LIE GROUPOIDS-II

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1. Frame Groupoid

Let $E \xrightarrow{q} M$ be a n-dimensional vector bundle. Consider the set

$$\Phi(E) = \{E_x \xrightarrow{\xi} E_y : x, y \in M, \xi \text{ a linear isomorphism}\}.$$

Define

$$\alpha : \Phi(E) \longrightarrow M \quad \text{by} \quad (E_x \xrightarrow{\xi} E_y) \mapsto x$$

$$\beta : \Phi(E) \longrightarrow M \quad \text{by} \quad (E_x \xrightarrow{\xi} E_y) \mapsto y$$

$$\epsilon : M \longrightarrow \Phi(E) \quad \text{by} \quad x \mapsto (E_x \xrightarrow{\text{id}} E_x)$$

and a partial multiplication $m : \Phi(E) \times_M \Phi(E) \longrightarrow \Phi(E)$ by

$$m : (E_x \xrightarrow{\xi} E_y, E_z \xrightarrow{\eta} E_x) \mapsto (E_z \xrightarrow{\xi \circ \eta} E_y).$$

Define $\iota : \Phi(E) \longrightarrow \Phi(E)$ by $(E_x \xrightarrow{\xi} E_y) \mapsto (E_y \xrightarrow{\xi^{-1}} E_x)$.
Then $\Phi(E) \rightrightarrows M$ is a set groupoid.
We describe a smooth structure on $\Phi(E)$ as follows:
Assume $\mathbb{R}^n$ has the standard basis. Let $\{\psi_i : U_i \times \mathbb{R}^n \longrightarrow q^{-1}(U_i)\}$ be an atlas for E. Then for all $x \in U_i$,

$$\psi_{i,x} : \mathbb{R}^n \longrightarrow q^{-1}(x) = E_x \qquad \text{defined by} \qquad t \mapsto \psi_i(x, t)$$

is a linear isomorphism.
For any i and j, define

$$\overline{\psi}{}_i^j : U_j \times GL_n(\mathbb{R}) \times U_i \longrightarrow \Phi(E)_{U_i}^{U_j}$$

by

$$(y, A, x) \mapsto \psi_{j,y} \circ A \circ \psi_{i,x}^{-1},$$

where $y \in U_j$, $x \in U_i$ and $A \in GL_n(\mathbb{R})$.
Then $\overline{\psi}{}_i^j$ is bijective with

$$(\overline{\psi}{}_i^j)^{-1}(\xi : E_x \longrightarrow E_y) = (y, \, \psi_{j,y}^{-1} \circ \xi \circ \psi_{i,x} \, , \, x)$$

Also for any $U_i \cap U_k = U_{ik} \neq \emptyset$ and $U_j \cap U_l = U_{jl} \neq \emptyset$ the map

$$U_{jl} \times GL_n(\mathbb{R}) \times U_{ik} \xrightarrow{\overline{\psi}{}_i^j} \Phi(E)_{U_{ik}}^{U_{jl}} \xrightarrow{(\overline{\psi}{}_k^l)^{-1}} U_{jl} \times GL_n(\mathbb{R}) \times U_{ik}$$

is given by

$$\begin{array}{rcl} (y, A, x) \;\mapsto\; \psi_{j,y} \circ A \circ \psi_{i,x}^{-1} & \mapsto & (y, \, \psi_{l,y}^{-1} \circ (\psi_{j,y} \circ A \circ \psi_{i,x}^{-1}) \circ \psi_{k,x} \, , \, x) \\ & = & (y, \, (\psi_{l,y}^{-1} \circ \psi_{j,y}) \circ A \circ (\psi_{i,x}^{-1} \circ \psi_{k,x}) \, , \, x). \end{array}$$

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Therefore $(\overline{\psi}{}^{l}_{k})^{-1} \circ \overline{\psi}{}^{j}_{i}$ is a diffeomorphism.
So, there is a well defined smooth structure on $\Phi(E)$ for which each $\overline{\psi}{}^{j}_{i}$ is a diffeomorphism. Identifying $\Phi(E)^{U_j}_{U_i}$ with $U_j \times GL_n(\mathbb{R}) \times U_i$, we get that α and β are described locally as

$$\mathrm{pr}_3 : U_j \times GL_n(\mathbb{R}) \times U_i \longrightarrow U_i \quad \text{ and } \quad \mathrm{pr}_1 : U_j \times GL_n(\mathbb{R}) \times U_i \longrightarrow U_j$$

respectively. So α and β are surjective submersions.
Also the local description of ϵ is $U_i \longrightarrow U_i \times GL_n(\mathbb{R}) \times U_i$, $x \mapsto (x, I, x)$ where I is the identity matrix of order n. Hence ϵ is smooth.
Finally the partial multiplication is defined locally as

$$(U_k \times GL_n(\mathbb{R}) \times U_j) \times_{U_j} (U_j \times GL_n(\mathbb{R}) \times U_i) \longrightarrow U_k \times GL_n(\mathbb{R}) \times U_i$$
$$((z, B, y), (y, A, x)) \mapsto (z, BA, x).$$

Hence the partial multiplication is smooth.
Thus $\Phi(E) \rightrightarrows M$ is a Lie groupoid. It is called the frame groupoid of the vector bundle $E \xrightarrow{q} M$.

2. Gauge Groupoid

Definition 2.1. Let P and M be smooth manifolds and G be a Lie group with a free right action on P. Let $\pi : P \longrightarrow M$ be a surjective submersion so that the orbits of the G-action coincide with the fibres of π and M is covered by the domains of local sections $\sigma : U \longrightarrow P$ of π where U is open in M. Then $P(M, G, \pi)$ is a *principal G-bundle.*

Let $P(M, G, \pi)$ be a principal G-bundle. Let $\Omega := \dfrac{P \times P}{G}$ be the set of orbits of the diagonal action of G on $P \times P$. Denote the orbit through $(u, v) \in P \times P$ by $\langle u, v\rangle$. Then Ω is a quotient manifold and the quotient map $p : P \times P \longrightarrow \Omega$ is a surjective submersion. We define a groupoid structure $\Omega \rightrightarrows M$ as follows:

$$\alpha : \langle u, v\rangle \mapsto \pi(v) \qquad \beta : \langle u, v\rangle \mapsto \pi(u) \qquad \langle u, v\rangle \in \Omega.$$

Define $\epsilon : M \longrightarrow \Omega$ by $\epsilon(x) = \langle u, u\rangle$, where $u \in \pi^{-1}(x)$.
To define the partial multiplication m take $\langle u, v\rangle, \langle v', w\rangle \in \Omega$ so that $\pi(v) = \pi(v')$. Then there is a $g \in G$ so that $v' = vg$. Hence $\langle v', w\rangle = \langle vg, w\rangle = \langle v, wg^{-1}\rangle$. Define

$$m(\langle u, v\rangle, \langle vg, w\rangle) = \langle u, wg^{-1}\rangle.$$

Finally define $\iota(\langle u, v\rangle) = \langle v, u\rangle$.
Note that the following diagram is commutative.

$$\begin{tikzcd}
P \times P \arrow[r, "p"] \arrow[d, "pr_2"] & \Omega \arrow[d, "\alpha"] \\
P \arrow[r, "\pi"] & M
\end{tikzcd}$$

Since π, pr_2 and p are surjective submersions, it follows that α is a surjective submersion.
Similarly, the following diagram is commutative.

$$\begin{tikzcd}
P \times P \arrow[r, "p"] \arrow[d, "pr_1"] & \Omega \arrow[d, "\beta"] \\
P \arrow[r, "\pi"] & M
\end{tikzcd}$$

Since π, pr_1 and p are surjective submersions, it follows that β is a surjective submersion.
Let $\Delta : P \longrightarrow P \times P$ be defined by $\Delta(u) = (u, u)$. Let (U, φ) be a chart on M and $\sigma : U \longrightarrow P$ be a local section of π. Then $\epsilon|_U$ is the composition $p \circ \Delta \circ \sigma$ and hence $\epsilon|_U$ is smooth. Thus ϵ is smooth.
Let $P \times_\pi P = \{(u, v) : \pi(u) = \pi(v)\} = \{(ug, u) : u \in P, g \in G\}$. Define $\delta : P \times_\pi P \longrightarrow G$ by $\delta(ug, u) = g$. Then δ is smooth. Now

$$p \times p : (P \times P) \times (P \times P) \longrightarrow \Omega \times \Omega$$

is a surjective submersion and $(p \times p)^{-1}(\Omega \times_M \Omega) = P \times (P \times_\pi P) \times P$. So

$$p \times p : P \times (P \times_\pi P) \times P \longrightarrow \Omega \times_M \Omega$$

is a surjective submersion. Next define

$$F : P \times (P \times_\pi P) \times P \longrightarrow P \times P$$

to be the composition

$$P \times (P \times_\pi P) \times P \longrightarrow P \times G \times P \longrightarrow P \times P \times G \longrightarrow P \times P$$

given by

$$(u_4, u_3, u_2, u_1) \mapsto (u_4, \delta(u_3, u_2), u_1) \mapsto (u_4, u_1, \delta(u_3, u_2)) \mapsto (u_4, u_1\delta(u_3, u_2)).$$

Since F is the composition of smooth maps, it is smooth.
Also the following diagram is commutative.

$$\begin{tikzcd}
P \times (P \times_\pi P) \times P \arrow[r, "F"] \arrow[d, "p \times p"] & P \times P \arrow[d, "p"] \\
\Omega \times_M \Omega \arrow[r, "m"] & \Omega
\end{tikzcd}$$

Thus m is smooth.
$\Omega \rightrightarrows M$ is called the gauge groupoid of the principal bundle $P(M, G, \pi)$.

3. Morphisms and Subgroupoids

Definition 3.1. Let $G_1 \rightrightarrows M_1$ and $G_2 \rightrightarrows M_2$ be two Lie groupoids with structure maps $\alpha_1, \beta_1, m_1, \epsilon_1, \iota_1$ and $\alpha_2, \beta_2, m_2, \epsilon_2, \iota_2$ respectively. A *morphism* from $G_1 \rightrightarrows M_1$ to $G_2 \rightrightarrows M_2$ is a pair of smooth maps $F : G_1 \longrightarrow G_2$, $f : M_1 \longrightarrow M_2$ so that $\alpha_2 \circ F = f \circ \alpha_1$, $\beta_2 \circ F = f \circ \beta_1$ and for any $(g, h) \in G_1 \times_{M_1} G_1$, $F(gh) = F(g)F(h)$.
Note that $\alpha_2(F(g)) = f(\alpha_1(g)) = f(\beta_1(h)) = \beta_2(F(h))$. So $(F(g), F(h)) \in G_2 \times_{M_2} G_2$.

Proposition 3.2. *Let (F, f) be a Lie groupoid morphism between the Lie groupoids $G_1 \rightrightarrows M_1$ and $G_2 \rightrightarrows M_2$. Then $F \circ \epsilon_1 = \epsilon_2 \circ f$ and $\iota_2 \circ F = F \circ \iota_1$.*

Definition 3.3. Let $G \rightrightarrows M$ be a Lie groupoid. A Lie groupoid $H \rightrightarrows N$ is a *subgroupoid* of $G \rightrightarrows M$ if there is a Lie groupoid morphism (i, i_0) from $H \rightrightarrows N$ to $G \rightrightarrows M$ where i and i_0 are both injective immersions.
If $N = M$ and $i_0 = \mathrm{id}_M$, then the subgroupoid $H \rightrightarrows M$ is called a *wide subgroupoid* of $G \rightrightarrows M$.

4. LOCALLY TRIVIAL LIE GROUPOIDS

Definition 4.1. A set groupoid $G \rightrightarrows M$ is *transitive* if $(\beta, \alpha) : G \longrightarrow M \times M$ is surjective. A Lie groupoid $G \rightrightarrows M$ is *locally trivial* if $(\beta, \alpha) : G \longrightarrow M \times M$ is a surjective submersion.

Remark 4.2. Let M be a smooth manifold. Let G be a Lie group. Let be The trivial groupoid on M with group G, $M \times G \times M \rightrightarrows M$ is clearly a locally trivial groupoid.

Proposition 4.3. *Let $G \rightrightarrows M$ be a Lie groupoid. The following statements are equivalent:*

- (i) *G is locally trivial.*
- (ii) *The map $\beta_x : G_x \longrightarrow M$ defined by $\beta_x(g) = \beta(g)$ is a a surjective submersion for all $x \in M$.*
- (iii) *The map $\delta_x : G_x \times G_x \longrightarrow G$ defined by $\delta_x(h, g) = hg^{-1}$ is a a surjective submersion for one and hence for all $x \in M$.*

Proof: *(i)* ⇒ *(ii)*: Let $x \in M$. Then $M \times \{x\}$ is a closed embedded submanifold of $M \times M$ and $(\beta, \alpha)^{-1}(M \times \{x\}) = G_x$. So the restriction of (β, α) to G_x i.e. the map β_x is a surjective submersion.

(ii) ⇒ *(i)*: Let $x \in M$. Then $(\beta, \alpha) \circ \delta_x = \beta_x \times \beta_x$. Now β_x is a surjective submersion implies that $\beta_x \times \beta_x$ is a surjective submersion and hence (β, α) is a surjective submersion.

(ii) ⇒ *(iii)*: Form the pullback

$$\begin{tikzcd}
G_x \times_\beta G \arrow[r, "\theta"] \arrow[d, "\mathrm{pr}_1"] & G \arrow[d, "\beta"] \\
G_x \arrow[r, "\beta_x"] & M
\end{tikzcd}$$

Since β_x is a surjective submersion, so is θ.
Define

$$\psi : G_x \times G_x \longrightarrow G_x \times_\beta G \qquad \text{by} \qquad (h, g) \mapsto (h, hg^{-1}).$$

Then ψ is a diffeomorphism with inverse

$$\psi^{-1} : G_x \times_\beta G \longrightarrow G_x \times G_x \qquad \text{by} \qquad (h, g) \mapsto (h, g^{-1}h).$$

Also $\delta_x = \theta \circ \psi$. Hence δ_x is a surjective submersion.

(iii) ⇒ *(ii)*: The following diagram commutes:

$$\begin{tikzcd}
G_x \times G_x \arrow[r, "\delta_x"] \arrow[d, "\mathrm{pr}_1"] & G \arrow[d, "\beta"] \\
G_x \arrow[r, "\beta_x"] & M
\end{tikzcd}$$

Now δ_x is a surjective submersion implies that $\beta \circ \delta_x$ is a surjective submersion i.e. $\beta_x \circ \mathrm{pr}_1$ is a surjective submersion which in turn implies that β_x is a surjective submersion. $\square$

Remark 4.4. Let $G \rightrightarrows M$ be a locally trivial Lie groupoid. Fix $x \in M$. For any $y \in M$ there is an open subset U of M such that $y \in U$ and a local section $\sigma : U \longrightarrow G_x$ of β_x. Thus there is an open cover $\{U_i\}$ of M by the domains of the local sections $\sigma_i : U_i \longrightarrow G_x$ of β_x. We get base preserving isomorphisms

$$\textstyle\sum_i : U_i \times G_x^x \times U_i \longrightarrow G_{U_i}^{U_i} \qquad (y, g, z) \mapsto \sigma_i(y) g \sigma_i(z)^{-1}$$

from a trivial groupoid to the restriction of G to $G_{U_i}^{U_i}$. Thus G is "locally isomorphic to trivial groupoids".

Remark 4.5. Let $P(M, G, \pi)$ be a principal G-bundle. Let $\Omega \rightrightarrows M$ be the corresponding gauge groupoid. Then the following diagram is commutative:

$$\begin{tikzcd}
P \times P \arrow[d, "p"] \arrow[dr, "\pi \times \pi"] & \\
\Omega \arrow[r, "(\beta, \alpha)"'] & M \times M
\end{tikzcd}$$

Since π is a surjective submersion, $\pi \times \pi = (\beta, \alpha) \circ p$ is a surjective submersion which implies that (β, α) is a surjective submersion. Hence $\Omega \rightrightarrows M$ is a locally trivial groupoid.

5. EXAMPLES

1. Let M be a smooth connected manifold. Let $\widetilde{M}$ be the universal cover of M. Then the fundamental groupoid $\Pi(M)$ of M is the gauge groupoid of the principal bundle $\widetilde{M}(M, \pi(M))$ where $\pi(M)$ is the fundamental group of M. Hence $\Pi(M)$ is locally trivial.
2. Let G be a Lie group. Let M be a smooth G-manifold where G has a left action on M. The corresponding action groupoid $G \ltimes M$ is transitive if and only if the G-action is transitive.
3. Let $E \xrightarrow{q} M$ be a vector bundle. The frame groupoid of E can be identified with the gauge groupoid of any frame bundle of E. Hence the frame groupoid of E is locally trivial.

REFERENCES

[1] Mackenzie K. C. H. *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press

6. EXERCISES

(1) Let M be a smooth manifold and G be a Lie group. Determine the vertex groups and orbits for
  (i) the base groupoid $M \rightrightarrows M$.
  (ii) the trivial groupoid $M \times G \times M \rightrightarrows M$.
  (iii) the action groupoid $G \ltimes M$, where M is a left G-space.

(2) Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Consider the Lie groupoid $T * G \rightrightarrows \mathfrak{g}^*$. For any $(g, \omega), (h, \theta) \in T^*G$ such that $\alpha(g, \omega) = \beta(h, \theta)$ and $X \in T_gG$, $Y \in T_hG$ show that

$$\langle (g, \omega) \cdot (h, \theta), X \cdot Y \rangle = \omega(X) + \theta(Y).$$

Also verify that

$$\alpha[(g, \omega) \cdot (h, \theta)] = \alpha(h, \theta) \qquad \text{and} \qquad \beta[(g, \omega) \cdot (h, \theta)] = \beta(g, \omega)$$

(3) Let $G \rightrightarrows M$ be a Lie groupoid. Then verify that $(\beta, \alpha) : G \longrightarrow M \times M$ is a base preserving morphism from $G \rightrightarrows M$ to the pair groupoid $M \times M \rightrightarrows M$.

(4) Let $G \rightrightarrows M$ be a Lie groupoid. Define $G \times_\alpha G = \{(h, g) : \alpha(h) = \alpha(g)\}$. Let the structure maps of $G \times_\alpha G \rightrightarrows G$ be the restriction of the structure maps of the pair groupoid $G \times G \rightrightarrows G$. Then verify that $G \times_\alpha G \rightrightarrows G$ is a wide subgroupoid of the pair groupoid $G \times G \rightrightarrows G$. Define $\delta : G \times_\alpha G \longrightarrow G$ by $\delta(h, g) = hg^{-1}$. Then verify that (δ, β) is a Lie groupoid morphism from $G \times_\alpha G \rightrightarrows G$ to $G \rightrightarrows M$. δ is called the *division map.*

(5) Let M_1 and M_2 be two smooth manifolds and G_1 and G_2 be two Lie groups.
Let $f : M_1 \longrightarrow M_2$ and $\theta : M_1 \longrightarrow G_2$ be smooth maps and $s : G_1 \longrightarrow G_2$ be a Lie group homomorphism. Define

$$F : M_1 \times G_1 \times M_1 \longrightarrow M_2 \times G_2 \times M_2$$

by

$$F(y, g, x) = (f(y),\ \theta(y)s(g)\theta(x)^{-1},\ f(x)).$$

Then verify that (F, f) is a morphism between the trivial groupoids $M_1 \times G_1 \times M_1 \rightrightarrows M_1$ and $M_2 \times G_2 \times M_2 \rightrightarrows M_2$.

(6) Let G_1 and G_2 be two Lie groups. Let M_1 be a left G_1-manifold and M_2 be a left G_2-manifold. Let $\psi : G_1 \longrightarrow G_2$ be a Lie group homomorphism. Let $f : M_1 \longrightarrow M_2$ be an equivariant smooth map. Then verify that $(\psi \times f, f)$ is a Lie groupoid morphism between the action groupoids $G_1 \ltimes M_1 \rightrightarrows M_1$ and $G_2 \ltimes M_2 \rightrightarrows M_2$.

(7) Let $G \rightrightarrows M$ be a Lie groupoid. Let $U \subseteq M$ be an open submanifold. Show that $G_U^U \rightrightarrows U$ is a Lie subgroupoid of $G \rightrightarrows M$. This subgroupoid is called the *restriction* of $G \rightrightarrows M$ to U.