LIE ALGEBROID -IV

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1. Derivations of a Lie algebroid

Let E be a vector bundle on a smooth manifold M. A smooth zeroth-order differential operator on E is a $C^\infty(M)$-linear endomorphism $\Gamma E \longrightarrow \gamma E$ and hence corresponds to a vector bundle endomorphism $E \longrightarrow E$.

A first order differential operator on E is an $\mathbb{R}$-linear map $D : \Gamma E \longrightarrow \Gamma E$ such that for each $u \in C^\infty(M)$, the map $\Gamma E \longrightarrow \Gamma E$, given by $\mu \mapsto D(u\mu) - uD(\mu)$ is a zeroth-order differential operator. The first order differential operators are sections of a vector bundle $Diff^1(E)$ on M.

A derivation on E is a first order differential operator D that satisfies condition (*).

(*) There exists a vector field D_M on M such that for all $u \in C^\infty(M)$ and $\mu \in \Gamma E$,

$$D(u\mu) = uD(\mu) + D_M(u)\mu.$$

Let α denote the assignment $D \mapsto D_M$ from the space of derivations on E to vector fields, that is $\alpha(D) = D_M$.

Exercise 1. Suppose D and D' are derivations on E. Prove that the bracket $[D, D'] = D \circ D' - D' \circ D$ is also a derivation with $\alpha([D, D']) = [\alpha(D), \alpha(D')]$.

Exercise 2. Verify that for all derivations D, D' on E and $u \in C^\infty(M)$ following holds,

$$[D, uD'] = u[D, D'] + \alpha(D)(u)D'.$$

Let me state a result due to Kosmann-Scwarzbach-Mackenzie.

Theorem 1. *Given a vector bundle E on M, there exists a transitive Lie algebroid $\mathcal{D}(E)$ on M (which may be realized as a subbundle of $Diff^1(E)$) for which the space of smooth sections $\Gamma\mathcal{D}(E)$ is precisely the space of derivations on E and for which the bracket is given by Exercise 1, and the anchor is given by the map α.*

2. Representation of a Lie algebroid

First, let us discuss a class of examples of Lie algebroid, called Lie algebra bundle (LAB). Let $q : L \longrightarrow M$ be a vector bundle. A section $x \mapsto [\ ,\ ]_x$ of the bundle $Alt^2(L, L)$ is called a field of Lie algebra brackets in L, if for each $x \in M$, $[\ ,\ ]_x : L_x \times L_x \longrightarrow L_x$ is a Lie algebra bracket, that is, satisfies the Jacobi identity.

Definition 2. *An LAB is a vector bundle $q : L \longrightarrow M$, together with a field of Lie algebra brackets $[\ ,\ ] : \Gamma L \times \Gamma L \longrightarrow \Gamma L$ such that L admits an atlas $\{\psi_i : U_i \times \mathfrak{g} \longrightarrow q^{-1}(U_i) = L_{U_i}\}$ in which for each $x \in U_i$, $\psi_{i,x}$ is a Lie algebra isomorphism.*

Note that any LAB can be viewed as a totally intransitive Lie algebroid. However, a totally intransitive lie algebroid may be merely a vector bundle with a field of Lie brackets and need not be an LAB.

Definition 3. *Suppose $q : L \longrightarrow M$ and $q' : L' \longrightarrow M'$ are Lie algebra bundles. A morphism $L \longrightarrow L'$ of Lie algebra bundles is a morphism of vector bundles $\phi : L \longrightarrow L'$, $\phi_0 : M \longrightarrow M'$ such that for each $x \in M$, $\phi_x : L_x \longrightarrow L'_{\phi_0(x)}$ is a Lie algebra homomorphism.*

Definition 4. *Let A be a transitive Lie algebroid on M. The kernel L of $a : A \longrightarrow TM$ is a sub vector bundle of A. Clearly, the space ΓL of sections of L, inherits the bracket of ΓA, because of the condition $a([X, Y]) = [a(X), a(Y)]$. Hence L is a totally intransitive Lie algebroid on M. We call L the adjoint bundle of A and denote it by the exact sequence*

$$0 \longrightarrow L \longrightarrow A \longrightarrow TM \longrightarrow 0.$$

Suppose $\phi : A \longrightarrow A'$ is a morphism of transitive Lie algebroids over M. Since $a' \circ \phi = a$, it is clear that ϕ induces a morphism of the adjoint bundles $\phi^+ : L \longrightarrow L'$.

Exercise 3. Prove that ϕ is surjective, injective or bijective if and only if ϕ^+ is respectively, surjective, injective or bijective.

Definition 5. *Let A be a Lie algebroid and E be a vector bundle on M. A representation of A on E is a morphism of Lie algebroid $\rho : A \longrightarrow \mathcal{D}(E)$.*

In particular, a representation ρ of A on a vector bundle E may be viewed as a bilinear map $\rho : \Gamma A \times \Gamma E \longrightarrow \Gamma E$ such that for each $X \in \Gamma A$, $\rho(X,\ ) : \Gamma E \longrightarrow \Gamma E$ is a derivation on E

Example 6. *Let $E \longrightarrow M$ be a smooth vector bundle. Recall that a connection on E is a bilinear map*

$$\nabla : \chi(M) \times \Gamma E \longrightarrow \Gamma E,\ (X, \mu) \mapsto \nabla_X \mu$$

satisfying

: $\nabla_{uX} \mu = u \nabla_X \mu$

: $\nabla_X (u\mu) = u \nabla_X \mu + (Xu)\mu$

for all $u \in C^\infty(M)$, $X \in \chi(M)$, $\mu \in \Gamma(E)$. $\nabla_X \mu$ is called the covariant derivative of μ with respect to X. Thus the covariant derivative ∇_X is a derivation.

Example 7. *Let A be a Lie algebroid on M. The representation of A on $M \times \mathbb{R} \longrightarrow M$ given by $\rho^0(X)(f) = a(X)(f)$ for all $f \in C^\infty(M)$ and $X \in \Gamma A$ is known as the trivial representation.*

Example 8. *Let A be a transitive Lie algebroid on M. We define the adjoint representation $ad : A \longrightarrow \mathcal{D}(L)$ on its adjoint bundle L as follows.*

$$ad(X)(V) = [X, V],\ X \in \Gamma A,\ V \in \Gamma L.$$

3. Cohomology

Let A be a Lie algebroid on M. Let $\rho : A \longrightarrow \mathcal{D}(E)$ be a representation of A on a vector bundle E. Define a complex $C^*(A, E)$ as follows. For $n \geq 0$, $C^n(A, E) = \mathrm{Alt}^n(A; E)$. Thus $C^n(A, E),\ n \geq 0$ is a sequence of vector bundles. Note that $C^n(A, E)$ is a sub-bundle of the vector bundle Hom $(A^{\otimes n}, E)$ consisting of all bundle morphisms $\phi : A^{\otimes n} \longrightarrow E$ such that ϕ restricted to a fibre $\phi_x : A_x^{\otimes n} \longrightarrow E_x$ is alternating. Clearly, a section of $C^n(A, E)$ is an alternating multi linear map $f : \Gamma A \times \Gamma A \times \cdots \times \Gamma A \longrightarrow \Gamma E$. Define coboundary operator $d : C^n(A, E) \longrightarrow C^{n+1}(A, E)$ by the formula

$$\begin{aligned} df(X_1, \cdots, X_{n+1}) &= \sum_{r=1}^{n+1} (-1)^{r+1} \rho(X_r)(f(X_1, \cdots, \widehat{X}_r, \cdots X_{n+1})) \\ &+ \sum_{r<s} (-1)^{r+s} f([X_r, X_s], X_1, \cdots, \widehat{X}_r, \cdots, \widehat{X}_s, \cdots, X_{n+1}), \end{aligned}$$

for $f \in C^n(A, E)$ and $X_1,\ X_2, \cdots, X_{n+1} \in \Gamma A$.

A standard calculation as in Chevalley-Eilenberg complex of a Lie algebra shows that $d \circ d = 0$. Let

$$\mathcal{Z}^n(A, E) := \ker\ d : \Gamma C^n(A, E) \longrightarrow \Gamma C^{n+1}(A, E),$$
$$\mathcal{B}^n(A, E) := \mathrm{im}\ d : \Gamma C^{n-1}(A, E) \longrightarrow \Gamma C^n(A, E),\ n \geq 1$$

and $\mathcal{B}^0(A, E) = \{0\}$. Then the cohomology spaces are defined by the quotient

$$\mathcal{H}^n(A, E) := \mathcal{Z}^n(A, E)/\mathcal{B}^n(A, E).$$

Remark 9.

- *When M is a point, then A is a finite dimensional Lie algebra, and, in that case, the above cohomology reduces to Chevally-Eilenberg cohomology of the Lie algebra A.*
- *When A is TM, then the above cohomology is precisely the de rham cohomology of M.*

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