

LIE ALGEBROIDS-I

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1. BASIC CONCEPTS

1.1. Lie algebroids: Examples.

Definition 1.1.1. A smooth vector bundle $A \rightarrow M$ over a manifold M together with a bundle homomorphism $\rho : A \rightarrow TM$ to the tangent bundle TM of M is said to be a Lie Algebroid if there is a \mathbb{R} -bilinear map $\Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$, $(X, Y) \mapsto [X, Y]$, satisfying the following properties:

- (1) For all $X, Y \in \Gamma(A)$, $[X, Y] = -[Y, X]$;
- (2) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \Gamma(A)$;
- (3) $[X, fY] = f[X, Y] + \rho(X)fY$ for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$;
- (4) $\rho[X, Y] = [\rho(X), \rho(Y)]$ for all $X, Y \in \Gamma(A)$, where the right hand side bracket denote the Lie bracket of vector fields.

The morphism ρ is called the anchor map of the Lie algebroid A .

Example 1.1.2. The tangent bundle of a manifold M is a Lie algebroid with anchor map Id. Also, any involutive distribution on M is a Lie algebroid.

Example 1.1.3. Every finite dimensional Lie algebra is a Lie algebroid over a point.

Example 1.1.4. Let M be a smooth manifold. For any finite dimensional Lie algebra \mathfrak{g} , $TM \oplus M \times \mathfrak{g}$ is a Lie algebroid. Define Lie bracket by the formula

$$[X \oplus V, Y \oplus W] = [X, Y] \oplus \{X(W) - Y(V) + [V, W]\},$$

where X, Y are vector fields on M and V, W are \mathfrak{g} valued functions on M . The anchor map is defined as the projection $TM \oplus M \times \mathfrak{g} \rightarrow TM$.

Example 1.1.5. Let $G \rightrightarrows M$ be a Lie groupoid with α as the source and β as the target map. Since β is a submersion, $\ker d\beta$ is a subbundle of TG . We shall denote it by $T^\beta G$. Let AG denote the pull-back vector bundle of $T^\beta G$ by the unit map $1 : M \rightarrow G$. Since the unit map is an embedding AG can be viewed as the restriction of $T^\beta G$ to 1_M . Let $i : AG \rightarrow T^\beta G$ denote the inclusion map. We shall show that AG has a Lie algebroid structure with $\rho = d\alpha \circ i$ as its anchor.

Recall that every element $g \in G$ defines a partial left translation on G :

$$\ell_g : G^{\alpha g} \rightarrow G^{\beta g}, \quad h \mapsto gh.$$

The element $1_{\alpha g}$ is the left identity of $G^{\alpha g}$ and the tangent space of $G^{\alpha g}$ at a point h is $\ker d\beta_h$. Both these maps are diffeomorphisms.

Definition 1.1.6. A vector field X on G is said to be left invariant if it satisfies the following condition:

$$X(g) = d\ell_g X(1_{\alpha g}) \text{ for all } g \in G$$

If X is a left invariant vector field then

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- (1) X is completely determined by its values on the unit manifold.
- (2) Moreover, if $h \in G^{\alpha g}$ then

$$X(\ell_g(h)) = X(gh) = d\ell_{gh}(1_{\alpha(gh)}) = d\ell_g(d\ell_h(1_{\alpha h})) = d\ell_g(X(h)).$$

- (3) X is a section of the subbundle $\ker d\beta$ of TG : since $\beta \circ \ell_g$ is the constant map at $1_{\beta g}$, $d\beta(X(g)) = d\beta \circ d\ell_g X(1_{\alpha g}) = 0$. Therefore, $X(g) \in \ker d\beta_g$.
- (4) Since $\ker d\beta_h = T_h G^x$ for every $h \in G^x$ and $x \in M$, $X(h) \in T_h G^x$. Therefore, $X|_{G^x}$ is a vector field on G^x .

Let us denote the set of all left invariant vector fields on G by $\mathcal{L}(G)$.

Proposition 1.1.7. *The elements of $\mathcal{L}(G)$ are in one-to-one correspondence with elements of $\Gamma(AG)$.*

Proof. First observe that a left invariant vector field is a section of the vector bundle $T^\beta G$. Hence, the section of AG that corresponds to \tilde{X} is given by $\tilde{X} \circ 1$. On the other hand, since a left invariant vector field is completely known from its values on 1_M , a section X of AG extends to a unique left invariant vector field \tilde{X} by the formula $\tilde{X}(g) = d\ell_g(X(1_{\alpha g}))$ for $g \in G$. We can recover X by restricting \tilde{X} to the submanifold 1_M , that is, $X = \tilde{X} \circ 1$. \square

Let $\chi(G)$ denote the Lie algebra of vector fields on G . We will show that $\mathcal{L}(G)$ is a subalgebra of $\chi(G)$ under the Lie bracket operation of vector fields. At this point we introduce the following definition.

Definition 1.1.8. Let $f : M \rightarrow N$ be a smooth map and X, Y be two smooth vector fields on M and N respectively. We say that X is f -related to Y if the following relation holds: $df_x(X(x)) = Y(f(x))$ for all $x \in X$. In notation, we write $X \sim_f Y$.

In order that X is f -related to a vector field on N , the restriction of df_x to $X|_{f^{-1}(x)}$ must be constant. If $f : M \rightarrow N$ is a surjective submersion, and X is f -related to some Y on N then we say that X is projectable. Note that such an Y is uniquely determined by X . We generally denote this vector field Y by $df(X)$ and call it the projection of X . We have the following result.

Lemma 1.1.9. *Let $f : M \rightarrow N$ be a smooth map. If $X, X' \in \chi(M)$ and $Y, Y' \in \chi(N)$ are such that $X \sim_f Y$ and $X' \sim_f Y'$, then $[X, X'] \sim_f [Y, Y']$.*

In particular, if $f : M \rightarrow N$ is a surjective submersion and X and Y are projectable vector fields on M then so is $[X, Y]$. Furthermore, $df([X, Y]) = [df(X), df(Y)]$.

Proposition 1.1.10. *The Lie bracket of two left invariant vector fields on a Lie groupoid G is again a left invariant vector field. Hence $\mathcal{L}(G)$ is a subalgebra of $\chi(G)$.*

Proof. We have already observed that if X is a left-invariant vector field on G , then $X|_{G^x}$ is a vector field on G^x . The left invariance condition on X further implies that

$$X|_{G^{\alpha g}} \text{ is } \ell_g \text{ related to } X|_{G^{\beta g}} \text{ for all } g \in G.$$

Let X and Y be two left invariant vector fields on G . Then, $X|_{G^{\alpha g}}$ is ℓ_g related to $X|_{G^{\beta g}}$ and $Y|_{G^{\alpha g}}$ is ℓ_g related to $Y|_{G^{\beta g}}$. Hence it follows from the above lemma that $[X|_{G^{\alpha g}}, Y|_{G^{\alpha g}}]$ is ℓ_g -related to $[X|_{G^{\beta g}}, Y|_{G^{\beta g}}]$ for all $g \in G$. Since $X|_{G^{\alpha g}}$ and $Y|_{G^{\alpha g}}$ are vector fields on the submanifold $G^{\alpha g}$, we have

$$[X|_{G^{\alpha g}}, Y|_{G^{\alpha g}}] = [X, Y]|_{G^{\alpha g}} \text{ and } [X|_{G^{\beta g}}, Y|_{G^{\beta g}}] = [X, Y]|_{G^{\beta g}}.$$

Therefore, $[X, Y]|_{G\alpha g}$ is ℓ_g -related to $[X, Y]|_{G\beta g}$ for all $g \in G$. This means that $[X, Y]$ is a left invariant vector field on G . \square

Theorem 1.1.11. *Let $G \rightrightarrows M$ be a Lie groupoid. Then $\Gamma(AG)$ is a Lie algebroid with the anchor map $\rho = d\alpha \circ i$ and the Lie bracket defined as follows:*

If $X, Y \in \Gamma(AG)$ then $[X, Y]$ is the unique section of AG satisfying $\widetilde{[X, Y]} = [\widetilde{X}, \widetilde{Y}]$.

Proof. In view of the last two propositions it is enough to verify that $[X, fY] = f[X, Y] + \rho(X)fY$ for any $X, Y \in \Gamma(AG)$ and a function $f \in C^\infty(M)$. First note that

$$\widetilde{fX}(g) = dl_g(fX)(1_{\alpha g}) = dl_g[(f \circ \alpha)(g)X(1_{\alpha g})] = (f \circ \alpha)(g)\widetilde{X}(g)$$

so that $\widetilde{fX} = (f \circ \alpha)\widetilde{X}$. Now,

$$\begin{aligned} [\widetilde{X}, \widetilde{fY}] &= [\widetilde{X}, (f \circ \alpha)\widetilde{Y}] \\ &= (f \circ \alpha)[\widetilde{X}, \widetilde{Y}] + \widetilde{X}(f \circ \alpha)\widetilde{Y} \\ &= (f \circ \alpha)[\widetilde{X}, \widetilde{Y}] + (\rho(X)f \circ \alpha)\widetilde{Y} \\ &= (f \circ \alpha)[X, Y] + \rho(X)fY \end{aligned}$$

since $\widetilde{X}(f \circ \alpha)(g) = (df \circ d\alpha)(\widetilde{X}(g)) = (df \circ d\alpha)(\widetilde{X}(1_{\alpha g})) = df(\rho(X)(\alpha g)) = (\rho(X)f \circ \alpha)(g)$. Hence $[X, fY] = f[X, Y] + \rho(X)fY$. \square

Example 1.1.12. The Lie algebroid of the pair groupoid over a manifold M is the tangent Lie algebroid TM . More generally, the Lie algebroid of the trivial groupoid $M \times G \times M$ is the trivial Lie algebroid $TM \oplus M \times \mathfrak{g}$.

Example 1.1.13. Let P be a principal G bundle. The Lie algebroid of the Gauge groupoid $P \times P/G \rightrightarrows M$ is TP/G .

Example 1.1.14. Let E be a vector bundle over M and let $D(E)$ denote the Lie algebroid of the frame groupoid $\Phi(E)$. The space of sections of $D(E)$ can be identified with the first order differential operators on $\Gamma(E)$.

1.2. Morphisms and sub-objects of Lie algebroids. The simplest example of a Lie algebroid is a Lie algebra which can be thought of as a Lie algebroid over a point. Let $f : M \rightarrow *$ be the constant map. The pull back bundle $M \times \mathfrak{g}$ over M , is however not a Lie algebroid as there is no natural anchor. In order to bring in the anchor map, it would be natural to consider the bundle $TM \oplus M \times \mathfrak{g}$ and take the projection onto TM as the anchor. We have already encountered this object as a trivial Lie algebroid over M . This brings us to following definition of pull-back Lie algebroid.

Definition 1.2.1. Let $A \rightarrow M$ be a Lie algebroid with anchor ρ and $f : M' \rightarrow M$ a continuous map. Consider the set $TM' \oplus_{TM} A$ consisting of all pairs $X' \oplus V$ such that $Df(X') = \rho(V)$, $X' \in TM'$ and $V \in A$.

$$\begin{array}{ccc}
& & A \\
& & \downarrow a \\
TM' & \xrightarrow{\quad} & TM \\
& & \downarrow Df
\end{array}$$

Suppose that the projection map $p : TM' \oplus_{TM} A \rightarrow M'$ defined by $p(X' \oplus V) = p'_T(X')$ is a vector bundle projection, where $p'_T : TM' \rightarrow M'$ is the tangent bundle of M' . A section of the pull-back has the expression

$$X' \oplus (\sum u'_i \otimes V_i), \text{ where } X' \in \chi(M'), u'_i \in C^\infty(M') \text{ and } V_i \in \Gamma A$$

such that $Df(X')(m') = \sum u'_i(m')a(V_i(f(m')))$ for all $m' \in M'$. We define the bracket of two sections of the pull-back as follows:

$$[X' \oplus (\sum u'_i \otimes V_i), Y' \oplus (\sum v'_j \otimes W_j)] = [X', Y'] \oplus (\sum u'_i v'_j \otimes [V_i, W_j]) + \sum a'(X') v'_j \otimes W_j - \sum a'(Y') u'_i \otimes V_i$$

Then $TM' \oplus_{TM} A$ is a Lie algebroid with anchor map ρ' given by

$$\rho'(X' \oplus (\sum u'_i \otimes V_i)) = X'.$$

The pull-back Lie algebroid will be denoted by $f^!(A)$.

Note that, in general, the pull-back construction does not even give a vector bundle. However, if f is a surjective submersion or A is a transitive Lie algebroid then the pull-back Lie algebroid exists.

Definition 1.2.2. Morphisms of Lie algebroids. Let $q' : A' \rightarrow M'$ and $q : A \rightarrow M$ be two Lie algebroids with anchor maps ρ and ρ' respectively. First consider the case $M = M'$. A vector bundle morphism $\phi : A' \rightarrow A$ takes a section X' of A' onto a section of $\phi(X') = \phi \circ X'$ of A . Then ϕ is a Lie algebroid morphism if $\rho \circ \phi = \rho'$ and $X' \mapsto \phi(X')$ is a Lie algebra homomorphism from $\Gamma(A')$ to $\Gamma(A)$.

$$\begin{array}{ccc}
A' & \xrightarrow{\quad \phi \quad} & A \\
& \searrow \rho' & \swarrow \rho \\
& & TM
\end{array}$$

However, when $M \neq M'$, a vector bundle morphism from $q' : A' \rightarrow M'$ to $q : A \rightarrow M$ does not in general take a section of A' onto a section of A . A morphism of vector bundles $(\phi, f) : A' \rightarrow A$ is a Lie algebroid morphism if $\rho \circ \phi = df \circ \rho'$ and $\phi^! : A' \rightarrow f^!A$ defined by $X' \mapsto \rho'(X') \oplus \phi(X')$ is a morphism of Lie algebroids (over the same base).

$$\begin{array}{ccccc}
& & TM' & \xrightarrow{df} & TM \\
& \nearrow \rho' & & \nearrow \rho & \\
A' & \xrightarrow{\phi} & A & & \\
\downarrow & & \downarrow & & \\
M' & \xrightarrow{f} & M & &
\end{array}$$

Example 1.2.3. The anchor map of a Lie algebroid $A \rightarrow M$ is a Lie algebroid homomorphism

Example 1.2.4. If $f : M \rightarrow N$ is a smooth map then $(df, f) : TM \rightarrow TN$ is a Lie algebroid morphism.

We shall now discuss some functorial properties of A . Let G and G' be Lie groupoids over M and M' respectively and $(F, f) : G' \rightarrow G$ a Lie groupoid morphism. Then $(F_*, F) : TG' \rightarrow TG$ is a Lie algebroid morphism, where F_* denotes the derivative of F . Moreover, since $\beta'F = f\beta$, F_* maps $\ker d\beta'$ into $\ker d\beta$ and $F_* \circ i' : AG' \rightarrow T^\beta G$ is a vector bundle morphism over $F \circ 1' = 1 \circ f$.

$$\begin{array}{ccccc}
AG' & \xrightarrow{i'} & T^{\beta'} G' & \xrightarrow{F_*} & T^\beta G \\
\downarrow & & \downarrow & & \downarrow \\
M' & \xrightarrow{1'} & G' & \xrightarrow{F} & G
\end{array}$$

This morphism factors through the Lie algebroid $AG \rightarrow M$ as $i \circ AF$, where $(AF, f) : AG' \rightarrow AG$ is a vector bundle morphism.

$$\begin{array}{ccccc}
AG' & \xrightarrow{AF} & AG & \xrightarrow{i} & T^\beta G \\
\downarrow & & \downarrow & & \downarrow \\
M' & \xrightarrow{f} & M & \xrightarrow{1} & G
\end{array}$$

If $M = M'$ and f is the identity map then $F_* \circ X$ is a section of AG for every $X \in \Gamma(AG')$. We shall denote this section by $AF(X)$. It follows that $F_* \circ \tilde{X} = \widehat{AF(X)} \circ F$ for all $X \in \Gamma(AG')$. Here, for any $Z \in \Gamma(AG)$, \tilde{Z} denotes the unique left invariant vector field on G extending Z .

Proposition 1.2.5. *If $(F, f) : G' \rightarrow G$ is a groupoid morphism then $(AF, f) : AG' \rightarrow AG$ is a Lie algebroid morphism.*

Proof. Since (F, f) is a groupoid morphism $\alpha \circ F = f \circ \alpha'$. Taking derivative of both sides we get $\alpha_* \circ F_* = f_* \circ \alpha'_*$. It then follows from the definition of the anchor maps that $\rho \circ AF = f_* \circ \rho'$. We now restrict ourselves to the case where both G and G' are Lie groupoids over the same base M . To prove that $AF : \Gamma(AG') \rightarrow \Gamma(AG)$ is a Lie algebra homomorphism, take any two sections $X, Y \in \Gamma(AG')$. Then $F_* \circ \tilde{X} = \widetilde{AF(X)} \circ F$ and $F_* \circ \tilde{Y} = \widetilde{AF(Y)} \circ F$. Hence we get

$$\begin{aligned} F_* \circ \widetilde{[X, Y]} &= F_* \circ \widetilde{[\tilde{X}, \tilde{Y}]} \\ &= \widetilde{[AF(X), AF(Y)]} \circ F. \end{aligned}$$

Precomposing with 1 we obtain $AF([X, Y]) = [AF(X), AF(Y)]$. We refer to [1] for the general case. \square

Example 1.2.6. If $G \rightrightarrows M$ is a Lie groupoid with source and target maps α and β respectively, then $(\alpha, \beta) : G \rightarrow M \times M$ is a morphism of Lie groupoids. Applying the functor A to this map we get the anchor map $\rho : AG \rightarrow TM$ of the Lie algebroid.

We end this lecture by defining the sub-objects of Lie algebroids.

Definition 1.2.7. Let $q : A \rightarrow M$ be a Lie algebroid with anchor ρ , and let M' be a closed embedded submanifold of M . A subbundle $L \rightarrow M'$ of $A|_{M'}$ is called a Lie subalgebroid of A if the following relations hold:

- (1) The anchor map ρ of A maps L into TM' ;
- (2) If $X, Y \in \Gamma(A)$ are such that $X|_{M'}$ and $Y|_{M'}$ are sections of L then $[X, Y]|_{M'}$ is also a section of L ;
- (3) If $X, Y \in \Gamma(A)$ are such that $X|_{M'} = 0$ and $Y|_{M'}$ is a section of L then $[X, Y]|_{M'} = 0$.

There is a unique Lie algebroid structure on L such that $i : L \rightarrow M$ is a Lie algebroid morphism. If x, y are two sections of L then the Lie bracket is defined by $[x, y] = [X, Y]|_{M'}$, where $X, Y \in \Gamma A$ are extensions of x and y respectively. The anchor is the restriction of ρ to L .

Example 1.2.8. If N be a closed embedded submanifold of M then TN is a Lie subalgebroid of TM .

REFERENCES

- [1] Mackenzie K. C. H. *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press