b_J(T)}\right] - \frac{\lambda T \mu_J}{b_J(T)} \xrightarrow{\operatorname{P}} 0. \tag{3.52}$$

We rewrite the left side of (3.52) as

$$P(T) \operatorname{E}\left[\frac{W_1^{(T)}}{b_J(T)}\right] - \frac{\lambda T \mu_J}{b_J(T)}$$

$$=\frac{P(T)-\lambda\widehat{F}_L(T)}{\sqrt{\lambda\widehat{F}_L(T)}}\sqrt{\lambda\widehat{F}_L(T)}\,\mathrm{E}\left[\frac{W_1^{(T)}}{b_J(T)}\right]-\frac{\lambda}{b_J(T)}\left(T\mu_J-\widehat{F}_L(T)\,\mathrm{E}\left[W_1^{(T)}\right]\right). \tag{3.53}$$

Now, we observe that, using (3.35)

$$\begin{aligned}\sqrt{\lambda\widehat{F}_L(T)}\,\mathrm{E}\left[\frac{W_1^{(T)}}{b_J(T)}\right] &= \sqrt{\lambda\widehat{F}_L(T)}\int_{w=0}^{\infty}\mathrm{P}\left[W_1^{(T)}>b_J(T)w\right]dw\\ &= \sqrt{\frac{\lambda}{\widehat{F}_L(T)}}\int_{w=0}^{\infty}\int_{\gamma=0}^{T}\mathrm{P}[b_J(T)w<J_1\leqslant A_1(\gamma)]\,d\gamma\,dw\\ &\leqslant \sqrt{\frac{\lambda}{\widehat{F}_L(T)}}T\int_{w=0}^{\infty}\mathrm{P}[J_1>b_J(T)w]\,dw\\ &\sim \frac{\sqrt{T\lambda}}{b_J(T)}\,\mathrm{E}[J_1]\to 0,\end{aligned}$$

since $\frac{\sqrt{T}}{b_J(T)}\in RV_{\frac{1}{2}-\frac{1}{\alpha_J}}$ and $\alpha_J<2$. Also, $P(T)$ having a Poisson distribution with mean $\lambda\widehat{F}_L(T)\sim\lambda T\to\infty$, we have, from the central limit theorem, that $\frac{P(T)-\lambda\widehat{F}_L(T)}{\sqrt{\lambda\widehat{F}_L(T)}}$ converges weakly to a standard normal distribution and hence is bounded in probability. Thus by Slutsky's theorem the first term on the right side of (3.53) goes to 0 in probability, i.e.,

$$\frac{P(T)-\lambda\widehat{F}_L(T)}{\sqrt{\lambda\widehat{F}_L(T)}}\sqrt{\lambda\widehat{F}_L(T)}\,\mathrm{E}\left[\frac{W_1^{(T)}}{b_J(T)}\right]\xrightarrow{\mathrm{P}}0. \tag{3.54}$$

Now we consider the second term on the right side of (3.53). Observe that

$$T\mu_J=\int_{j=0}^{\infty}\int_{\gamma=0}^{T}\mathrm{P}[J_1>j]\,d\gamma\,dj$$

and

$$\widehat{F}_L(T)\,\mathrm{E}\left[W_1^{(T)}\right]=\widehat{F}_L(T)\int_{j=0}^{\infty}\mathrm{P}\left[W_1^{(T)}>j\right]dj=\int_{j=0}^{\infty}\int_{\gamma=0}^{T}\mathrm{P}[j<J_1\leqslant A_1(\gamma)]\,d\gamma\,dj$$

by (3.35). Thus, we have,

$$T\mu_J - \widehat{F}_L(T)\,\mathrm{E}\left[W_1^{(T)}\right] = \int\limits_{j=0}^{\infty}\int\limits_{\gamma=0}^{T} \mathrm{P}[J_1 > j, J_1 > A_1(\gamma)]\,d\gamma\,dj \tag{3.55}$$

$$= \int\limits_{j=0}^{\infty}\int\limits_{\gamma=0}^{T} \mathrm{P}[J_1 > j, L_1 > \gamma]\,d\gamma\,dj$$

$$\leqslant \int\limits_{\gamma=0}^{T} \left[\gamma\overline{F}_L(\gamma) + \int\limits_{j=\gamma}^{\infty} \overline{F}_J(j)\,dj\right] d\gamma.$$

Hence,

$$\frac{\lambda}{b_J(T)}\left(T\mu_J - \widehat{F}_L(T)\,\mathrm{E}\left[W_1^{(T)}\right]\right) = \frac{\lambda}{b_J(T)}\int\limits_{\gamma=0}^{T} \left[\gamma\overline{F}_L(\gamma) + \int\limits_{j=\gamma}^{\infty} \overline{F}_J(j)\,dj\right] d\gamma.$$

Now, $\overline{F}_J \in RV_{-\alpha_J}$ with $1 < \alpha_J < 2$, and, therefore, using Karamata's theorem twice in succession, we obtain that

$$\frac{\lambda}{b_J(T)}\int\limits_{\gamma=0}^{T}\int\limits_{j=\gamma}^{\infty} \overline{F}_J(j)\,dj \sim \frac{\lambda}{(2-\alpha_J)(\alpha_J-1)}\frac{T^2\overline{F}_J(T)}{b_J(T)} \in RV_{2-\alpha_J-\frac{1}{\alpha_J}}$$

and because $2-\alpha_J-\frac{1}{\alpha_J} = -\frac{(\alpha_J-1)^2}{\alpha_J} < 0$, we have

$$\frac{\lambda}{b_J(T)}\int\limits_{\gamma=0}^{T}\int\limits_{j=\gamma}^{\infty} \overline{F}_J(j)\,dj \to 0.$$

Also, since $\overline{F}_L \in RV_{-H\alpha_J}$, if $H\alpha_J \leqslant 2$, using Karamata's theorem again, we have

$$\frac{\lambda}{b_J(T)}\int\limits_{\gamma=0}^{T} \gamma\overline{F}_L(\gamma)\,d\gamma \sim \frac{\lambda}{2-\alpha_J}\frac{T^2\overline{F}_L(T)}{b_J(T)} \in RV_{2-H\alpha_J-\frac{1}{\alpha_J}},$$

and because $2-H\alpha_J-\frac{1}{\alpha_J} \leqslant 2-\alpha_J-\frac{1}{\alpha_J} < 0$ (recall $H \geqslant 1$ necessarily), we again have

$$\frac{\lambda}{b_J(T)}\int\limits_{\gamma=0}^{T} \gamma\overline{F}_L(\gamma)\,d\gamma \to 0.$$

If $H\alpha_J > 2$, then we know that L_1 has finite second moment and from $b_J(T) \to \infty$, we conclude that

$$\frac{\lambda}{b_J(T)} \int\limits_{\gamma=0}^{T} \gamma \overline{F}_L(\gamma)\, d\gamma \to 0 \cdot \int\limits_{\gamma=0}^{\infty} \gamma \overline{F}_L(\gamma)\, d\gamma = 0 \cdot \frac{1}{2}\, \mathrm{E}[L_1^2] = 0.$$

Thus,

$$\frac{\lambda}{b_J(T)} \left(T\mu_J - \widehat{F}_L(T)\, \mathrm{E}\left[W_1^{(T)} \right] \right) \to 0, \tag{3.56}$$

which along with (3.54) gives (3.52) and hence completes the proof of Lemma 3.6.5. □

Now, we are ready to complete the proof of one-dimensional convergence provided we show the negligibility of the part due to X_2, which we do in the next lemma.

Lemma 3.6.6. *Under assumptions and notations as used in Theorem 3.4.2, we have*

$$\frac{X_2(T)}{b_J(T)} \xrightarrow{\mathrm{P}} 0, \tag{3.57}$$

where X_2 is defined by (3.31).

Proof. First observe that,

$$X_2(T) = \iiint\limits_{\mathcal{R}_2(T)} a(T-\gamma)\, M(d\gamma, da, dj) \leqslant \iiint\limits_{\mathcal{R}_2(T)} j\, M(d\gamma, da, dj), \tag{3.58}$$

since $a(T-\gamma) < j$ on $\mathcal{R}_2(T)$. Also, since M is a Poisson point process with mean measure $\lambda d\gamma \times \mathrm{P}[A_1 \in da, J_1 \in dj]$, we have that

$$\mathrm{E}\left[\iiint\limits_{\mathcal{R}_2(T)} j\, M(d\gamma, da, dj) \right] = \lambda \iiint\limits_{\mathcal{R}_2(T)} j\, d\gamma\, \mathrm{P}[A_1 \in da, J_1 \in dj]$$

$$= \lambda \int_{\gamma=0}^{T} \iint_{j>a(t-\gamma)} j \, d\gamma \, \mathrm{P}[A_1 \in da, J_1 \in dj]$$

$$= \lambda \int_{\gamma=0}^{T} \iint_{j>a(t-\gamma)} \int_{x<j} dx \, d\gamma \, \mathrm{P}[A_1 \in da, J_1 \in dj]$$

$$= \lambda \int_{\gamma=0}^{T} \int_{x=0}^{\infty} \iint_{\substack{j>x \\ j>a(T-\gamma)}} \mathrm{P}[A_1 \in da, J_1 \in dj] \, dx \, d\gamma$$

$$= \lambda \int_{\gamma=0}^{T} \int_{x=0}^{\infty} \mathrm{P}[J_1 > x, J_1 > A_1(T-\gamma)] \, dx \, d\gamma$$

$$= \lambda \int_{\gamma=0}^{T} \int_{j=0}^{\infty} \mathrm{P}[J_1 > j, J_1 > A_1(\gamma)] \, dj \, d\gamma$$

Then we have,

$$\begin{aligned} \mathrm{E}\left[\frac{1}{b_J(T)} \iiint_{\mathcal{R}_2(T)} j \, M(d\gamma, da, dj)\right] &= \frac{\lambda}{b_J(T)} \int_{j=0}^{\infty} \int_{\gamma=0}^{T} \mathrm{P}[J_1 > j, J_1 > A_1(\gamma)] \, d\gamma \, dj \\ &= \frac{\lambda}{b_J(T)} \left(T\mu_J - \widehat{F}_L(T) \, \mathrm{E}\left[W_1^{(T)}\right]\right) \quad \text{by (3.55)} \\ &\to 0 \quad \text{by (3.56)} \end{aligned}$$

Thus, we have,

$$\frac{1}{b_J(T)} \iiint_{\mathcal{R}_2(T)} j \, M(d\gamma, da, dj) \xrightarrow{\mathrm{P}} 0. \tag{3.59}$$

Hence, combining (3.58) and (3.59), we conclude that

$$\frac{X_2(T)}{b_J(T)} \leqslant \frac{1}{b_J(T)} \iiint_{\mathcal{R}_2(T)} j \, M(d\gamma, da, dj) \xrightarrow{\mathrm{P}} 0.$$

□

Finally, we complete the proof of Theorem 3.4.2 in the one-dimensional case.

Proof of Theorem 3.4.2. (for the one-dimensional case) Recall from Lemma 3.6.5 that

$$\frac{X_1(T) - \lambda T \mu_J}{b_J(T)} \Rightarrow Z_{\alpha_J}(1),$$

and from Lemma 3.6.6 that

$$\frac{X_2(T)}{b_J(T)} \xrightarrow{\text{P}} 0.$$

Adding them we get,

$$\frac{X(T) - \lambda T \mu_J}{b_J(T)} \Rightarrow Z_{\alpha_J}(1).$$

Thus, since $b_J \in RV_{\frac{1}{\alpha_J}}$, we have, for all $t > 0$,

$$Y_T(t) = \frac{X(Tt) - \lambda Tt\mu_J}{b_J(T)} = \frac{b_J(Tt)}{b_J(T)} \frac{X(T) - \lambda T \mu_J}{b_J(Tt)} \Rightarrow t^{\frac{1}{\alpha_J}} Z_{\alpha_J}(1) = Z_{\alpha_J}(t). \quad (3.60)$$

□

3.6.4 Finite dimensional convergence

We complete the proof of Theorem 3.4.2 by showing finite dimensional convergence.

Proof of Theorem 3.4.2. Let $0 < s < t$. Observe that

$$X_1(Tt) - X_1(Ts) = \iiint\limits_{\mathcal{R}_1(Tt)\setminus\mathcal{R}_1(Ts)} j\, M(d\gamma, da, dj)$$

is independent of

$$X_1(Tu) = \iiint\limits_{\mathcal{R}_1(Tu)} j\, M(d\gamma, da, dj) \quad \forall u \leqslant s,$$

since they are the functions of Poisson point process restricted to disjoint sets. Hence $X_1(T\cdot)$ has independent increments. Also, let us define

$$B_T(s,t) = \iiint\limits_{\substack{Ts<\gamma\leqslant Tt\\ j\leqslant a(Tt-\gamma)}} j\, M(d\gamma, da, dj) = \iiint\limits_{\mathcal{R}_1(T(t-s))+(Ts,0,0)} j\, M(d\gamma, da, dj)$$

and

$$C_T(s,t) = \iiint\limits_{\substack{\gamma\leqslant Ts\\ a(Ts-\gamma)<j\leqslant a(Tt-\gamma)}} j\, M(d\gamma, da, dj).$$

Observe that $X_1(Tt) - X_1(Ts) = B_T(s,t) + C_T(s,t)$. Now, setting $N(\cdot) = M(\cdot + (Ts, 0, 0))$, we get

$$B_T(s,t) = \iiint\limits_{\mathcal{R}_1(T(t-s))} j\, N(d\gamma, da, dj) \stackrel{\mathrm{d}}{=} \iiint\limits_{\mathcal{R}1(T(t-s))} j\, M(d\gamma, da, dj) = X_1(T(t-s)),$$

where the equality in distribution follows from the fact that, by invariance of Lebesgue measure under translation, M and N have same mean measure and hence the same distribution. So, by (3.60), we have,

$$\frac{B_T(s,t) - \lambda T(t-s)\mu_J}{b_J(T)} \Rightarrow Z_{\alpha_J}(t-s) \stackrel{\mathrm{d}}{=} Z_{\alpha_J}(t) - Z_{\alpha_J}(s). \tag{3.61}$$

Also, we have,

$$C_T(s,t) \leqslant \iiint\limits_{\mathcal{R}_2(Ts)} j\, M(d\gamma, da, dj)$$

and (3.59) implies

$$\frac{C_T(s,t)}{b_J(T)} \leqslant \frac{1}{b_J(T)} \iiint\limits_{\mathcal{R}_2(T)} j\, M(d\gamma, da, dj) = \frac{b_J(Ts)}{b_J(T)} \frac{1}{b_J(Ts)} \iiint\limits_{\mathcal{R}_2(T)} j\, M(d\gamma, da, dj) \stackrel{\mathrm{P}}{\to} 0. \tag{3.62}$$

Then adding (3.61) and (3.62), we get

$$Y_1^{(T)}(t) - Y_1^{(T)}(s) \Rightarrow Z_{\alpha_J}(t) - Z_{\alpha_J}(s),$$

where

$$Y_1^{(T)}(t) = \frac{X_1(Tt) - \lambda T t \mu_J}{b_J(T)}.$$

By the independent increment property of $Y_1^{(T)}$ and Z_{α_J}, coordinatewise convergence of increments implies joint convergence of increments. Thus,

$$Y_1^{(T)} \stackrel{\text{fidi}}{\to} Z_{\alpha_J}. \tag{3.63}$$

Also, by (3.57),

$$\frac{X_2(Tt)}{b_J(T)} = \frac{X_2(Tt)}{b_J(Tt)} \cdot \frac{b_J(Tt)}{b_J(T)} \stackrel{\text{P}}{\to} 0.$$

Thus, we have, for all $0 \leqslant t_1 < \cdots < t_k$,

$$\frac{X_2(Tt_1), \cdots, X_2(Tt_k)}{b_J(T)} \stackrel{\text{P}}{\to} \mathbf{0}. \tag{3.64}$$

Adding, (3.63) and (3.64), we get

$$Y^{(T)} \stackrel{\text{fidi}}{\to} Z_{\alpha_J}.$$

□

Chapter 4

Gaussian and Multifractal Nature of a Network Traffic Model

4.1 Introduction

The model proposed in Chapter 3 succeeded to integrate the empirically observed behaviors for both large and small time scales. However, the result for large time scale was not completely satisfactory. Due to lack of empirical evidence for heavy tailed traffic rates, a Gaussian limit will be more useful. Among other sources, Riedi and Willinger (2000), Willinger et al. (1997) argue for a Gaussian approximation both from the empirical point of view as well as heuristically. Mikosch et al. (2002) considered a family of $M/G/\infty$ models with increasing input rate. They showed that the possible limit for large time scale depends on the growth of the input rate and may be either fractional Brownian motion (fBm) or a stable Lévy motion.

However, they considered a deterministic linear transmission schedule, which is unrealistic and does not allow for mutlifractal behavior at small time scale.

This chapter tries to generalize the results obtained in Riedi and Willinger (2000), Willinger et al. (1997). We propose a sequence of $M/G/\infty$ model with random transmission schedule. The model gives the multifractal behavior at the microscopic level. At the macroscopic level, for the slow growth of the input rate, we get a stable Lévy limit, whereas the fast growth gives a Gaussian limit.

This chapter is arranged as follows: Section 4.2 describes the model used in this chapter, as well as discusses the critical input rate. In Sections 4.3 and 4.4, we collect useful results for further analyses. In Section 4.5 and 4.6, we consider the slow and the fast growth cases respectively.

4.2 The Model

In this section, we describe the assumptions of the model, which is analogous to the model considered in Chapter 3. We quickly recollect the important features and point out the changes.

1. We denote the time when k-th transmission begins by Γ_k. $\{\Gamma_k\}$ is a sequence strictly increasing to ∞.

2. The size of the file transmitted is J_k and we assume $J_k > 0$.

3. The transmission schedule is denoted by $A_k(\cdot)$, where $A_k(t)$ denotes the amount of data transmitted in time t after the kth transmission has begun. It

is a non-decreasing càdlàg function starting at 0 and increasing to ∞, which vanishes on the negative real axis.

The quantity of interest is the cumulative input traffic defined as

$$X(t) = \sum_{k=1}^{\infty} A_k(t - \Gamma_k) \wedge J_k. \tag{4.1}$$

The length of k-th transmission is defined as

$$L_k = \inf\{t : A_k(t) \geqslant J_k\} = A_k^{\leftarrow}(J_k).$$

F_L denotes the marginal distribution of the transmission lengths, and satisfies

$$F_L(x) = \mathrm{P}[L_1 \leqslant x] = \mathrm{P}[A_1(x) \geqslant J_1]$$

by right continuity of paths of A_1.

4.2.1 Small time scale behavior

To study the behavior of the cumulative traffic process $X(\cdot)$ for small time scales, we need to make the following further minimal assumptions on the transmission schedule $\{A_k\}$:

4. We assume $\{A_k\}$ are identically distributed and have stationary increments.

5. The multifractal spectrum of $A_k(\cdot)$ is not degenerated to a single point, which ensures that we consider processes with paths that show real multifractal behavior.

6. The multifractal spectrum of $A_k(\cdot)$ restricted to any (non-random) interval is non-random.

For the definitions and discussion regarding multifractal spectrum, we invite the reader to consult Riedi (2001) and Section 3.3.

Remark 4.2.1. If A_k is, for example, an increasing Lévy process, then, restricted to any interval, it has a non-random multifractal spectrum for the Hölder exponent based on exponential growth rate. (Jaffard, 1999, cf. Section 3.5 and).

In Section 3.5, it has been shown that under the assumptions (1)-(6), the multifractal spectrum of the process X coincide with that of A_1.

4.2.2 Large time scale behavior

For large time scale analysis, we need to consider a family of models indexed by T, and put distributional assumptions on them. The dependence of the models on T appears through the transmission initiation points, which we now denote by $\left\{\Gamma_k^{(T)}, k \geqslant 1\right\}$. The distributional assumptions are:

7. We assume $\left\{\Gamma_k^{(T)}, k \geqslant 1\right\}$ form a homogeneous Poisson process with intensity parameter $\lambda(T)$, called the *input rate*. We assume $\lambda(T)$ to be non-decreasing.

8. We assume $\{A_k, k \geqslant 1\}$ and $\{J_k, k \geqslant 1\}$ are independent of each other and are i.i.d. sequences independent of $\left\{\Gamma_k^{(T)}, k \geqslant 1\right\}$.

9. We assume the tail of the distribution of J_1 is regularly varying of index of $-\alpha_J$, where $\alpha_J \in (1, 2)$, i.e.,

$$\overline{F}_J \in RV_{-\alpha_J}.$$

   Hence J_1 has finite first moment denoted by μ_J.

10. The transmission schedule A_1 is H-self-similar (H-ss), where H satisfies:

    (a) $H\alpha_J > 1$.

    (b) $H < \frac{1}{\alpha_J - 1}$.

    (c) $H < \frac{1}{2-\alpha_J}$.

11. We also put the following moment conditions on A_1:

$$\mathrm{E}\left[A_1(1)^{-\alpha_J}\right] < \infty \text{ and } \mathrm{E}\left[A_1(1)^{2-\alpha_J+\delta}\right] < \infty,$$

    for some $\delta > 0$.

Remark 4.2.2. The path by path assumptions (4)-(6) for small time scale analysis can still hold for each of the models under consideration, since the microscopic analysis does not require any distributional assumptions on the transmission initiation times. So the result about multifractal spectrum continues to hold for each of these models. Further, assume A_1 is a non-decreasing H-ss process with stationary, independent increments, where H satisfies the assumption (10). Since A_1 has increasing paths and is H-ss with stationary increment which is not identically zero, from Theorem 2.1 of Vervaat (1994), we must have $H \geqslant 1$. Then $A_1(1)$ is a positive stable random variable of index $\frac{1}{H}$ and hence has a density which decays exponentially near 0 (cf. Theorem 2.5.2 of Zolotarev (1986)) and so has all negative moments finite. Also $A_1(1)$ has all positive moments smaller than $1/H$ finite. Hence, assumption (10c) guarantees the other moment condition in assumption (11). Also Remark 4.2.1 shows that a Lévy process satisfies the conditions for the multifractal analysis. Thus a $\frac{1}{H}$-stable Lévy process satisfies the requirements of the transmission schedule.

Remark 4.2.3. From Proposition 4.7.2, assumptions (8), (9) and (11) imply that $\overline{F}_L \in RV_{-H\alpha_J}$, and hence by assumption (10a) L has finite mean denoted by μ_L. Actually, a closer look at the proof of that proposition gives us that

$$\overline{F}_L(T) \sim \overline{F}_J(T^H) \tag{4.2}$$

If we further assume that L_1 has infinite variance, i.e., $H\alpha_J < 2$, then $\alpha_J < 2$ implies

$$\frac{\alpha_J}{\alpha_J - 1} = 1 + \frac{1}{\alpha_J - 1} > 2 > H\alpha_J,$$

and so assumption (10b) holds.

4.2.3 Critical input rate

The results of large time scale analysis depends on the input rate. Depending on whether the input rate is *slow* or *fast* - a concept made precise in the following - we can get either a stable or a Gaussian limit.

As described, the input rate is called slow, if

$$\text{(S)} \qquad \lim_{T\to\infty} \frac{b_J\left(\lambda(T)T^{1/H}\right)}{T} = 0,$$

and it is called fast, if

$$\text{(F)} \qquad \lim_{T\to\infty} \frac{b_J\left(\lambda(T)T^{1/H}\right)}{T} = \infty.$$

The following lemmas provide alternate approaches to the above conditions.

Lemma 4.2.1. *The slow growth condition (S) is equivalent to*

$$\lim_{T\to\infty} \lambda(T)T^{1/H}\overline{F}_J(T) = 0. \tag{4.3}$$

On the other hand, the fast growth condition (F) is equivalent to

$$\lim_{T\to\infty} \lambda(T)T^{1/H}\overline{F}_J(T) = \infty. \tag{4.4}$$

Proof. First we prove the conditions (S) and (F) imply (4.3) and (4.4) respectively. We define $0 < \varepsilon(T) := \frac{b_J(\lambda(T)T^{1/H})}{T}$ and we have $T\varepsilon(T) \to \infty$, since $b_J(T) \to \infty$. Then

$$\lambda(T)T^{1/H}\overline{F}_J(T) \sim \frac{\overline{F}_J(T)}{\overline{F}_J(T\varepsilon(T))},$$

which converges to 0 or ∞ according as the condition (S) or (F) holds, since $\overline{F}_J \in RV_{-\alpha_J}$.

Conversely, define

$$\delta(T) := \lambda(T)T^{1/H}\overline{F}_J(T) \sim \frac{\lambda(T)T^{1/H}}{b_J^{\leftarrow}(T)}.$$

Then

$$\frac{b_J(\lambda(T)T^{1/H})}{T} \sim \frac{b_J(\delta(T)b_J^{\leftarrow}(T))}{b_J(b_J^{\leftarrow}(T))},$$

which converges to 0 or ∞ according as $\delta(T)$ goes to 0 or ∞, since $b_J \in RV_{1/\alpha_J}$. □

The following lemma gives implications of the growth conditions which come out to be useful for analyzing the slow growth condition.

Lemma 4.2.2. *The slow growth condition (S) implies*

$$\lim_{T\to\infty} \frac{\lambda(T)T^{1/H}T\overline{F}_J(T)}{b_J(\lambda(T)T^{1/H})} = 0. \tag{4.5}$$

The limit is ∞ when the condition (F) holds.

Proof. Define $\varepsilon(T) = b_J(\lambda(T)T^{1/H})/T$ as before. Then $\lambda(T)T^{1/H} \sim 1/\overline{F}_J(T\varepsilon(T))$. Thus,

$$\frac{\lambda(T)T^{1/H}T\overline{F}_J(T)}{b_J(\lambda(T)T^{1/H})} \sim \frac{\overline{F}_J(T)}{\varepsilon(T)\overline{F}_J(T\varepsilon(T))} = \frac{T\overline{F}_J(T)}{T\varepsilon(T)\overline{F}_J(T\varepsilon(T))},$$

which goes to 0 or ∞ according as the condition (S) or (F) holds, since $T\overline{F}_J(T) \in RV_{1-\alpha_J}$ and $\alpha_J > 1$. $\square$

4.3 Basic Decomposition

We consider the following Poisson point process to facilitate the analysis:

$$M_T = \sum_{k=1}^{\infty} \varepsilon_{\left(\Gamma_k^{(T)}, A_k, J_k\right)},$$

which has mean measure $\lambda(T)d\gamma \times \mathrm{P}[A_1 \in da] \times \mathrm{P}[J_1 \in dj]$ on $(0,\infty) \times \mathbb{D}_\uparrow \times (0,\infty)$. The random variable $X_T(t)$ is the following function of M restricted to $\mathcal{R}(t) = \{(\gamma, a, j) \in (0,\infty) \times \mathbb{D}_\uparrow \times (0,\infty) : \gamma < t\}$:

$$X_T(t) = \sum_{k=1}^{\infty} \left[J_k \wedge A_k\left(t - \Gamma_k^{(T)}\right)\right] \mathbf{1}_{\mathcal{R}(t)}\left(\Gamma_k^{(T)}, A_k, J_k\right).$$

It helps to split $\mathcal{R}(t)$ in two disjoint sets

$$\mathcal{R}_1(t) = \{(\gamma, a, j) \in (0,\infty) \times \mathbb{D}_\uparrow \times (0,\infty) : \gamma < t, j \leqslant a(t-\gamma)\}$$

and

$$\mathcal{R}_2(t) = \{(\gamma, a, j) \in (0,\infty) \times \mathbb{D}_\uparrow \times (0,\infty) : \gamma < t, j > a(t-\gamma)\}.$$

$\mathcal{R}_1(t)$ and $\mathcal{R}_2(t)$ correspond to the regions where transmission has ended or is continuing respectively, by time t. Correspondingly, the input process X_T breaks into two sums:

$$X_T^{(1)}(t) = \sum_{k=1}^{\infty} J_k \mathbf{1}_{\mathcal{R}_1(t)} \left(\Gamma_k^{(T)}, A_k, J_k\right) \tag{4.6}$$

and

$$X_T^{(2)}(t) = \sum_{k=1}^{\infty} A_k \left(t - \Gamma_k^{(T)}\right) \mathbf{1}_{\mathcal{R}_2(t)} \left(\Gamma_k^{(T)}, A_k, J_k\right). \tag{4.7}$$

Since $X_T^{(i)}(t)$, $i = 1, 2$ are functions of $M_T|_{\mathcal{R}_i(t)}$, $i = 1, 2$ respectively with $\mathcal{R}_1(t) \cap \mathcal{R}_2(t) = \emptyset$, we have $X_T^{(1)}(t)$ and $X_T^{(2)}(t)$ are independent.

For any $t > 0$, we also observe the following facts about the regions $\mathcal{R}_i\left((Tt)^{\frac{1}{H}}\right)$, $i = 1, 2$, as $T \to \infty$:

$$\begin{aligned} m_1(Tt) &=: \frac{1}{\lambda(T)} \mathrm{E}\left[M_T\left(\mathcal{R}_1\left((Tt)^{\frac{1}{H}}\right)\right)\right] = \int_{\gamma=0}^{(Tt)^{\frac{1}{H}}} \mathrm{P}\left[L_1 \leqslant (Tt)^{\frac{1}{H}} - \gamma\right] d\gamma \\ &=: \widehat{F}_L\left((Tt)^{\frac{1}{H}}\right) \sim (Tt)^{\frac{1}{H}}, \end{aligned}$$

and

$$m_2(Tt) =: \frac{1}{\lambda(T)} \mathrm{E}\left[M_T\left(\mathcal{R}_1\left((Tt)^{\frac{1}{H}}\right)\right)\right] = \int_{\gamma=0}^{(Tt)^{\frac{1}{H}}} \mathrm{P}\left[L_1 > (Tt)^{\frac{1}{H}} - \gamma\right] d\gamma \sim \mu_L.$$

So the mean measure restricted to $\mathcal{R}_1\left((Tt)^{\frac{1}{H}}\right)$ or $\mathcal{R}_2\left((Tt)^{\frac{1}{H}}\right)$ is finite and we can have the Poisson representations

$$M_T^{(i)}|_{\mathcal{R}_i\left((Tt)^{\frac{1}{H}}\right)} \stackrel{\mathrm{d}}{=} \sum_{k=1}^{P_i(T)} \varepsilon_{\tau_{k,i}^{(T)}, S_{k,i}^{(T)}, W_{k,i}^{(T)}}, \tag{4.8}$$

where $P_i(T)$ is a Poisson random variable with parameter $\lambda(T) m_i(Tt)$, which is independent of $\left(\tau_{k,i}^{(T)}, S_{k,i}^{(T)}, W_{k,i}^{(T)}\right)$, which has distribution

$$\left.\frac{d\gamma\, \mathrm{P}[A_1 \in da] F_J(dw)}{m_i(Tt)}\right|_{\mathcal{R}_i\left((Tt)^{\frac{1}{H}}\right)}.$$

Note t is fixed in this argument and sometimes suppressed in the notations. Also we define random variables $\left(\tau_i^{(T)}, S_i^{(T)}, W_i^{(T)}\right)$ independent of $P_i(T)$ and distributed identically as $\left(\tau_{1,i}^{(T)}, S_{1,i}^{(T)}, W_{1,i}^{(T)}\right)$, for $i = 1, 2$. Then we can rewrite $X_T^{(i)}, i = 1, 2$ in terms of the above Poisson representation (4.8) as:

$$X_T^{(1)}\left((Tt)^{\frac{1}{H}}\right) \stackrel{\mathrm{d}}{=} \sum_{k=1}^{P_1(T)} W_{k,1}^{(T)}, \tag{4.9}$$

$$X_T^{(2)}\left((Tt)^{\frac{1}{H}}\right) \stackrel{\mathrm{d}}{=} \sum_{k=1}^{P_2(T)} S_{k,2}^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_{k,2}^{(T)}\right). \tag{4.10}$$

4.4 Moment Behavior

For further analysis, it is useful to study the moments of the above summands. But before that, we need to consider the following extension of Potter's bound (Potter, 1940, cf.) for regularly varying functions cf. (Resnick, 1987, cf.).

Lemma 4.4.1. *If ϕ is a regularly varying function of index $\rho > -1$. Then given $\varepsilon > 0$, there exists T_0 such that for $T > T_0$, and $x > 1$,we have*

$$(1-\varepsilon)\phi(T)x^{\rho+1-\varepsilon} < (\rho+1)\int_0^x \phi(Tu)du < (1+\varepsilon)\phi(T)x^{\rho+1+\varepsilon}.$$

Proof. Since $\rho > -1$, from Karamata's theorem, we have

$$(\rho+1)\frac{\int_0^T \phi(u)du}{T\phi(T)} \to 1$$

and so, given $\varepsilon > 0$, there exists T_0 such that for all $T > T_0$, we have the following bounds:

$$\sqrt{1-\varepsilon} < (\rho+1)\frac{\int_0^T \phi(u)du}{T\phi(T)} < \sqrt{1+\varepsilon} \tag{4.11}$$

and using Potter's bound, for all $x > 1$,

$$\sqrt{1-\varepsilon}\, x^{\rho+1-\varepsilon} < \frac{Tx\phi(Tx)}{T\phi(T)} < \sqrt{1+\varepsilon}\, x^{\rho+1+\varepsilon}. \tag{4.12}$$

From (4.11), we have for all $T > T_0$ and all $x > 1$,

$$\sqrt{1-\varepsilon} < (\rho+1)\frac{\int_0^T x\phi(u)du}{Tx\phi(Tx)} < \sqrt{1+\varepsilon}. \tag{4.13}$$

Multiplying (4.12) and (4.13), we get for all $T > T_0$ and all $x > 1$, the bounds become

$$\sqrt{1-\varepsilon}\, x^{\rho+1-\varepsilon} < (\rho+1)\frac{\int_0^T x\phi(u)du}{T\phi(T)} < \sqrt{1+\varepsilon}\, x^{\rho+1+\varepsilon}$$

and thus the result follows. □

Now, we are to study the moments of the summands in the representations (4.9) and (4.10). Recall from assumption (11), we have $\mathrm{E}[A_1(1)^{2-\alpha_J+\delta}] < \infty$. Then, for $2 \leqslant l \leqslant 2 + \frac{\delta}{2}$, we have

$$\begin{aligned}
\mathrm{E}\left[\left(W_1^{(T)}\right)^l\right] &\sim \frac{1}{(Tt)^{\frac{1}{H}}} \int_{\gamma=0}^{(Tt)^{\frac{1}{H}}} \int_{w=0}^{\infty} l w^{l-1}\, \mathrm{P}\left[w < J_1 \leqslant A_1\left((Tt)^{\frac{1}{H}} - \gamma\right)\right] dw\, d\gamma \\
&= (Tt)^l \int_{\gamma=0}^{1} \int_{w=0}^{\infty} l w^{l-1}\, \mathrm{P}\left[w < \frac{J_1}{Tt} \leqslant \frac{A_1\left((Tt)^{\frac{1}{H}}\gamma\right)}{Tt}\right] dw\, d\gamma
\end{aligned}$$

$$= (Tt)^l \overline{F}_J(Tt) \int\limits_{\gamma=0}^{1} \int\limits_{w=0}^{\infty} l w^{l-1} \frac{1}{\overline{F}_J(Tt)} \, \mathrm{P}\left[w < \frac{J_1}{Tt} \leqslant A_1(\gamma) \right] \, dw \, d\gamma$$

since A_1 is H-ss

$$= (Tt)^l \overline{F}_J(Tt) \int\limits_{\gamma=0}^{1} \mathrm{E} \int\limits_{w=0}^{\infty} l w^{l-1} \frac{\overline{F}_J(Ttw) - \overline{F}_J(TtA_1(\gamma))}{\overline{F}_J(Tt)} \mathbf{1}_{[w<A_1(\gamma)]} \, dw \, d\gamma. \tag{4.14}$$

Now, the integrand on the right side of (4.14) is bounded by

$$l w^{l-1} \frac{\overline{F}_J(Ttw)}{\overline{F}_J(Tt)} \left[\mathbf{1}_{[A_1(1)>w\vee 1]} + \mathbf{1}_{[w\leqslant 1]} \right] \to l w^{l-\alpha_J-1} \left[\mathbf{1}_{[A_1(1)>w\vee 1]} + \mathbf{1}_{[w\leqslant 1]} \right] \tag{4.15}$$

and

$$\int\limits_{\gamma=0}^{1} \mathrm{E} \int\limits_{w=0}^{\infty} l w^{l-1} \frac{\overline{F}_J(Ttw)}{\overline{F}_J(Tt)} \left[\mathbf{1}_{[A_1(1)>w\vee 1]} + \mathbf{1}_{[w\leqslant 1]} \right] \, dw \, d\gamma$$

$$= \mathrm{E} \int\limits_{w=0}^{A_1(1)} l w^{l-1} \, dw \frac{\overline{F}_J(Ttw)}{\overline{F}_J(Tt)} \mathbf{1}_{[A_1(1)>1]} + \int\limits_{w=0}^{1} l w^{l-1} \frac{\overline{F}_J(Ttw)}{\overline{F}_J(Tt)} \, dw. \tag{4.16}$$

Now, by Karamata's theorem, the second term on the right side of (4.16) converges to

$$\frac{l}{l-\alpha_J} = \int\limits_{\gamma=0}^{1} \mathrm{E} \int\limits_{w=0}^{\infty} l w^{l-\alpha_J-1} \mathbf{1}_{[w\leqslant 1]} \, dw \, d\gamma. \tag{4.17}$$

For the first term on the right side of (4.16), observe that $T^{l-1}\overline{F}_J(T)$ is regularly varying with index $l-\alpha_J-1 > -1$, and hence by the upper bound from Lemma 4.4.1, we obtain that a T_0, which is independent of ω, such that for all $T > T_0$, we have

$$\int\limits_{w=0}^{A_1(1)} l w^{l-1} \, dw \frac{\overline{F}_J(Ttw)}{\overline{F}_J(Tt)} \mathbf{1}_{[A_1(1)>1]} < \left(1 + \frac{\delta}{2}\right) \frac{l}{l-\alpha_J} A_1(1)^{l+\frac{\delta}{2}-\alpha_J} \mathbf{1}_{[A_1(1)>1]},$$

which is integrable by assumption (11). Then, by the Dominated Convergence Theorem, the first term on the right side of (4.16) converges to

$$\mathrm{E}\left[\frac{l}{l-\alpha_J}A_1(1)^{l-\alpha_J}\mathbf{1}_{[A_1(1)>1]}\right] = \int_{\gamma=0}^{1} \mathrm{E}\int_{w=0}^{\infty} lw^{l-\alpha_J-1}\mathbf{1}_{[A_1(1)>w\vee 1]}\,dw\,d\gamma,$$

which along with (4.17) shows

$$\int_{\gamma=0}^{1} \mathrm{E}\int_{w=0}^{\infty} lw^{l-1}\frac{\overline{F}_J(Ttw)}{\overline{F}_J(Tt)}\left[\mathbf{1}_{[A_1(1)>w\vee 1]}+\mathbf{1}_{[w\leqslant 1]}\right]\,dw\,d\gamma$$

$$\to \int_{\gamma=0}^{1} \mathrm{E}\int_{w=0}^{\infty} lw^{l-\alpha_J-1}\left[\mathbf{1}_{[A_1(1)>w\vee 1]}+\mathbf{1}_{[w\leqslant 1]}\right]\,dw\,d\gamma \quad (4.18)$$

Then, from (4.15) and (4.18), using Pratt's lemma (Resnick, 1998, cf.), we are allowed to take the limit under the integral sign in the right side of (4.14) to get, for $2 \leqslant l < 2+\delta$,

$$\begin{aligned}
\mathrm{E}\left[\left(W_1^{(T)}\right)^l\right] &\sim T^l\overline{F}_J(T)t^{l-\alpha_J}\int_{\gamma=0}^{1}\mathrm{E}\int_{w=0}^{\infty} lw^{l-1}[w^{-\alpha_J}-A_1(\gamma)^{-\alpha_J}]_+\,dw\,d\gamma\\
&= T^l\overline{F}_J(T)t^{l-\alpha_J}\int_{\gamma=0}^{1}\mathrm{E}\int_{w=0}^{A_1(\gamma)} lw^{l-1}[w^{-\alpha_J}-A_1(\gamma)^{-\alpha_J}]_+\,dw\,d\gamma\\
&= \frac{\alpha_J}{l-\alpha_J}T^l\overline{F}_J(T)t^{l-\alpha_J}\int_{\gamma=0}^{1}\mathrm{E}\left[A_1(\gamma)^{l-\alpha_J}\right]\,d\gamma\\
&= \frac{\alpha_J}{(l-\alpha_J)[H(l-\alpha_J)+1]}T^l\overline{F}_J(T)t^{l-\alpha_J}\,\mathrm{E}\left[A_1(1)^{l-\alpha_J}\right]. \qquad (4.19)
\end{aligned}$$

Also, using monotone convergence, we observe that as $T\to\infty$,

$$\mathrm{E}\left[W_1^{(T)}\right] \sim \int_{\gamma=0}^{1}\int_{w=0}^{\infty} \mathrm{P}\left[w<J_1, L_1 \leqslant (Tt)^{\frac{1}{H}}\gamma\right]\,dw\,d\gamma \uparrow \mu_J. \qquad (4.20)$$

For future reference, note from (4.19) and (4.20) that

$$\lim_{T\to\infty} \frac{\operatorname{Var}\left[W_1^{(T)}\right]}{T^2\overline{F}_J(T)} = \frac{\alpha_J}{(2-\alpha_J)[H(2-\alpha_J)+1]} t^{2-\alpha_J}\,\mathrm{E}\left[A_1(1)^{2-\alpha_J}\right] =: \sigma_1^2 t^{2-\alpha_J} \quad (4.21)$$

and

$$\limsup_{T\to\infty} \frac{\mathrm{E}\left[\left|W_1^{(T)} - \mathrm{E}\left[W_1^{(T)}\right]\right|^{2+\frac{\delta}{2}}\right]}{T^{2+\frac{\delta}{2}}\overline{F}_J(T)} \leqslant \limsup_{T\to\infty} \frac{\mathrm{E}\left[\left(W_1^{(T)}\right)^{2+\frac{\delta}{2}}\right] + \left(\mathrm{E}\left[W_1^{(T)}\right]\right)^{2+\frac{\delta}{2}}}{T^{2+\frac{\delta}{2}}\overline{F}_J(T)}, \quad (4.22)$$

which is a constant.

Next we study the moments of the summands of (4.10). First we consider the moments of order l, with $2 \leqslant l \leqslant 2+\delta/2$. We have, using self-similarity of A_1, that

$$\begin{aligned}
&\mathrm{E}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right]^l \\
\sim&\frac{1}{\mu_L}\int_{\gamma=0}^{(Tt)^{\frac{1}{H}}}\int_{w=0}^{\infty} lw^{l-1}\,\mathrm{P}\left[w < A_1\left((Tt)^{\frac{1}{H}}-\gamma\right) < J_1\right] dw\,d\gamma \\
=&\frac{1}{\mu_L}(Tt)^{l+\frac{1}{H}}\int_{\gamma=0}^{1}\int_{w=0}^{\infty} lw^{l-1}\,\mathrm{P}\left[w < \frac{A_1\left((Tt)^{\frac{1}{H}}\gamma\right)}{Tt} < \frac{J_1}{Tt}\right] dw\,d\gamma \\
=&\frac{1}{\mu_L}(Tt)^{l+\frac{1}{H}}\overline{F}_J(Tt)\int_{\gamma=0}^{1}\int_{w=0}^{\infty} lw^{l-1}\frac{1}{\overline{F}_J(Tt)}\,\mathrm{P}\left[w < A_1(\gamma) < \frac{J_1}{Tt}\right] dw\,d\gamma \\
=&\frac{1}{\mu_L}(Tt)^{l+\frac{1}{H}}\overline{F}_J(Tt)\int_{\gamma=0}^{1}\mathrm{E}\int_{w=0}^{\infty} lw^{l-1}\frac{\overline{F}_J(TtA_1(\gamma))}{\overline{F}_J(Tt)}\mathbf{1}_{[A_1(\gamma)>w]}\,dw\,d\gamma,
\end{aligned}$$

and bounding the integrand above as in the case of l-th moments of $W_1^{(T)}$ with $2 \leqslant l < 2+\delta$, we justify the interchange of the limit and integral to obtain

$$\mathrm{E}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right]^l \sim \frac{1}{\mu_L}T^{l+\frac{1}{H}}\overline{F}_J(T)t^{l-\alpha_J+\frac{1}{H}}\int_{\gamma=0}^{1}\mathrm{E}\left[A_1(\gamma)^{l-\alpha_J}\right]\,d\gamma$$

$$= \frac{1}{\mu_L[H(l-\alpha_J)+1]} T^{l+\frac{1}{H}} \overline{F}_J(T) t^{l-\alpha_J+\frac{1}{H}} \,\mathrm{E}[A_1(1)^{l-\alpha_J}]. \tag{4.23}$$

The first moment requires more careful analysis in this case. As for the higher moments, we again have, using self-similarity of A_1,

$$\mathrm{E}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right] \tag{4.24}$$

$$\begin{aligned}
&\sim \frac{1}{\mu_L}(Tt)^{1+\frac{1}{H}}\overline{F}_J(Tt) \int_{\gamma=0}^{1} \mathrm{E} \int_{w=0}^{\infty} \frac{\overline{F}_J(TtA_1(\gamma))}{\overline{F}_J(Tt)} \mathbf{1}_{[A_1(\gamma)>w]} \, dw \, d\gamma \\
&= \frac{1}{\mu_L}(Tt)^{1+\frac{1}{H}}\overline{F}_J(Tt) \int_{\gamma=0}^{1} \mathrm{E} \int_{w=0}^{\infty} \frac{\overline{F}_J\left(TtA_1(1)\gamma^H\right)}{\overline{F}_J(Tt)} \mathbf{1}_{[A_1(1)\gamma^H>w]} \, dw \, d\gamma \\
&= \frac{1}{\mu_L}(Tt)^{1+\frac{1}{H}}\overline{F}_J(Tt) \int_{\gamma=0}^{1} \mathrm{E}\, A_1(1)\gamma^H \frac{\overline{F}_J\left(TtA_1(1)\gamma^H\right)}{\overline{F}_J(Tt)} \, d\gamma \\
&= \frac{1}{\mu_L}(Tt)^{1+\frac{1}{H}}\overline{F}_J(Tt) \frac{1}{H} \mathrm{E}\left[A_1(1)^{-\frac{1}{H}}\right] \int_{\nu=0}^{A_1(1)} \frac{(Tt\nu)^{\frac{1}{H}}\overline{F}_J(Tt\nu)}{(Tt)^{\frac{1}{H}}\overline{F}_J(Tt)} \, d\nu,
\end{aligned} \tag{4.25}$$

where we substitute $\nu = A_1(1)\gamma^H$ in the last step. Now, by Karamata's theorem, as $T \to \infty$,

$$\int_{\nu=0}^{A_1(1)} \frac{(Tt\nu)^{\frac{1}{H}}\overline{F}_J(Tt\nu)}{(Tt)^{\frac{1}{H}}\overline{F}_J(Tt)} \, d\nu \to \frac{A_1(1)^{\frac{1}{H}-\alpha_J+1}}{\frac{1}{H}-\alpha_J+1}, \tag{4.26}$$

since by assumption (10b) $H < \frac{1}{\alpha_J - 1}$. Also, there exists a non-random T_0, such that for all $T > T_0$, we have,

$$\begin{aligned}
\int_{\nu=0}^{A_1(1)} \frac{(Tt\nu)^{\frac{1}{H}}\overline{F}_J(Tt\nu)}{(Tt)^{\frac{1}{H}}\overline{F}_J(Tt)} \, d\nu &\leqslant \int_{\nu=0}^{1} \frac{(Tt\nu)^{\frac{1}{H}}\overline{F}_J(Tt\nu)}{(Tt)^{\frac{1}{H}}\overline{F}_J(Tt)} \, d\nu + \int_{\nu=0}^{A_1(1)} \frac{(Tt\nu)^{\frac{1}{H}}\overline{F}_J(Tt\nu)}{(Tt)^{\frac{1}{H}}\overline{F}_J(Tt)} \, d\nu \mathbf{1}_{[A_1(1)>1]} \\
&< \frac{1+\frac{\alpha_J-1}{2}}{\frac{1}{H}-(\alpha_J-1)} \left[1 + A_1(1)^{\frac{1}{H}-\frac{\alpha_J-1}{2}} \mathbf{1}_{[A_1(1)>1]}\right],
\end{aligned}$$

where we bound the first term using Karamata's theorem and the second term using the upper bound from Lemma 4.4.1. Thus,

$$A_1(1)^{-\frac{1}{H}} \int_{\nu=0}^{A_1(1)} \frac{(Tt\nu)^{\frac{1}{H}} \overline{F}_J(Tt\nu)}{(Tt)^{\frac{1}{H}} \overline{F}_J(Tt)} \, d\nu < \frac{\alpha_J + 1}{2\left(\frac{1}{H} - (\alpha_J - 1)\right)} \left[A_1(1)^{-\frac{1}{H}} + 1\right],$$

which is integrable, since $\mathrm{E}[A_1(1)^{-\alpha_J}] < \infty$ by assumption (11) and $1/H < \alpha_J$. So by the Dominated Convergence Theorem and (4.26), we have from (4.25),

$$\mathrm{E}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right] \sim \frac{1}{\mu_L} T^{1+\frac{1}{H}} \overline{F}_J(T) t^{\frac{1}{H} - \alpha_J + 1} \frac{\mathrm{E}[A_1(1)^{1-\alpha_J}]}{1 - H(\alpha_J - 1)}. \tag{4.27}$$

We again, collect some results about variance and other centered moments using (4.27) and (4.23), for future reference:

$$\lim_{T\to\infty} \frac{\mathrm{Var}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right]}{T^{2+\frac{1}{H}} \overline{F}_J(T)} = \frac{1}{\mu_L[H(2-\alpha_J)+1]} t^{2-\alpha_J+\frac{1}{H}} \, \mathrm{E}\left[A_1(1)^{2-\alpha_J}\right] \tag{4.28}$$

$$=: \frac{1}{\mu_L} \sigma_2^2 t^{2-\alpha_J+\frac{1}{H}}$$

and

$$\limsup_{T\to\infty} \frac{\mathrm{E}\left[\left|S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right) - \mathrm{E}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right]\right|^{2+\frac{\delta}{2}}\right]}{T^{2+\frac{1}{H}+\frac{\delta}{2}} \overline{F}_J(T)}$$
$$\leqslant \limsup_{T\to\infty} \frac{\mathrm{E}\left[\left(S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right)^{2+\frac{\delta}{2}}\right] + \left(\mathrm{E}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right]\right)^{2+\frac{\delta}{2}}}{T^{2+\frac{1}{H}+\frac{\delta}{2}} \overline{F}_J(T)}, \tag{4.29}$$

which is a constant.

4.5 Heavy Tailed Approximation Under the Slow Growth Condition

To study the slow growth condition, we need to look at the tail behavior of $W_1^{(T)}$ as well, which we summarize in the following proposition:

Proposition 4.5.1. *Under the slow growth condition (S), the random variable* $W_1^{(T)}$ *given in* (4.8) *satisfies*

$$\lim_{T\to\infty} \zeta(T)\,\mathrm{P}\left[W_1^{(T)} > b_J\left(\zeta(T)\right)w\right] = w^{-\alpha_J}, \tag{4.30}$$

where $\zeta(T) = \lambda(T)T^{\frac{1}{H}}$.

Proof. Observe that

$$\begin{aligned}\zeta(T)\,\mathrm{P}\left[W_1^{(T)} > b_J\left(\lambda(T)T^{\frac{1}{H}}\right)w\right] &\sim \frac{\zeta(T)}{(Tt)^{\frac{1}{H}}}\int_0^{(Tt)^{\frac{1}{H}}} \mathrm{P}\left[b_J\left(\zeta(T)\right)w < J_1 \leqslant A_1(\gamma)\right]d\gamma \\ &\leqslant \zeta(T)\overline{F}_J\left(b_J\left(\zeta(T)\right)w\right) \to w^{-\alpha_J}.\end{aligned}$$

Hence,

$$\limsup_{T\to\infty} \zeta(T)\,\mathrm{P}\left[W_1^{(T)} > b_J(\zeta(T))w\right] \leqslant w^{-\alpha_J}. \tag{4.31}$$

On the other hand,

$$\zeta(T)\,\mathrm{P}\left[W_1^{(T)} > b_J\left(\zeta(T)\right)w\right] \sim \frac{\zeta(T)}{(Tt)^{\frac{1}{H}}}\int_0^{(Tt)^{\frac{1}{H}}} \mathrm{P}[b_J\left(\zeta(T)\right)w < J_1 \leqslant A_1(\gamma)]\,d\gamma$$

$$= \left(\frac{b_J(\zeta(T))}{T}\right)^{\frac{1}{H}} \int_0^{\left(\frac{T}{b_J(\zeta(T))}\right)^{\frac{1}{H}}} \zeta(T)\,\mathrm{P}\left[b_J(\zeta(T))w < J_1 \leqslant A_1\left((b_J(\zeta(T))t)^{\frac{1}{H}}\gamma\right)\right]d\gamma$$

$$\geqslant \left(\frac{b_J(\zeta(T))}{T}\right)^{\frac{1}{H}} \int\limits_{N}^{\left(\frac{T}{(b_J(\zeta(T)))}\right)^{\frac{1}{H}}} \zeta(T)\,\mathrm{P}\left[b_J(\zeta(T))w < J_1 \leqslant A_1\left((b_J(\zeta(T))t)^{\frac{1}{H}}N\right)\right]\, d\gamma \tag{4.32}$$

$$= \left(1 - N\left(\frac{b_J(\zeta(T))}{T}\right)^{\frac{1}{H}}\right)\zeta(T)\,\mathrm{P}\left[b_J(\zeta(T))w < J_1 \leqslant A_1\left((b_J(\zeta(T))t)^{\frac{1}{H}}N\right)\right]$$

$$\sim \zeta(T)\,\mathrm{P}\left[b_J(\zeta(T))w < J_1 \leqslant A_1\left((b_J(\zeta(T))t)^{\frac{1}{H}}N\right)\right] \tag{4.33}$$

$$= \zeta(T)\,\mathrm{P}\left[w < \frac{J_1}{b_J(\zeta(T))} \leqslant tA_1(N)\right] \tag{4.34}$$

$$\to \mathrm{E}[w^{-\alpha_J} - (tA_1(N))^{-\alpha_J}]_+, \tag{4.35}$$

where the inequality (4.32) holds for any natural number N for sufficiently large T, since, by the slow growth condition (S), we have $b_J(\zeta(T))/T \to 0$. The equivalence (4.33) holds for the same reason. The equality in (4.34) follows from the H-self-similarity of A_1. Finally the convergence (4.35) holds by the regular variation of the tail of J_1 and Dominated Convergence Theorem. Then letting N go to ∞, using the fact $A_1(\infty) = \infty$) and Dominated Convergence Theorem, we have

$$\liminf_{T\to\infty} \zeta(T)\,\mathrm{P}\left[W_1^{(T)} > b_J(\zeta(T))w\right] \geqslant w^{-\alpha_J}. \tag{4.36}$$

The inequalities (4.31) and (4.36) together complete the proof. □

Using the tail behavior in the above Proposition 4.5.1 and an analysis based on the point process as in, for example, Chapter 3 or Exercise 4.4.2.8 in Resnick (1987), we can conclude that

$$\frac{\sum_{k=1}^{P_1(T)} W_{k,1}^{(T)}}{b_J(\lambda(T)T^{\frac{1}{H}})} - \frac{P_1(T)\,\mathrm{E}[W_1^{(T)}]}{b_J(\lambda(T)T^{\frac{1}{H}})} \Rightarrow Z_{\alpha_J}(t^{\frac{1}{H}}), \tag{4.37}$$

where Z_α is α-stable Lévy motion with mean 0, skewness 1 and scale $C_\alpha^{\frac{1}{\alpha}}$ and

$$C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}.$$

Now observe that

$$\begin{aligned} &\frac{P_1(T)\,\mathrm{E}\left[W_1^{(T)}\right] - \lambda(Tt)(Tt)^{\frac{1}{H}}\mu_J}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)} \\ =&\frac{P_1(T) - \lambda(T)\widehat{F}_L\left((Tt)^{\frac{1}{H}}\right)}{\sqrt{\lambda(T)\widehat{F}_L\left((Tt)^{\frac{1}{H}}\right)}}\sqrt{\lambda(T)\widehat{F}_L\left((Tt)^{\frac{1}{H}}\right)}\frac{\mathrm{E}\left[W_1^{(T)}\right]}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)} \\ &-\frac{\lambda(T)}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)}\left((Tt)^{\frac{1}{H}}\mu_J - \widehat{F}_L\left((Tt)^{\frac{1}{H}}\right)\mathrm{E}\left[W_1^{(T)}\right]\right). \end{aligned} \tag{4.38}$$

Since we know from (4.20) that

$$\sqrt{\lambda(T)\widehat{F}_L\left((Tt)^{\frac{1}{H}}\right)}\frac{\mathrm{E}\left[W_1^{(T)}\right]}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)} \sim \frac{\sqrt{\lambda(T)T^{\frac{1}{H}}}}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)}\sqrt{t}\mu_J \to 0,$$

and $P_1(T)$ is a Poisson random variable with parameter $\lambda(T)\widehat{F}_L\left((Tt)^{\frac{1}{H}}\right)$, which goes to ∞, the first term on the right side of (4.38) is probabilistically negligible. As for the second term, observe that

$$\begin{aligned} &\frac{\lambda(T)}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)}\left((Tt)^{\frac{1}{H}}\mu_J - \widehat{F}_L\left((Tt)^{\frac{1}{H}}\right)\mathrm{E}\left[W_1^{(T)}\right]\right) \\ =&\frac{\lambda(T)}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)}\int_{j=0}^{\infty}\int_{\gamma=0}^{(Tt)^{\frac{1}{H}}}\mathrm{P}[J_1 > j, L_1 > \gamma]\,d\gamma\,dj \\ \leqslant&\frac{\lambda(T)}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)}\left[\int_{\gamma=0}^{(Tt)^{\frac{1}{H}}}\gamma^H\overline{F}_L(\gamma)\,d\gamma + \int_{\gamma=0}^{(Tt)^{\frac{1}{H}}}\int_{j=\gamma^H}^{\infty}\overline{F}_J(j)\,dj\,d\gamma\right] \end{aligned}$$

$$\sim \text{constant } \frac{\lambda(T)T^{\frac{1}{H}} T\overline{F}_J(T)}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)} \to 0,$$

where the equivalence follows using Karamata's theorem, the fact $1+H(1-\alpha_J) > 0$ from assumption (10b) and the relation (4.2) between the tails of the distributions of L_1 and J_1. Thus the left side of (4.38) is probabilistically negligible. This fact combined with the convergence (4.37) gives us the central limit theorem for the contribution of the first region under slow growth: for all $t > 0$,

$$\frac{X_T^{(1)}\left((Tt)^{\frac{1}{H}}\right) - \lambda(T)(Tt)^{\frac{1}{H}}}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)} \Rightarrow Z_{\alpha_J}(t^{\frac{1}{H}}). \tag{4.39}$$

For the contribution of the second region, observe from the representation (4.10) and (4.27), we have

$$\begin{aligned} \frac{\mathrm{E}\left[X_T^{(2)}\left((Tt)^{\frac{1}{H}}\right)\right]}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)} &= \frac{\mathrm{E}[P_2(T)]\,\mathrm{E}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right]}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)} \\ &\sim \text{ constant } \frac{\lambda(T)T^{\frac{1}{H}} T\overline{F}_J(T)}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)} \to 0 \end{aligned}$$

and hence $X_T^{(2)}(t) = o_P\left(b_J\left(\lambda(T)T^{\frac{1}{H}}\right)\right)$. Combining this fact with (4.39), we get for all $t > 0$,

$$\frac{X_T\left((Tt)^{\frac{1}{H}}\right) - \lambda(T)(Tt)^{\frac{1}{H}}}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)} \Rightarrow Z_{\alpha_J}(t^{\frac{1}{H}}).$$

We can check the finite dimensional convergence as in Chapter 3 using stationarity and independence of the process X_T. Thus, finally, we have

Theorem 4.5.1. *Under the assumptions* (7) - (11) *and slow growth condition (S),* *we have,*

$$\frac{X_T\left((T\cdot)^{\frac{1}{H}}\right)-\lambda(T)(T\cdot)^{\frac{1}{H}}}{b_J\left(\lambda(T)T^{\frac{1}{H}}\right)}\overset{\text{fidi}}{\to} Z_{\alpha_J}\left((\cdot)^{\frac{1}{H}}\right),$$

where the convergence is in the sense of weak convergence of finite dimensional distributions and Z_α *is* α*-stable Lévy motion with mean 0, skewness 1 and scale* $C_\alpha^{\frac{1}{\alpha}}$.

4.6 Asymptotic Normality Under the Fast Growth Condition

4.6.1 One-dimensional convergence

We use the moment conditions (4.21), (4.22), (4.28), (4.29) along with Lyapunov's Central Limit Theorem to study the behavior of $X_T\left((Tt)^{\frac{1}{H}}\right)$. We define

$$\eta(T)=\sqrt{\lambda(T)T^{\frac{1}{H}}\overline{F}_J(T)} \tag{4.40}$$

Using (4.21) and (4.22) we get that

$$\limsup_{T\to\infty}\frac{\left\lfloor\lambda(T)(Tt)^{\frac{1}{H}}\right\rfloor \mathrm{E}\left[\left|W_1^{(T)}-\mathrm{E}\left[W_1^{(T)}\right]\right|^{2+\frac{\delta}{2}}\right]}{\left(\left\lfloor\lambda(T)(Tt)^{\frac{1}{H}}\right\rfloor \mathrm{Var}\left[W_1^{(T)}\right]\right)^{\frac{2+\delta/2}{2}}}$$
$$\leqslant \text{constant }\limsup_{T\to\infty}\frac{\lambda(T)(Tt)^{\frac{1}{H}}T^{2+\frac{\delta}{2}}\overline{F}_J(T)}{\left(\lambda(T)(Tt)^{\frac{1}{H}}T^2\overline{F}_J(T)\right)^{\frac{2+\delta/2}{2}}}=\left(\lambda(T)(Tt)^{\frac{1}{H}}\overline{F}_J(T)\right)^{-\frac{\delta}{4}}\to 0$$

by the fast growth condition (4.4). Also, since, by (4.21),

$$\left\lfloor\lambda(T)(Tt)^{\frac{1}{H}}\right\rfloor \mathrm{Var}\left[W_1^{(T)}\right]\sim\sigma_1^2 t^{2-\alpha_J+\frac{1}{H}}(T\eta(T))^2,$$

we have by Lyapunov's Central Limit Theorem,

$$\frac{\sum_{k=1}^{\left\lfloor \lambda(T)T^{\frac{1}{H}} \right\rfloor} \left(W_{1,k}^{(T)} - \mathrm{E}\left[W_1^{(T)}\right]\right)}{T\eta(T)} \Rightarrow \sigma_1 N\left(0, t^{2-\alpha_J+\frac{1}{H}}\right),$$

where $N(0,t)$ is a normal random variable with mean 0 and variance t. Also, we know that $P_1(T)$ is a Poisson random variable with parameter $\lambda(T)\widehat{F}_L\left((Tt)^{\frac{1}{H}}\right) \sim \left\lfloor \lambda(T)(Tt)^{\frac{1}{H}} \right\rfloor \to \infty$, we have $P_1(T)/\left\lfloor \lambda(T)(Tt)^{\frac{1}{H}} \right\rfloor \xrightarrow{\mathrm{P}} 1$. Hence by Theorem 4.1.2 of Gnedenko and Korolev (1996), we have,

$$\frac{\sum_{k=1}^{P_1(T)} W_{1,k}(T) - P_1(T)\,\mathrm{E}\left[W_1^{(T)}\right]}{T\eta(T)} \Rightarrow \sigma_1 N\left(0, t^{2-\alpha_J+\frac{1}{H}}\right).$$

Finally, we observe that,

$$\frac{(P_1(T) - \mathrm{E}[P_1(T)])\,\mathrm{E}\left[W_1^{(T)}\right]}{T\eta(T)} \sim \frac{(P_1(T) - \mathrm{E}[P_1(T)])}{\sqrt{\mathrm{E}[P_1(T)]}}\,\mathrm{E}\left[W_1^{(T)}\right]\left(T^2\overline{F}_J(T)\right)^{-\frac{1}{2}}$$

is $o_P(1)$. Combining, we get,

$$\frac{\sum_{k=1}^{P_1(T)} W_{1,k}(T) - \mathrm{E}[P_1(T)]\,\mathrm{E}\left[W_1^{(T)}\right]}{T\eta(T)} \stackrel{\mathrm{d}}{=} \frac{X_T^{(1)}(t) - \mathrm{E}\left[X_T^{(1)}(t)\right]}{T\eta(T)} \Rightarrow \sigma_1 N\left(0, t^{2-\alpha_J+\frac{1}{H}}\right). \tag{4.41}$$

For the second region, we consider the equations (4.28) and (4.29) and get

$$\limsup_{T\to\infty} \frac{\lfloor \lambda(T)\mu_L \rfloor\,\mathrm{E}\left[\left|S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right) - \mathrm{E}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right]\right|^{2+\frac{\delta}{2}}\right]}{\left(\lfloor \lambda(T)\mu_L \rfloor\,\mathrm{Var}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right]\right)^{\frac{2+\delta/2}{2}}}$$

$$\leqslant \text{constant } \limsup_{T\to\infty} \frac{\lambda(T)(Tt)^{\frac{1}{H}}T^{2+\frac{\delta}{2}}\overline{F}_J(T)}{\left(\lambda(T)(Tt)^{\frac{1}{H}}T^2\overline{F}_J(T)\right)^{\frac{2+\delta/2}{2}}} = \left(\lambda(T)(Tt)^{\frac{1}{H}}\overline{F}_J(T)\right)^{-\frac{\delta}{4}} \to 0.$$

Also, from (4.28), we have,

$$\lfloor \lambda(T)\mu_L \rfloor \operatorname{Var}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right] \sim \sigma_2^2 t^{2-\alpha_J+\frac{1}{H}}(T\eta(T))^2,$$

and $P_2(T)/(\lfloor \lambda(T)\mu_L \rfloor) \xrightarrow{\mathrm{P}} 1$. So again, using Lyapunov's Central Limit Theorem and Theorem 4.1.2 of Gnedenko and Korolev (1996), we get

$$\frac{\sum_{k=1}^{P_2(T)} S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right) - P_2(T)\,\mathrm{E}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right]}{T\eta(T)} \Rightarrow \sigma_1 2N\left(0, t^{2-\alpha_J+\frac{1}{H}}\right).$$

To change the centering to a non-random one, observe from (4.27),

$$\begin{aligned}
&\frac{P_2(T) - \mathrm{E}[P_2(T)]}{T\eta(T)}\,\mathrm{E}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right] \\
&\sim \text{constant}\;\frac{P_2(T) - \mathrm{E}[P_2(T)]}{\sqrt{\mathrm{E}[P_2(T)]}}\,\frac{\mathrm{E}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right]}{T^{1+\frac{1}{H}}\overline{F}_J(T)}\sqrt{T^{\frac{1}{H}}\overline{F}_J(T)} = o_P(1),
\end{aligned}$$

since $1/H < \alpha_J$. Combining, we get,

$$\begin{aligned}
&\frac{\sum_{k=1}^{P_1(T)} S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right) - \mathrm{E}[P_1(T)]\,\mathrm{E}\left[S_2^{(T)}\left((Tt)^{\frac{1}{H}} - \tau_2^{(T)}\right)\right]}{T\eta(T)} \\
&\stackrel{\mathrm{d}}{=} \frac{X_T^{(2)}(t) - \mathrm{E}\left[X_T^{(2)}(t)\right]}{T\eta(T)} \Rightarrow \sigma_2 N\left(0, t^{2-\alpha_J+\frac{1}{H}}\right). \qquad (4.42)
\end{aligned}$$

Finally, since, $X_T^{(1)}$ and $X_T^{(1)}$ are independent, adding (4.41) and (4.42), we get

Theorem 4.6.1. *Under the assumptions* (7) - (11) *and fast growth condition (F), we have, for each* $t > 0$,

$$\frac{X_T\left((Tt)^{\frac{1}{H}}\right) - \mathrm{E}\left[X_T\left((Tt)^{\frac{1}{H}}\right)\right]}{T\eta(T)} \Rightarrow \sigma N\left(0, t^{2-\alpha_J+\frac{1}{H}}\right),$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2 = \frac{2}{(2-\alpha_J)[H(2-\alpha_J)+1]}\,\mathrm{E}\left[A_1(1)^{2-\alpha_J}\right]$.

4.6.2 Finite-dimensional convergence

To study the finite dimensional convergence, it is helpful to prove the following lemmas:

Lemma 4.6.1. *Under the given assumptions, for* $0 < l < 2+\delta$, *we have*

$$\mathrm{E}[(uA_1(1))^l \overline{F}_J(uA_1(1))] \sim \mathrm{E}\left[A_1(1)^{l-\alpha_J}\right] u^l \overline{F}_J(u), \tag{4.43}$$

and hence $\mathrm{E}[(uA_1(1))^l \overline{F}_J(uA_1(1))]$, *as a function of* u, *is regularly varying of index* $l - \alpha_J$.

Proof. Fix $\rho > 0$. Choose $\varepsilon > 0$, such that $l + \varepsilon < 2 + \delta$. Using Potter's bound, choose anon-random U such that for all $u > U$, we have

$$\frac{A_1(1)^l \overline{F}_J(uA_1(1)) \mathbf{1}_{[A_1(1) \geqslant \rho]}}{\overline{F}_J(u)} \leqslant (1+\varepsilon) A_1(1)^{l-\alpha_J+\varepsilon}$$

and the right side is integrable by assumption (11). Hence by Dominated Convergence Theorem, we have

$$\mathrm{E}\left[\frac{A_1(1)^l \overline{F}_J(uA_1(1))}{\overline{F}_J(u)}\right] \geqslant \mathrm{E}\left[\frac{A_1(1)^l \overline{F}_J(uA_1(1)) \mathbf{1}_{[A_1(1) \geqslant \rho]}}{\overline{F}_J(u)}\right] \to \mathrm{E}\left[A_1(1)^{l-\alpha_J} \mathbf{1}_{[A_1(1) \geqslant \rho]}\right] \tag{4.44}$$

and letting $\rho \to 0$, we have

$$\liminf_{u \to \infty} \frac{\mathrm{E}\left[(uA_1(1))^l \overline{F}_J(uA_1(1))\right]}{u^l \overline{F}_J(u)} \geqslant \mathrm{E}\left[A_1(1)^{l-\alpha_J}\right]. \tag{4.45}$$

On the other hand, we have,

$$\mathrm{E}\left[\frac{A_1(1)^l \overline{F}_J(uA_1(1)) \mathbf{1}_{[A_1(1) < \rho]}}{\overline{F}_J(u)}\right]$$

$$
\begin{aligned}
&= \mathrm{E}\left[\frac{A_1(1)^l \mathbf{1}_{[J_1>uA_1(1)]}\mathbf{1}_{[A_1(1)<\rho]}\mathbf{1}_{[J_1\leqslant u\rho]}}{\overline{F}_J(u)}\right] + \mathrm{E}\left[\frac{A_1(1)^l \mathbf{1}_{[J_1>uA_1(1)]}\mathbf{1}_{[A_1(1)<\rho]}\mathbf{1}_{[J_1>u\rho]}}{\overline{F}_J(u)}\right] \\
&\leqslant \mathrm{E}\left[\frac{J_1^l \mathbf{1}_{[J_1\leqslant u\rho]}\mathbf{1}_{\left[A_1(1)^{-\alpha_J}>\left(\frac{J_1}{u}\right)^{-\alpha_J}\right]}}{u^l\overline{F}_J(u)}\right] + \mathrm{E}\left[\frac{\rho^l \mathbf{1}_{[J_1>u\rho]}\mathbf{1}_{\left[A_1(1)^{-\alpha_J}>\rho^{-\alpha_J}\right]}}{\overline{F}_J(u)}\right] \\
&\leqslant \frac{\mathrm{E}\left[J_1^{l+\alpha_J}\mathbf{1}_{[J\leqslant u\rho]}\right]}{u^{l+\alpha_J}\overline{F}_J(u)} \mathrm{E}\left[A_1(1)^{-\alpha_J}\right] + \rho^{l+\alpha_J}\frac{\overline{F}_J(\rho u)}{\overline{F}_J(u)} \mathrm{E}\left[A_1(1)^{-\alpha_J}\right] \\
&\to \frac{\alpha_J}{l}\rho^l \mathrm{E}\left[A_1(1)^{-\alpha_J}\right] + \rho^l \mathrm{E}\left[A_1(1)^{-\alpha_J}\right].
\end{aligned}
$$

Combining with (4.44), we have,

$$
\begin{aligned}
&\limsup_{u\to\infty} \mathrm{E}\left[\frac{(uA_1(1))^l\overline{F}_J(uA_1(1))}{u^l\overline{F}_J(u)}\right] \\
&\leqslant \limsup_{u\to\infty} \mathrm{E}\left[\frac{A_1(1)^l\overline{F}_J(uA_1(1))\mathbf{1}_{[A_1(1)\geqslant\rho]}}{\overline{F}_J(u)}\right] + \limsup_{u\to\infty} \mathrm{E}\left[\frac{A_1(1)^l\overline{F}_J(uA_1(1))\mathbf{1}_{[A_1(1)<\rho]}}{\overline{F}_J(u)}\right] \\
&\leqslant \mathrm{E}\left[A_1(1)^{l-\alpha_J}\mathbf{1}_{[A_1(1)\geqslant\rho]}\right] + \frac{\alpha_J}{l}\rho^l \mathrm{E}\left[A_1(1)^{-\alpha_J}\right] + \rho^l \mathrm{E}\left[A_1(1)^{-\alpha_J}\right].
\end{aligned}
$$

Letting $\rho \to 0$, we have

$$
\limsup_{u\to\infty} \frac{\mathrm{E}\left[(uA_1(1))^l\overline{F}_J(uA_1(1))\right]}{u^l\overline{F}_J(u)} \leqslant \mathrm{E}\left[A_1(1)^{l-\alpha_J}\right]. \tag{4.46}
$$

Combining (4.45) and (4.46), we get

$$
\mathrm{E}\left[(uA_1(1))^l\overline{F}_J(uA_1(1))\right] \sim \mathrm{E}\left[A_1(1)^{l-\alpha_J}\right] u^l\overline{F}_J(u)
$$

and hence is regularly varying of index $l - \alpha_J$. □

The other lemma involves an integrated version of the above lemma:

Lemma 4.6.2. *Under the given assumptions, for $\alpha_J < l < 2 + \delta$, we have*

$$
\mathrm{E}\left[\int_{w=0}^{A_1(u)} w^l\overline{F}_J(dw)\right] \sim \frac{\alpha_J}{l-\alpha_J} \mathrm{E}\left[A_1(1)^{l-\alpha_J}u^{Hl}\overline{F}_J(u^H)\right]
$$

and hence $\mathrm{E}\left[\int_{w=0}^{A_1(u)} w^l \overline{F}_J(dw)\right]$*, as a function of* u*, is regularly varying of order* $H(l-\alpha_J)$.

Proof. First observe that

$$\begin{aligned}\mathrm{E}\left[\int\limits_{w=0}^{A_1(u)} w^l \overline{F}_J(dw)\right] &= \mathrm{E}\left[\int\limits_{w=0}^{A_1(u)} l w^{l-1} \overline{F}_J(w)\,dw\right] - \mathrm{E}\left[A_1(u)^l \overline{F}_J(A_1(u))\right] \\ &\sim \mathrm{E}\left[\int\limits_{w=0}^{u^H A_1(1)} l w^{l-1} \overline{F}_J(w)\,dw\right] - \mathrm{E}\left[u^H A_1(1)^l \overline{F}_J(u^H A_1(1))\right].\end{aligned} \tag{4.47}$$

The second term can be handled using Lemma 4.6.1. For the first term we substitute $u^H = v$. Choose $\varepsilon > 0$, such that $l+\varepsilon < 2+\delta$. Let $\rho > 0$. Then using Lemma 4.4.1, we can find a non-random V, such that for all $v > V$,

$$\frac{\int\limits_{w=0}^{vA_1(1)} l w^{l-1} \overline{F}_J(w)\,dw \mathbf{1}_{[A_1(1)>\rho]}}{v^l \overline{F}_J(v)} \leqslant (1+\varepsilon)\frac{l}{l-\alpha_J} A_1(1)^{l-\alpha_J+\varepsilon}.$$

Since the upper bound is integrable by the assumption (11), we have from Dominated Convergence Theorem,

$$\begin{aligned}\mathrm{E}\left[\frac{\int\limits_{w=0}^{A_1(1)v} l w^{l-1} \overline{F}_J(w)\,dw}{v^l \overline{F}_J(v)}\right] &\geqslant \mathrm{E}\left[\frac{\int\limits_{w=0}^{A_1(1)v} l w^{l-1} \overline{F}_J(w)\,dw \mathbf{1}_{[A_1(1)>\rho]}}{v^l \overline{F}_J(v)}\right] \\ &\to \mathrm{E}\left[A_1(1)^{l-\alpha_J} \mathbf{1}_{[A_1(1)>\rho]}\right].\end{aligned} \tag{4.48}$$

Letting $\rho \to 0$, we have

$$\liminf_{v\to\infty} \mathrm{E}\left[\frac{\int\limits_{w=0}^{A_1(1)v} l w^{l-1} \overline{F}_J(w)\,dw}{v^l \overline{F}_J(v)}\right] \geqslant \frac{l}{l-\alpha_J}\,\mathrm{E}\left[A_1(1)^{l-\alpha_J}\right]. \tag{4.49}$$

Again, we have

$$\begin{aligned}
&\mathrm{E}\left[\frac{\int_{w=0}^{A_1(1)v} l w^{l-1}\overline{F}_J(w)\,dw\mathbf{1}_{[A_1(1)\leqslant\rho]}}{v^l\overline{F}_J(v)}\right]\\
&=\mathrm{E}\left[\frac{\int_{w=0}^{(A_1(1)v)\wedge J_1} l w^{l-1}\,dw\mathbf{1}_{[A_1(1)\leqslant\rho]}}{v^l\overline{F}_J(v)}\right]\\
&=\mathrm{E}\left[\frac{((A_1(1)v)\wedge J_1)^l\,\mathbf{1}_{[A_1(1)\leqslant\rho]}}{v^l\overline{F}_J(v)}\right]\\
&=\mathrm{E}\left[\frac{J_1^l\mathbf{1}_{\left[\frac{J_1}{v}<A_1(1)\leqslant\rho\right]}}{v^l\overline{F}_J(v)}\right]+\mathrm{E}\left[\frac{A_1(1)^l\mathbf{1}_{[A_1(1)\leqslant\rho]}\mathbf{1}_{[J_1>A_1(1)v]}}{\overline{F}_J(v)}\right]\\
&\leqslant\mathrm{E}\left[\frac{J_1^l\mathbf{1}_{[J_1<v\rho]}}{v^l\overline{F}_J(v)}\right]+\mathrm{E}\left[\frac{(A_1(1)v)^l\overline{F}_J(A_1(1)v)\mathbf{1}_{[A_1(1)\leqslant\rho]}}{v^l\overline{F}_J(v)}\right].
\end{aligned}$$

As $v\to\infty$, using Karamata's theorem, the first term converges to $\frac{\alpha_J}{l-\alpha_J}\rho^{l-\alpha_J}$. For the second term, observe that

$$\frac{(A_1(1)v)^l\overline{F}_J(A_1(1)v)}{v^l\overline{F}_J(v)}\to A_1(1)^{l-\alpha_J}$$

uniformly on $[A_1(1)\leqslant\rho]$. So by Dominated Convergence Theorem, the second term converges to $\mathrm{E}\left[A_1(1)^{l-\alpha_J}\mathbf{1}_{[A_1(1)\leqslant\rho]}\right]$. Since $l>\alpha_J$, we have, letting $\rho\to 0$,

$$\limsup_{v\to\infty}\mathrm{E}\left[\frac{\int_{w=0}^{A_1(1)v} l w^{l-1}\overline{F}_J(w)\,dw\mathbf{1}_{[A_1(1)\leqslant\rho]}}{v^l\overline{F}_J(v)}\right]=0$$

and hence, using (4.48)

$$\limsup_{v\to\infty}\mathrm{E}\left[\frac{\int_{w=0}^{A_1(1)v} l w^{l-1}\overline{F}_J(w)\,dw}{v^l\overline{F}_J(v)}\right]\leqslant\frac{l}{l-\alpha_J}\,\mathrm{E}\left[A_1(1)^{l-\alpha_J}\right]. \tag{4.50}$$

Combining (4.49) and (4.50), we have

$$\mathrm{E}\left[\int_{w=0}^{A_1(1)v} l w^{l-1}\overline{F}_J(w)\,dw\right] \sim \frac{l}{l-\alpha_J}\,\mathrm{E}\left[A_1(1)^{l-\alpha_J}\right] v^l \overline{F}_J(v).$$

Then from (4.47) and (4.43), we have the required result. □

Let N_T be the counting process corresponding to the initiation times $\{\Gamma_k\}$. Observe that

$$X_T(t) = \sum_{k=1}^{N_T(t)} A_k(t-\Gamma_k^{(T)}) \wedge J_k.$$

Now fix $0 < t_1 < t_2 < \cdots < t_n$. Then, using stationarity and independence of the increments and the order statistic property of a Poisson process, we have

$$\begin{aligned}
\begin{pmatrix} X_T(t_1) \\ X_T(t_2) \\ \vdots \\ X_T(t_n) \end{pmatrix} &= \begin{pmatrix} \sum_{k=1}^{N_T(t_1)} A_k\left(t_1-\Gamma_k^{(T)}\right)\wedge J_k \\ \sum_{k=1}^{N_T(t_2)} A_k\left(t_2-\Gamma_k^{(T)}\right)\wedge J_k \\ \vdots \\ \sum_{k=1}^{N_T(t_n)} A_k\left(t_n-\Gamma_k^{(T)}\right)\wedge J_k \end{pmatrix} \\
&\overset{\mathrm{d}}{=} \begin{pmatrix} \sum_{k=1}^{N_T^{(1)}(t_1)} A_{k,1}\left(t_1 U_{k,1}\right)\wedge J_{k,1} \\ \sum_{k=1}^{N_T^{(1)}(t_1)} A_{k,1}\left(t_2-t_1+t_1 U_{k,1}\right)\wedge J_{k,1} \\ \vdots \\ \sum_{k=1}^{N_T^{(1)}(t_1)} A_{k,2}\left(t_n-t_1+t_1 U_{k,1}\right)\wedge J_{k,1} \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ \sum_{k=1}^{N_T^{(2)}(t_2-t_1)} A_{k,2}\left(\left(t_2-t_1\right)U_{k,2}\right)\wedge J_{k,2} \\ \vdots \\ \sum_{k=1}^{N_T^{(2)}(t_n-t_2)} A_{k,2}\left(t_n-t_2+\left(t_2-t_1\right)U_{k,2}\right)\wedge J_{k,2} \end{pmatrix}
\end{aligned} \tag{4.51}$$

$$+\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \sum_{k=1}^{N_T^{(n)}(t_n-t_{n-1})} A_{k,n}\left(\left(t_n-t_{n-1}\right)U_{k,n}\right)\wedge J_{k,n} \end{pmatrix},$$

where $\{N_T^{(i)}, i \geqslant 1\}$ are i.i.d. copies of N_T, $U_{k,i}$ are i.i.d. copies of Uniform(0,1) random variables, $J_{k,i}$ and $A_{k,i}$ are i.i.d. copies of J_1 and A_1 respectively. So the finite dimensional convergence is determined by the behavior of the vectors of the form

$$\sum_{k=1}^{N_T\left((Tt_2)^{\frac{1}{H}}-(Tt_1)^{\frac{1}{H}}\right)} \begin{pmatrix} A_k\left(\left((Tt_2)^{\frac{1}{H}}-(Tt_1)^{\frac{1}{H}}\right)U_k\right)\wedge J_k \\ A_k\left((Tt_3)^{\frac{1}{H}}-(Tt_2)^{\frac{1}{H}}+\left((Tt_2)^{\frac{1}{H}}-(Tt_1)^{\frac{1}{H}}\right)U_k\right)\wedge J_k \\ \vdots \\ A_k\left((Tt_n)^{\frac{1}{H}}-(Tt_2)^{\frac{1}{H}}+\left((Tt_2)^{\frac{1}{H}}-(Tt_1)^{\frac{1}{H}}\right)U_k\right)\wedge J_k \end{pmatrix},$$

which we denote by Y_T. We first deal with the case when the above sum is made up to $\lfloor \lambda(T)T^{\frac{1}{H}} \rfloor$, and denoted by Y_T^*. To further study the central limit behavior of the sum, we study the moments of the coordinates of the summands. Observe a typical coordinate of the summand looks like, for $0 \leqslant r < s < t$,

$$\Upsilon_T = A_1\left((Tt)^{\frac{1}{H}}-(Ts)^{\frac{1}{H}}+\left((Ts)^{\frac{1}{H}}-(Tr)^{\frac{1}{H}}\right)U_1\right)\wedge J_1.$$

Also define

$$V_T(r,s,t,u) = A_1\left((Tt)^{\frac{1}{H}}-(Ts)^{\frac{1}{H}}+\left((Ts)^{\frac{1}{H}}-(Tr)^{\frac{1}{H}}\right)u\right).$$

Then

$$\mathrm{E}\left[(\Upsilon_T)^l\right] = \int_{u=0}^{1}\int_{w=0}^{\infty} \mathrm{E}\left[\left(V_T(r,s,t,u)\wedge w\right)^l\right] F_J(dw)\,du$$

$$= \int_{u=0}^{1} \mathrm{E}\left[V_T(r,s,t,u)^l\, \overline{F}_J(V_T(r,s,t,u))\right]\, du + \int_{u=0}^{1} \mathrm{E}\left[\int_{w=0}^{V_T(r,s,t,u)} w^l\, F_J(dw)\right] du. \tag{4.52}$$

Now, observe that, for $1 \leqslant \alpha_J < 2+\delta$,

$$\begin{aligned}
&\int_{u=0}^{1} \mathrm{E}\left[V_T(r,s,t,u)^l\, \overline{F}_J(V_T(r,s,t,u))\right]\, du \\
=&\frac{1}{(Ts)^{\frac{1}{H}} - (Tr)^{\frac{1}{H}}} \int_{u=(Tt)^{\frac{1}{H}}-(Ts)^{\frac{1}{H}}}^{(Tt)^{\frac{1}{H}}-(Tr)^{\frac{1}{H}}} \mathrm{E}\left[A_1(u)^l\, \overline{F}_J(A_1(u))\right]\, du \\
=&\frac{1}{(Ts)^{\frac{1}{H}} - (Tr)^{\frac{1}{H}}} \int_{u=(Tt)^{\frac{1}{H}}-(Ts)^{\frac{1}{H}}}^{(Tt)^{\frac{1}{H}}-(Tr)^{\frac{1}{H}}} \mathrm{E}\left[(u^H A_1(1))^l\, \overline{F}_J(u^H A_1(1))\right]\, du \\
\sim&\frac{1}{H(l-\alpha_J)+1} \frac{\left(t^{\frac{1}{H}} - r^{\frac{1}{H}}\right)^{H(l-\alpha_J)+1} - \left(t^{\frac{1}{H}} - s^{\frac{1}{H}}\right)^{H(l-\alpha_J)+1}}{s^{\frac{1}{H}} - r^{\frac{1}{H}}} \mathrm{E}\left[A_1(1)^{l-\alpha_J}\right] T^l \overline{F}_J(T),
\end{aligned} \tag{4.53}$$

using Karamata's theorem and Lemma 4.6.1, since, by assumption (10b), $H(l - \alpha_J) + 1 > 0$. For the second term of (4.52), observe that

$$\begin{aligned}
&\int_{u=0}^{1} \mathrm{E}\left[\int_{w=0}^{V_T(r,s,t,u)} w^l\, F_J(dw)\right] du \\
=&\frac{1}{(Ts)^{\frac{1}{H}} - (Tr)^{\frac{1}{H}}} \int_{u=(Tt)^{\frac{1}{H}}-(Ts)^{\frac{1}{H}}}^{(Tt)^{\frac{1}{H}}-(Tr)^{\frac{1}{H}}} \mathrm{E}\left[\int_{w=0}^{A_1(u)} w^l\, F_J(dw)\, du\right] \\
=&\frac{1}{(Ts)^{\frac{1}{H}} - (Tr)^{\frac{1}{H}}} \int_{u=(Tt)^{\frac{1}{H}}-(Ts)^{\frac{1}{H}}}^{(Tt)^{\frac{1}{H}}-(Tr)^{\frac{1}{H}}} \mathrm{E}\left[\int_{w=0}^{u^H A_1(1)} w^l\, F_J(dw)\, du\right],
\end{aligned}$$

which converges to $\mathrm{E}\left[J_1^l\right]$ for $0 < l < \alpha_J$ by Monotone Convergence Theorem and for $\alpha_J < l < 2+\delta$ grows like

$$\frac{\alpha_J}{l-\alpha_J}\frac{\left(t^{\frac{1}{H}}-r^{\frac{1}{H}}\right)^{H(l-\alpha_J)+1}-\left(t^{\frac{1}{H}}-s^{\frac{1}{H}}\right)^{H(l-\alpha_J)+1}}{[H(l-\alpha_J)+1]\left(s^{\frac{1}{H}}-r^{\frac{1}{H}}\right)}\,\mathrm{E}\left[A_1(1)^{l-\alpha_J}\right]T^l\overline{F}_J(T), \tag{4.54}$$

by Karamata's theorem and Lemma 4.6.2. Hence, combining (4.53) and (4.54), we have from (4.52) for $0 < l < \alpha_J$, we have $\mathrm{E}\left[\Upsilon_T^l\right] \to \mathrm{E}\left[J_1^l\right]$ for $0 < l < \alpha_J$ and $\mathrm{E}\left[\Upsilon_T^l\right] = O(T^l\overline{F}_J(T))$. Thus, we have

$$\mathrm{Var}\left[\Upsilon_T\right] \sim \frac{2}{(2-\alpha_J)}\frac{\left(t^{\frac{1}{H}}-r^{\frac{1}{H}}\right)^{H(2-\alpha_J)+1}-\left(t^{\frac{1}{H}}-s^{\frac{1}{H}}\right)^{H(2-\alpha_J)+1}}{[H(2-\alpha_J)+1]\left(s^{\frac{1}{H}}-r^{\frac{1}{H}}\right)}T^2\overline{F}_J(T) \tag{4.55}$$

and

$$\mathrm{E}\left[|\Upsilon_T-\mathrm{E}[\Upsilon_T]|^{2+\delta/2}\right] = O(T^{2+\delta/2}\overline{F}_J(T)).$$

Thus, we have

$$\limsup_{T\to\infty}\frac{\lfloor\lambda(T)T^{\frac{1}{H}}\rfloor\,\mathrm{E}\left[|\Upsilon_T-\mathrm{E}[\Upsilon_T]|^{2+\delta/2}\right]}{[T\eta(T)]^{2+\delta/2}} \leqslant \text{ constant } \limsup_{T\to\infty}\left(\eta(T)\right)^{-\delta/4} = 0$$

and hence Lyapunov's condition holds coordinate by coordinate. Also, if Σ_T is the covariance matrix of the summand of Y_T, then from (4.55), $\lim_{T\to\infty}\Sigma_T/(T^2\overline{F}_J(T))$ exists and let us denote the limit by Σ. Thus, by Lyapunov's Central Limit Theorem, we have Y_T^* centered by its mean and scaled by $T\eta(T)$ converges weakly to a normal random vector with zero mean and covariance matrix Σ. Also, we have

$$\frac{N_T\left((Tt_2)^{\frac{1}{H}}-(Tt_1)^{\frac{1}{H}}\right)}{\lfloor\lambda(T)T^{\frac{1}{H}}\rfloor} \xrightarrow{\mathrm{P}} t_2^{\frac{1}{H}}-t_1^{\frac{1}{H}}.$$

Hence, by using a multivariate version of Theorem 4.1.2 of Gnedenko and Korolev (1996), we have $(Y_T - \mathrm{E}[Y_T])/(T\eta(T))$ converges weakly to a normal random vector with zero mean and covariance matrix $\sqrt{t_2^{\frac{1}{H}} - t_1^{\frac{1}{H}}}\Sigma$. Note that we can change the centering as in one-dimensional case, since the changes are probabilistically negligible coordinate by coordinate. Since, the finite dimensional marginals of $X_T\left((Tt)^{\frac{1}{H}}\right)$ are linear combination of independent vectors of type Y_T (appended with zeros), we have the following theorem:

Theorem 4.6.2. *Under the assumptions* (7) - (11) *and fast growth condition (F), we have,*

$$\frac{X_T\left((T\cdot)^{\frac{1}{H}}\right) - \mathrm{E}\left[X_T\left((T\cdot)^{\frac{1}{H}}\right)\right]}{T\eta(T)} \overset{\text{fidi}}{\to} \sigma G(\cdot),$$

where G is a zero mean Gaussian process.

To understand the Gaussian process better, it helps to look at the structure of the covariance process. From the one dimensional convergence we know that $\mathrm{Var}[G(t)] = t^{2-\alpha_J+\frac{1}{H}}$. So we only need to find out $\mathrm{cov}[G(s), G(t)]$. For that it is enough to consider the two-dimensional convergence only. Recall from the representation (4.51) that,

$$\begin{pmatrix} X_T\left((Ts)^{\frac{1}{H}}\right) \\ X_T\left((Tt)^{\frac{1}{H}}\right) \end{pmatrix}$$

$$\overset{\mathrm{d}}{=} \begin{pmatrix} \sum_{k=1}^{N_T^{(1)}\left((Ts)^{\frac{1}{H}}\right)} A_{k,1}\left((Ts)^{\frac{1}{H}}U_{k,1}\right) \wedge J_{k,1} \\ \sum_{k=1}^{N_T^{(1)}\left((Ts)^{\frac{1}{H}}\right)} A_{k,1}\left((Tt)^{\frac{1}{H}} - (Ts)^{\frac{1}{H}} + (Ts)^{\frac{1}{H}}U_{k,1}\right) \wedge J_{k,1} \end{pmatrix}$$

$$+\begin{pmatrix} 0 \\ \sum_{k=1}^{N_T^{(2)}\left((Tt)^{\frac{1}{H}}-(Ts)^{\frac{1}{H}}\right)} A_{k,2}\left(\left((Tt)^{\frac{1}{H}}-(Ts)^{\frac{1}{H}}\right)U_{k,2}\right)\wedge J_{k,2} \end{pmatrix}$$

Thus, only the first term will contribute to the limiting covariance and, since $\mathrm{E}[\Upsilon_T] \to \mathrm{E}[J_1^l]$, we have,

$$\begin{aligned}
&\operatorname{cov}[G(s), G(t)] \\
=&s^{\frac{1}{H}} \lim_{T\to\infty} \frac{\operatorname{cov}\left[A_{k,1}\left((Ts)^{\frac{1}{H}}U_{k,1}\right)\wedge J_{k,1}, A_{k,1}\left((Tt)^{\frac{1}{H}}-(Ts)^{\frac{1}{H}}+(Ts)^{\frac{1}{H}}U_{k,1}\right)\wedge J_{k,1}\right]}{T^2\overline{F}_J(T)} \\
=&s^{\frac{1}{H}} \lim_{T\to\infty} \frac{\mathrm{E}\left[\left(A_{k,1}\left((Ts)^{\frac{1}{H}}U_{k,1}\right)\wedge J_{k,1}\right)\left(A_{k,1}\left((Tt)^{\frac{1}{H}}-(Ts)^{\frac{1}{H}}+(Ts)^{\frac{1}{H}}U_{k,1}\right)\wedge J_{k,1}\right)\right]}{T^2\overline{F}_J(T)} \\
=&s^{\frac{1}{H}} \lim_{T\to\infty} \frac{1}{T^2\overline{F}_J(T)} \int_{u=0}^{1}\int_{w=0}^{\infty} \mathrm{E}\left[(V_T(0,s,s,u)\wedge w)(V_T(0,s,t,u)\wedge w)\right] F_J(dw)\,du \\
=&s^{\frac{1}{H}} \lim_{T\to\infty} \int_{u=0}^{1}\int_{w=0}^{\infty} \mathrm{E}\left[(V_1(0,s,s,u)\wedge w)(V_1(0,s,t,u)\wedge w)\right] \frac{F_J(Tdw)}{\overline{F}_J(T)}\,du,
\end{aligned}$$

since $V_T(r,s,t,u) \overset{\mathrm{d}}{=} TV_1(r,s,t,u)$. Thus, for $a > 0$, we have, using the fact that $V_T(ar,as,at,u) \overset{\mathrm{d}}{=} aV_T(r,s,t,u)$,

$$\begin{aligned}
&\operatorname{cov}[G(as), G(at)] \\
=&(as)^{\frac{1}{H}} \lim_{T\to\infty} \int_{u=0}^{1}\int_{w=0}^{\infty} \mathrm{E}[(V_1(0,as,as,u)\wedge w)(V_1(0,as,at,u)\wedge w)]\frac{\overline{F}_J(T\,dw)}{\overline{F}_J(T)}\,du \\
=&(as)^{\frac{1}{H}} \lim_{T\to\infty} \frac{\overline{F}_J(Ta)}{\overline{F}_J(T)} \\
&\lim_{T\to\infty} \int_{u=o}^{1}\int_{w=0}^{\infty} \mathrm{E}[a^2(V_1(0,as,as,u)\wedge w)(V_1(0,as,at,u)\wedge w)]\frac{\overline{F}_J(Ta\,dw)}{\overline{F}_J(Ta)}\,du
\end{aligned}$$

$$=a^{2-\alpha_J+\frac{1}{H}} s^{\frac{1}{H}} \lim_{T\to\infty} \int\limits_{u=0}^{1} \int\limits_{w=0}^{\infty} \mathrm{E}\left[(V_1(0,s,s,u)\wedge w)(V_1(0,s,t,u)\wedge w)\right] \frac{\overline{F}_J(T\,dw)}{\overline{F}_J(T)}\,dw\,du$$

$$=a^{2-\alpha_J+\frac{1}{H}} \operatorname{cov}[G(s),G(t)].$$

Remark 4.6.1. Hence the index of self-similarity in the fast growth condition is $(2-\alpha_J+\frac{1}{H})/2$, which is between 1/2 and 1 by the assumptions (10a) and (10b). Also, the index of self-similarity is $\frac{1}{H\alpha_J} \in (1/2,1)$ for the slow growth condition, if we further assume that $H\alpha_J < 2$, that is L_1 has infinite variance.

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