LIE ALGEBROID-V

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The cochain groups defining cohomology of a Lie algebroid admit operators satisfying properties identical to those which hold in the calculus of vector-valued forms on a manifold. Let A be a Lie algebroid and fix a section $X \in \Gamma A$.

Definition 1.

- *The Lie derivative* $\mathcal{L}_X : \Gamma C^n(A,E) \longrightarrow \Gamma C^n(A,E)$ *is defined by*

$$\mathcal{L}_X(f)(X_1,\cdots,X_n) = \rho(X)(f(X_1,\cdots,X_n)) - \sum_{r=1}^{n} f(X_1,\cdots,[X,X_r],\cdots,X_n),$$

- *The interior multiplication* $i_X : \Gamma C^{n+1}(A,E) \longrightarrow \Gamma C^n(A,E)$, *for* $n > 0$, *is defined by*

$$i_X(f)(X_1,\cdots,X_n) = f(X,X_1,\cdots,X_n),$$

where $f \in \Gamma C^n(A,E),\ X_r \in \Gamma A,\ 1 \le r \le n.$

The operators $\mathcal{L}_X,\ i_X$ and the coboundary d satisfy the following formulas. The proofs of theses formulas are easy and analogous to the proofs of the corresponding formulas in the context of calculus of forms on a manifold.

Proposition 2. (1) *For* $X,\ Y \in \Gamma A$, $u \in C^\infty(M)$, *and* $f \in \Gamma C^n(A,E)$,

$$i_X(uf) = u i_X(f),\ i_{uX}(f) = u i_X(f),\ \text{and},\ i_X i_Y = -i_Y i_X,$$

(2) *for* $X \in \Gamma A,\ u \in C^\infty(M),\ f \in \Gamma C^n(A,E)$,

$$\mathcal{L}_X(uf) = u\mathcal{L}_X(f) + a(X)(u)(f),$$

(3) *for* $X, X_1, \cdots, X_n \in \Gamma A,\ u \in C^\infty(M),\ f \in \Gamma C^n(A,E)$,

$$\mathcal{L}_{uX}(f)(X_1,X_2,\cdots,X_n) = u\mathcal{L}_X(f)(X_1,X_2,\cdots,X_n)$$
$$+\sum_{r=1}^{n}(-1)^{r-1} a(X_r)(u) i_X(f)(X_1,\cdots,\widehat{X}_r,\cdots,X_n),$$

(4) *for* $X, Y \in \Gamma A$

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X,$$

(5) $\mathcal{L}_X = i_X \circ d + d \circ i_X$, *for* $X \in \Gamma A$,

(6) *for* $X \in \Gamma A$, $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$,

(7) *for* $X,\ Y \in \Gamma A$,

$$[\mathcal{L}_X, i_X] = \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X = i_{[X,Y]}.$$

Proof: The proof of (1) is easy and follows from the definition of the interior multiplication.

For (2), note that

$$(u\mathcal{L}_X(f) + a(X)(u)(f))(X_1, X_2, \cdots, X_n) = u\rho(X)(f(X_1, X_2, \cdots, X_n)$$

$$-\sum_{r=1}^{n} uf(X_1, \cdots, [X, X_r], \cdots, X_n) + a(X)(u)(f(X_1, \cdots, X_n)). \cdots (i)$$

On the other hand,

$$\mathcal{L}_X(uf)(X_1, \cdots, X_n) = \rho(X)(uf(X_1, \cdots, X_n)) - \sum_{r=1}^{n} uf(X_1, \cdots, [X, X_r], \cdots, X_n).$$

Note that $\rho(X) : \Gamma E \longrightarrow \Gamma E$ is a derivation. Therefore,

$$\rho(X)(uf(X_1, \cdots, X_n) = u\rho(X)(f(X_1, \cdots, X_n)) + \alpha(\rho(X))(u)f(X_1, \cdots, X_n).$$

Since $\rho : A \longrightarrow \mathcal{D}(E)$ is a Lie algebroid homomorphism, we have $\alpha \circ \rho = a$. Thus the first and the third term of the right hand side of (i) combined together yield the result.

To prove (3) observe that

$$\mathcal{L}_{uX}(f)(X_1, \cdots, X_n) = \rho(uX)(f(X_1, \cdots, X_n)) - \sum_{r=1}^{n} f(X_1, \cdots, [uX, X_r], \cdots, X_n). \cdots (i)$$

and

$$u\mathcal{L}_X(f)(X_1, \cdots, X_n) = u\rho(X)(f(X_1, \cdots, X_n) - \sum_{r=1}^{n} uf(X_1, \cdots [X, X_r], \cdots X_n). \cdots (ii)$$

Now observe that $\rho : A \longrightarrow \mathcal{D}(E)$ is a vector bundle morphism and therefore, the assignment

$$\Gamma A \longrightarrow \Gamma\mathcal{D}(E),\ X \mapsto \rho(X)$$

is $C^\infty(M)$-linear. Moreover, we have the identity

$$[uX, Y] = u[X, Y] - a(Y)(u)X,\ u \in C^\infty(M),\ X, Y \in \Gamma A.$$

Using these we get from (i)

$$\mathcal{L}_{uX}(f)(X_1, \cdots, X_n) = u\rho(X)(f(X_1, \cdots, X_n)) - \sum_{r=1}^{n} f(X_1, \cdots, u[X, X_r] - a(X_r)(u)X, \cdots, X_n).$$

$$= u\rho(X)(f(X_1, \cdots, X_n) - \sum_{r=1}^{n} uf(X_1, \cdots, [X, X_r], \cdots, X_n)$$

$$+\sum_{r=1}^{n} a(X_r)(u)f(X_1, \cdots, X_{r-1}, X, X_{r+1}, \cdots, X_n). \cdots (iii)$$

Therefore, using (iii), we get from (i) and (ii),

$$\mathcal{L}_{uX}(f)(X_1, \cdots, X_n) - u\mathcal{L}_X(f)(X_1, \cdots, X_n)$$

$$= \sum_{r=1}^{n} a(X_r)(u) f(X_1, \cdots, X_{r-1}, X, X_{r+1}, \cdots, X_n)$$

$$= \sum_{r=1}^{n} (-1)^{r-1} a(X_r)(u) f(X, X_1, \cdots, X_{r-1}, X_{r+1}, \cdots, X_n)$$

$$= \sum_{r=1}^{n} (-1)^{r-1} a(X_r)(u) i_X f(X_1, \ldots, \hat{X}_r, \cdots, X_n).$$

To prove (7), let $X,\ Y \in \Gamma A$, $n > 0$. Then

$$\mathcal{L}_X \circ_Y (f)(X_1, \cdots, X_{n-1}) = \rho(X)(i_Y(f)(X_1, \cdots, X_{n-1})$$
$$- \sum_{r=1}^{n-1} i_Y(f)(X_1, \cdots, [X, X_r], \cdots, X_{n-1})$$
$$= \rho(X)(f(Y, X_1, \cdots, X_{n-1})) - \sum_{r=1}^{n-1} f(Y, X_1, \cdots, [X, X_r], \cdots, X_{n-1}).$$

On the other hand,

$$i_Y \circ \mathcal{L}_X(f)(X_1, \cdots, X_{n-1}) = \mathcal{L}_X(f)(Y, X_1, \cdots, X_{n-1})$$
$$= \rho(X)(f(Y, X_1, \cdots, X_{n-1})) - \sum_{r=1}^{n-1} f(Y, X_1, \cdots, [X, X_r], \cdots, X_{n-1})$$
$$- f([X, Y], X_1, \cdots, X_{n-1}).$$

Therefore,

$$(\mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X)(f)(X_1, \cdots, X_{n-1})$$
$$= f([X, Y], X_1, \cdots, X_{n-1}) = i_{[X,Y]}(f)(X_1, \cdots, X_{n-1}).$$

This proves the result.

Next, we prove (5). By definition of i_X, we have

$$i_X \circ d(f)(X_1, \cdots, X_n) = d(f)(X, X_1, \cdots, X_n) = \rho(X)(f(X_1, \cdots, X_n))$$
$$+ \sum_{r=1}^{n} (-1)^r \rho(X_r) f(X, X_1, \cdots, \hat{X}_r, \cdots, X_n) + \sum_{j=1}^{n} (-1)^j f([X, X_j], X_1, \cdots, \hat{X}_j, \cdots, X_n)$$
$$+ \sum_{r<s} (-1)^{r+s} f([X_r, X_s], X_1, \cdots, \hat{X}_r, \cdots, \hat{X}_s, \cdots, X_n).$$

On the other hand,

$$d \circ i_X(f)(X_1, \cdots, X_n) = \sum_{r=1}^{n} (-1)^{r+1} \rho(X_r) i_X(f)(X_1, \cdots, \hat{X}_r, \cdots, X_n)$$
$$+ \sum_{r<s} (-1)^{r+s} i_X(f)([X_r, X_s], X_1, \cdots, \hat{X}_r, \cdots, \hat{X}_s, \cdots, X_n).$$
$$= \sum_{r=1}^{n} (-1)^{r+1} \rho(X_r)(f)(X, X_1, \cdots, \hat{X}_r, \cdots, X_n)$$

$$+\sum_{r<s}(-1)^{r+s}(f)(X,[X_r,X_s],X_1,\cdots,\hat{X}_r,\cdots,\hat{X}_s,\cdots,X_n).$$

Therefore,

$$\begin{aligned}&(i_X\circ d+d\circ i_X)(f)(X_1,\cdots,X_n)\\ &=\rho(X)(f(X_1,\cdots,X_n))+\sum_{j=1}^{n}(-1)^j f([X,X_j],X_1,\cdots,\hat{X}_j,\cdots,X_n)\\ &=\rho(X)(f(X_1,\cdots,X_n))-\sum_{j=1}^{n}f(X_1,\cdots,[X,X_j],\cdots,X_n)=\mathcal{L}_X(f)(X_1,\cdots,X_n).\end{aligned}$$

The property (6) follows immediately from (5) as $d^2=0$.

Remark 3. *Properties* (5) *and* (7) *are known as Cartan formulas.*

Property (4) may be proved by an induction argument. Note that for $n=0$, $C^0(A,E)=\Gamma E$, the space of all smooth sections of E and by definition, for any $X\in\Gamma A$, $\mathcal{L}_X:\Gamma E\longrightarrow\Gamma E$ is given by $\mathcal{L}_X(f)=\rho(X)(f)$. Therefore,

$$\begin{aligned}\mathcal{L}_{[X,Y]}(f)=\rho([X,Y])(f)=[\rho(X),\rho(Y)]f=(\rho(X)\rho(Y)-\rho(Y)\rho(X))f\\ \rho(X)(\rho(Y)f)-\rho(Y)(\rho(X)f)=\mathcal{L}_X\mathcal{L}_Y(f)-\mathcal{L}_Y\mathcal{L}_X(f).\end{aligned}$$

Thus the identity holds for $n=0$. Assume that the identity holds for $n\geq 0$. We will prove it for $n+1$.

Let $f\in C^{n+1}(A,E)$. Then for any arbitrary $Z\in\Gamma A$, $i_Z(f)\in C^n(A,E)$. Therefore, by induction hypothesis, for any $X,\ Y\in\Gamma A$ we have

$$\mathcal{L}_{[X,Y]}(i_Z(f))=(\mathcal{L}_X\mathcal{L}_Y-\mathcal{L}_Y\mathcal{L}_X)i_Z(f).\cdots(i)$$

On the other hand, by property (7)

$$\mathcal{L}_{[X,Y]}(i_Z(f))=(i_Z\mathcal{L}_{[X,Y]}+i_{[[X,Y],Z]})(f).$$

Again, by (7),

$$\mathcal{L}_Y i_Z(f)=(i_Z\mathcal{L}_Y+i_{[Y,Z]})(f).$$

Therefore,

$$\begin{aligned}&\mathcal{L}_X\mathcal{L}_Y i_Z(f)=\mathcal{L}_X(i_Z\mathcal{L}_Y+i_{[Y,Z]})(f)\\ &=\{(i_Z\mathcal{L}_X+i_{[X,Z]})\mathcal{L}_Y+(i_{[Y,Z]}\mathcal{L}_X+i_{[X,[Y,Z]]})\}(f)\\ &=\{i_Z\mathcal{L}_X\mathcal{L}_Y+i_{[X,Z]}\mathcal{L}_Y+i_{[Y,Z]}\mathcal{L}_X+i_{[X,[Y,Z]]}\}(f).\end{aligned}$$

Similarly, changing the role of X and Y we get,

$$\mathcal{L}_Y\mathcal{L}_X i_Z(f)=\{i_Z\mathcal{L}_Y\mathcal{L}_X+i_{[Y,Z]}\mathcal{L}_X+i_{[X,Z]}\mathcal{L}_Y+i_{[Y,[X,Z]]}\}(f).$$

Therefore,

$$(\mathcal{L}_X\mathcal{L}_Y-\mathcal{L}_Y\mathcal{L}_X)i_Z(f)=i_Z(\mathcal{L}_X\mathcal{L}_Y-\mathcal{L}_Y\mathcal{L}_X)(f)+(i_{[X,[Y,Z]]}-i_{[Y,[X,Z]]})(f).$$

Thus,

$$\begin{aligned}&\{\mathcal{L}_{[X,Y]}-(\mathcal{L}_X\mathcal{L}_Y-\mathcal{L}_Y\mathcal{L}_X)\}i_Z(f)\\ &=i_Z\{\mathcal{L}_{[X,Y]}-(\mathcal{L}_X\mathcal{L}_Y-\mathcal{L}_Y\mathcal{L}_X)\}(f)+(i_{[[X,Y],Z]}+i_{[[Y,Z],X]}+i_{[[Z,X],Y]})(f)\end{aligned}$$

$$= i_Z\{\mathcal{L}_{[X,Y]} - (\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X)\}(f).$$

Note that by Jacobi identity, the last term is 0.

Now observe that by (i),

$$i_Z\{\mathcal{L}_{[X,Y]} - (\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X)\}(f) = 0.$$

Since $Z \in \Gamma A$ is arbitrary, we must have,

$$(\mathcal{L}_{[X,Y]} - (\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X))(f) = 0.$$

This completes the proof.

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