LIE ALGEBROID -III

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1. INTRODUCTION

Let M be a smooth manifold. Recall that a Lie algebroid on M is a vector bundle $A \longrightarrow M$ equipped with a vector bundle map $a : A \longrightarrow TM$, called the anchor of A, and, a bracket $[\ ,\ ] : \Gamma A \times \Gamma A \longrightarrow \Gamma A$ which makes ΓA a Lie algebra such that for all $X, Y \in \Gamma A$, and $u \in C^\infty(M)$ following hold.

(1) $[X, uY] = u[X, Y] + a(X)(u)Y$,
(2) $a([X, Y]) = [a(X), a(Y)]$.

A Lie algebroid is called

(1) transitive, if the anchor a is fibrewise surjective;
(2) regular if a is of locally constant rank;
(3) totally intransitive, if $a = 0$.

If $A' \longrightarrow M$ is another Lie algebroid with anchor a', then a morphism of Lie algebroid is a vector bundle morphism $\phi : A \longrightarrow A'$, such that

(1) $a' \circ \phi = a$,
(2) $\phi([X, Y]) = [\phi(X), \phi(Y)]$, for all $X,\ Y \in \Gamma A$.

There is a nice connection between Lie algebroid structures on a vector bundle and Schouten brackets. The plan of my first talk is to describe this correspondence.

2. Correspondence between Lie algebroid and Schouten algebra

Definition 1. *Let R be a ring and C be an R-algebra. A Schouten algebra is a $\mathbb{Z}$- graded module $\mathcal{A} = \oplus_{i \in \mathbb{Z}} \mathcal{A}^i$ over C, equipped with*

(1) *a bilinear operation*

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A},\ (a, b) \mapsto a \wedge b,\ a \in \mathcal{A}^i, b \in \mathcal{A}^j,$$

satisfying
: $a \wedge b \in \mathcal{A}^{i+j},\ a \in \mathcal{A}^i,\ b \in \mathcal{A}^j$,
: $b \wedge a = (-1)^{ij} a \wedge b,\ a \in \mathcal{A}^i,\ b \in \mathcal{A}^j$,
: $a \wedge (b \wedge c) = (a \wedge b) \wedge c,\ a \in \mathcal{A}^i,\ b \in \mathcal{A}^j, c \in A^k$

(2) *and another bilinear operation*

$$[\ ,\ ]:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{A},\ (a,b)\mapsto[a,b],$$

verifying
- : $[a,b]\in\mathcal{A}^{i+j-1}$,
- : $[a,b]=-(-1)^{(i-1)(j-1)}[b,a]$,
- : $(-1)^{(i-1)(k-1)}[[a,b],c]+(-1)^{(j-1)(i-1)}[[b,c],a]+(-1)^{(k-1)(j-1)}[[c,a],b]=0$ *where* $a\in\mathcal{A}^i,\ b\in\mathcal{A}^j,\ c\in\mathcal{A}^k$,

(3) *and the following derivation condition holds*

$$[a,b\wedge c]=[a,b]\wedge c+(-1)^{(i-1)j}b\wedge[a,c],\ a\in\mathcal{A}^i,\ b\in\mathcal{A}^j,c\in A^k.$$

Let $T^*(V)=\mathbb{R}\oplus V\oplus V^{\otimes 2}\oplus\cdots\oplus V^{\otimes p}\oplus\cdots$ denote the tensor algebra on a vector space V over $\mathbb{R}$ with multiplication given by juxtaposition

$$(v_1\otimes v_2\otimes\cdots\otimes v_p,v_{p+1}\otimes v_{p+2}\otimes\cdots\otimes v_{p+q})\mapsto v_1\otimes v_2\otimes\cdots\otimes v_{p+q}.$$

Recall that the exterior algebra $\Lambda^*(V)=\mathbb{R}\oplus\Lambda(V)\oplus\Lambda^2(V)\oplus\cdots\oplus\Lambda^p(V)\oplus\cdots$ on V is the quotient of $T^*(V)$ by the ideal I generated by the homogeneous elements $v\otimes w+w\otimes v\in V^{\otimes 2},\ v,\ w\in V$. The graded product in $\Lambda^*(V)$ is given by

$$(v_1\wedge v_2\wedge\cdots\wedge v_p,v_{p+1}\wedge v_{p+2}\wedge\cdots\wedge v_{p+q})\mapsto v_1\wedge v_2\wedge\cdots\wedge v_{p+q},$$

where $v_1\wedge v_2\wedge\cdots\wedge v_p=v_1\otimes v_2\otimes\cdots\otimes v_p+I$. If $\{v_1,\ v_2,\cdots,v_n\}$ is a basis of V, then the subspace $\Lambda^p(V)$–the p^{th} homogeneous part –of $\Lambda^*(V)$ has a basis of the form $\{v_{i_1}\wedge\cdots\wedge v_{i_p};1\le i_1<\cdots<i_p\le n\}$.

Suppose now that $q:A\longrightarrow M$ be a Lie algebroid. Performing the above construction fibrewise, we get a vector bundle $\Lambda^p(A)$ over M for each p. A section of $\Lambda^p(A)$ is called a p-multi section.

We may now define wedge product of multisections as follows. Let $\phi\in\Lambda^p(A)$ and $\psi\in\Lambda^q(A)$. Define $\phi\wedge\psi\in\Lambda^{p+q}(A)$ by $(\phi\wedge\psi)(x)=\phi(x)\wedge\psi(x)\in\Lambda^{p+q}(A_x)$, $x\in M$.

This makes $\Gamma\Lambda^*(A):=\oplus_{p\ge 0}\Gamma\Lambda^p(A)$ a graded commutative associative algebra.

Note that $\Gamma\Lambda^0(A)=C^\infty(M)$ and $\Gamma\Lambda^1(A)=\Gamma A$ and each $\Gamma\Lambda^k(A)$ is a $C^\infty(M)$-module with the module structure

$$C^\infty(M)\times\Gamma\Lambda^k(A)\longrightarrow\Gamma\Lambda^k(A),\ (f,\phi)\mapsto f\phi,$$

where $f\phi(x)=f(x)\phi(x)$.

We define a bracket on $\Gamma\Lambda^*(A)$ extending the Lie algebra bracket on ΓA as follows. For $X\in\Gamma\Lambda^1(A)=\Gamma A$ and $f\in\Gamma\Lambda^0(A)=C^\infty(M)$, define $[X,f]=a(X)f$. Then use (3) and the graded commutativity in the condition (2) of the definition of a Schouten algebra inductively to define all other brackets. It follows that for any Lie algebroid A over M, $\Gamma\Lambda^*(A)$ with the above structures is a Schouten algebra.

Conversely, let $q:A\longrightarrow M$ be a vector bundle. Consider the $C^\infty(M)$-graded module $\Gamma\Lambda^*(A):=\oplus_{p\ge 0}\Gamma\Lambda^p(A)$. This is already a graded commutative associative algebra with respect to the wedge product. Assume that $\Gamma\Lambda^*(A)$ admits a bracket $[\ ,\ ]$ making it a

Schouten algebra. Since $\Gamma\Lambda^1(A) = \Gamma A$, it is clear that we have a bracket $[\ ,\ ] : \Gamma A \times \Gamma A \longrightarrow \Gamma A$ which by assumption is skew-symmetric and satisfies Jacobi identity. Thus ΓA is a Lie algebra.

Define a map $\Gamma A \longrightarrow \chi(M)$ as follows. Let $X \in \Gamma A$ and $f,\ g \in C^\infty(M) = \Gamma\Lambda^0(A)$. Then by condition (3) we have

$$[X, fg] = [X, f\Lambda g] = [X, f]\Lambda g + f\Lambda[X, g] = [X, f]g + f[X, g].$$

Thus $[X,\ ]$ is a vector field. Define $a : \Gamma A \longrightarrow \chi(M)$ by $a(X) := [X,\ ]$. Note that for $g \in C^\infty(M)$,

$$a(gX)(f) = [gX, f] = [g\Lambda X, f] = -[f, g\Lambda X] = -\{[f, g]\Lambda X + g\Lambda[f, X]\}.$$

Thus,

$$a(gX)(f) = g[X, f] - [f, g]\Lambda X = g[X, f] = ga(X)(f).$$

(Note that by definition of $[\ .\ ]$, $[f, g] = 0$.) Thus a is $C^\infty(M)$-linear and hence we have a vector bundle morphism $a : TA \longrightarrow TM$.

Our aim is to prove that $A \longrightarrow M$ is Lie algebroid with a as its anchor.

To verify this, let $X,\ Y \in \Gamma A$ and $f \in C^\infty(M)$. Then by definition of a, we have $a([X, Y])(f) = [[X, Y], f]$. By graded Jacobi identity (with $i = 1,\ j = 1,\ k = 0$) we have $[[X, Y], f] + [[Y, f], X] + [[f, X], Y] = 0$. Therefore,
$[[X, Y], f] = -[[Y, f], X] - [[f, X], Y] = [X, [Y, f]] + [Y, [f, X]]$
$= [X, [Y, f]] - [Y, [X, f]] = [X, a(Y)(f)] - [Y, a(X)(f)] = a(X)a(Y)(f) - a(Y)a(X)(f)$
$= [a(X), a(Y)](f)$.

As a consequence, $a([X, Y]) = [a(X), a(Y)]$.

Finally, note that for $X,\ Y \in \Gamma A = \Gamma\Lambda^1(A)$ and $u \in C^\infty(M) = \Gamma\Lambda^0(A)$, we have by condition (3) of the definition of a Schouten algebra (with $i = 1,\ j = 0,\ k = 1$)

$$[X, uY] = [X, u \wedge Y] = [X, u] \wedge Y + u \wedge [X, Y] = u[X, Y] + a(X)(u)Y.$$

This proves that A is a Lie algebroid. Thus we have proved the following theorem.

Theorem 2. *Let A be a vector bundle on M and let $\mathcal{A} = \Gamma\Lambda^*(A)$ denote the algebra of multisections with standard wedge product. Then there is a one-to-one correspondence between Lie algebroid structures on A and Schouten brackets on $\mathcal{A}$, which together with the exterior algebra structure make $\mathcal{A}$ a Schouten algebra.*

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