LIE GROUPOIDS-III

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1. Bisections

Definition 1.1. A *bisection* of a Lie groupoid $G \rightrightarrows M$ is a smooth map $\sigma : M \longrightarrow G$ so that $\alpha \circ \sigma = \mathrm{id}_M$ and $\beta \circ \sigma$ is a diffeomorphism.
The set of all bisections of $G \rightrightarrows M$ will be denoted by $\mathcal{B}(G)$.

Proposition 1.2. *Let $G \rightrightarrows M$ be a Lie groupoid. Then $\mathcal{B}(G)$ is a group with respect to the operation*

$$(\sigma\tau)(x) = \sigma[(\beta \circ \tau)(x)]\tau(x) \qquad \sigma, \tau \in \mathcal{B}(G),\ x \in M.$$

with ϵ as the identity element and for any $\sigma \in \mathcal{B}(G)$, the inverse of σ being defined by

$$\sigma^{-1}(x) = \iota[\sigma\{(\beta \circ \sigma)^{-1}(x)\}] \qquad x \in M.$$

Proposition 1.3. *Let $\sigma \in \mathcal{B}(G)$. Define $L_\sigma : G \longrightarrow G$ by $L_\sigma(g) = \sigma(\beta(g)) \cdot g$ for all $g \in G$. Then L_σ is a diffeomorphism. Moreover, for any $x \in M$, $L_\sigma(g) = \sigma(x) \cdot g$ for all $g \in G^x$. The map L_σ is called the* left translation on G associated with σ.

Proposition 1.4. *Let $\sigma \in \mathcal{B}(G)$. Define $R_\sigma : G \longrightarrow G$ by $R_\sigma(g) = g \cdot \sigma[(\beta \circ \sigma)^{-1}(\alpha(g))]$ for all $g \in G$. Then R_σ is a diffeomorphism. Moreover, for any $x \in M$, $R_\sigma(g) = g \cdot \sigma[(\beta \circ \sigma)^{-1}(x)]$ for all $g \in G_x$. The map R_σ is called the* right translation on G associated with σ.

Definition 1.5. Let $G \rightrightarrows M$ be a Lie groupoid. Let U be an open subset of M. A *local bisection* of G on U is a map $\sigma : U \longrightarrow G$ such that $\alpha \circ \sigma = \mathrm{id}_U$ and $\beta \circ \sigma$ is a diffeomorphism from U to the open set $\beta \circ \sigma(U)$ in M.
Let $\sigma : U \longrightarrow G$ be a local bisection of G on U. Let $\beta \circ \sigma(U) = V$.
The *local left translation induced by* σ is the diffeomorphism

$$L_\sigma : G^U \longrightarrow G^V \quad \text{defined by} \quad L_\sigma(g) = \sigma(\beta(g)) \cdot g, \quad g \in G^U.$$

The *local right translation induced by* σ is the diffeomorphism

$$R_\sigma : G_V \longrightarrow G_U \quad \text{defined by} \quad R_\sigma(g) = g \cdot \sigma[(\beta \circ \sigma)^{-1}(\alpha(g))], \quad g \in G_V.$$

Remark 1.6. Let $G \rightrightarrows M$ be a Lie groupoid. Let $g \in G$ and let $\alpha(g) = x$, $\beta(g) = y$. Then $\ker T(\alpha)_g = T_g G_x$ and $\ker T(\beta)_g = T_g G^y$.

Proposition 1.7. *Let $G \rightrightarrows M$ be a Lie groupoid. Let $g \in G$. Then there is a local bisection σ of $G \rightrightarrows M$ with $\sigma(\alpha(g)) = g$.*

Proof: Let $\alpha(g) = x$ and $\beta(g) = y$. Let V be a subspace of $T_g G$ such that

$$T_g G_x \oplus V = T_g G^y \oplus V = T_g G.$$

Since α is a surjective submersion, there is a local section $\sigma : U \longrightarrow G$ so that $\sigma(x) = g$. Use a local α-preserving diffeomorphism in G to arrange that image of $T(\sigma)_x$ is V. Then $T(\beta \circ \sigma)_x$ is an isomorphism. Hence by shrinking U if necessary, we get that $\beta \circ \sigma$ is a diffeomorphism onto its image. □

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Corollary 1.8. *Let $G \rightrightarrows M$ be a Lie groupoid. Let $x \in M$. Then $\beta_x : G_x \longrightarrow M$ is of constant rank.*

Proof: Let $g, h \in G_x$. Let $k = gh^{-1}$. Then there is a local bisection σ of $G \rightrightarrows M$ over an open subset U of M with $\sigma(\alpha(k)) = k$. Let $\beta \circ \sigma(U) = V$. Note that $h \in G_x^U$. Let the local left translation induced by σ be $L_\sigma : G^U \longrightarrow G^V$ defined by $L_\sigma(j) = \sigma(\beta(j)) \cdot j$, for $j \in G^U$. Then $L_\sigma : G_x^U \longrightarrow G_x^V$. Also

$$L_\sigma(h) = \sigma(\beta(h)) \cdot h = \sigma(\alpha(h^{-1})) \cdot h = \sigma(\alpha(gh^{-1})) \cdot h = \sigma(\alpha(k)) \cdot h = kh = g.$$

So $g \in G_x^V$. Again for any $j \in G_x^U$,

$$\beta_x \circ L_\sigma(j) = \beta_x(\sigma(\beta(j)) \cdot j) = \beta(\sigma(\beta(j))) = (\beta \circ \sigma) \circ \beta_x(j)$$

i.e the following diagram is commutative:

$$\begin{array}{ccc} G_x^U & \xrightarrow{\;L_\sigma\;} & G_x^V \\ \big\downarrow \beta_x & & \big\downarrow \beta_x \\ U & \xrightarrow{\;\beta \circ \sigma\;} & V \end{array}$$

Since L_σ and $\beta \circ \sigma$ are diffeomorphisms, the result follows. □

Corollary 1.9. *Let $G \rightrightarrows M$ be a Lie groupoid. Let $x, y \in M$. Then G_x^y is a closed embedded submanifold of G. In particular, the vertex groups G_x^x are Lie groups.*

Proof: We have $G_x^y = \beta_x^{-1}(y)$ and β_x has constant rank. Hence G_x^y is a closed embedded submanifold of G_x and hence of G. It follows that $G_x^x \times G_x^x$ is a closed embedded submanifold of $G \times_M G$. Hence the restriction of the partial multiplication m to $G_x^x \times G_x^x$ is smooth. □

Theorem 1.10. *Let $G \rightrightarrows M$ be a Lie groupoid. Then for each $x \in M$, the orbit $\mathcal{O}_x = \beta(G_x)$ is a submanifold of M.*

Proof: The restriction of the groupoid multiplication to $G_x \times G_x^x \longrightarrow G_x$ is a smooth action of a Lie group on a manifold.
Let K, L be compact subsets of G_x. Then $K \times L$ is compact and $K \times_{\beta_x} L = (K \times L) \cap (G_x \times_{\beta_x} G_x)$ is a closed subset of $K \times L$. Consider the map $\Phi : G_x \times_{\beta_x} G_x \longrightarrow G_x^x$ defined by $(h, g) \mapsto h^{-1}g$. If $(h, g) \in K \times_{\beta_x} L$ then $hh^{-1}g = g \in Kh^{-1}g \cap L$. Also if $g \in G_x^x$ is such that $Kg \cap L \neq \emptyset$, then there is a $k \in K$ and $l \in L$ so that $kg = l$. This implies that $\beta(k) = \beta(kg) = \beta(l)$ and $g = k^{-1}l$. Hence

$$\Phi(K \times_{\beta_x} L) = \{g \in G_x^x : Kg \cap L \neq \emptyset\}.$$

So $\{g \in G_x^x : Kg \cap L \neq \emptyset\}$ is compact.
Thus the above action is proper. It is also free. Hence $\{(gh, g) : g \in G_x, h \in G_x^x\}$ is a closed embedded submanifold of $G_x \times G_x$. So there is a quotient manifold structure on the orbit space G_x/G_x^x. Define $i : G_x/G_x^x \longrightarrow M$ by $i(gG_x^x) = \beta_x(g)$ for all $g \in G_x$. Then i is smooth and injective. Also $G_x \longrightarrow G_x/G_x^x$ is a surjective submersion. So for all $g \in G_x$, $\text{rank}_{gG_x^x}(i) = \text{rank}_g(\beta_x)$. Thus i is of constant rank. Hence i is an immersion. Also Image $i = \mathcal{O}_x$. □

Theorem 1.11. *Let $G \rightrightarrows M$ be a Lie groupoid. Let $x \in M$. Then there is a manifold structure on $G_{\mathcal{O}_x}^{\mathcal{O}_x}$ with respect to which it is a submanifold of G and a Lie groupoid on $\mathcal{O}_x$. Further $G_{\mathcal{O}_x}^{\mathcal{O}_x} \rightrightarrows \mathcal{O}_x$ is locally trivial.*

Proof: By the above result, $G_x(\mathcal{O}_x, G_x^x, \beta_x)$ is a principal bundle. Let $\Omega \rightrightarrows \mathcal{O}_x$ be the gauge groupoid of this principal bundle.
The set mapping $\delta_x : \Omega \longrightarrow G_{\mathcal{O}_x}^{\mathcal{O}_x}$, $\delta_x(\langle h, g\rangle) = hg^{-1}$ is bijective.
Again β_x is of constant rank implies that δ_x is of constant rank. Form the pullback

$$\begin{array}{ccc} G_x \times_\beta G_{\mathcal{O}_x}^{\mathcal{O}_x} & \xrightarrow{\theta} & G_{\mathcal{O}_x}^{\mathcal{O}_x} \\ \downarrow \mathrm{pr}_1 & & \downarrow \beta \\ G_x & \xrightarrow{\beta_x} & \mathcal{O}_x \end{array}$$

Since β_x is of constant rank, so is θ.
Define

$$\psi : \Omega \longrightarrow G_x \times_\beta G_{\mathcal{O}_x}^{\mathcal{O}_x} \qquad \text{by} \qquad \langle h, g\rangle \mapsto (h, hg^{-1})$$

Then ψ is a diffeomorphism with inverse

$$\psi^{-1} : G_x \times_\beta G_{\mathcal{O}_x}^{\mathcal{O}_x} \longrightarrow \Omega \qquad \text{by} \qquad (h, g) \mapsto \langle h, g^{-1}h\rangle.$$

Also $\delta_x = \theta \circ \psi$. Hence δ_x is of constant rank.
Therefore δ_x is an immersion. So, $G_{\mathcal{O}_x}^{\mathcal{O}_x}$ is a submanifold of G. Hence etc. □

Corollary 1.12. *Let $G \rightrightarrows M$ be a transitive Lie groupoid. Then $G \rightrightarrows M$ is a locally trivial Lie groupoid.*

Proof: Let $x \in M$. Since $G \rightrightarrows M$ is transitive, for any $y \in M$, there is a $g \in G$ such that $\alpha(g) = x$ and $\beta(g) = y$. Hence $\mathcal{O}_x = M$. So, $G_{\mathcal{O}_x}^{\mathcal{O}_x} = G$. □

Remark 1.13. Let $G \rightrightarrows M$ be a Lie groupoid. Let $\sigma \in \mathcal{B}(G)$. Define $\sigma_2 = \sigma \circ (\beta \circ \sigma)^{-1}$. Then $\beta \circ \sigma_2 = \mathrm{id}_M$ and $\sigma(M) = \sigma_2(M)$. Conversely, if $\sigma_1, \sigma_2 : M \longrightarrow G$ be smooth maps such that $\alpha \circ \sigma_1 = \mathrm{id}_M$, $\beta \circ \sigma_2 = \mathrm{id}_M$ and $\sigma_1(M) = \sigma_2(M)$ then $\sigma_1 \in \mathcal{B}(G)$.

We saw earlier that the gauge groupoid of a principal bundle is locally trivial. We also have

Remark 1.14. Let $\Omega \rightrightarrows M$ be a locally trivial groupoid. Let $x \in M$. Then the action of Ω_x^x on Ω_x

$$\Omega_x \times \Omega_x^x \longrightarrow \Omega_x \quad \text{defined by} \quad (g, h) \mapsto gh$$

is smooth and free. Let $g \in \Omega_x$ and Θ_g be the orbit of g under this action. Then $\Theta_g = \beta_x^{-1}(\beta(g))$.
Therefore $\Omega_x(M, \Omega_x^x, \beta_x)$ is a principal Ω_x^x-bundle called the *vertex bundle* at x.

Proposition 1.15. *Let $P(M, G, \pi)$ be a principal G-bundle. Let $\Omega \rightrightarrows M$ be the corresponding gauge groupoid. Choose $u \in P$ and $\pi(u) = x$. Define*

$$\Psi_x : P \longrightarrow \Omega_x \quad \textit{by} \quad v \mapsto \langle v, u\rangle$$
$$\psi_x : G \longrightarrow \Omega_x^x \quad \textit{by} \quad g \mapsto \langle ug, u\rangle$$

Then Ψ is a diffeomorphism and ψ is an isomorphism of Lie groups. Together they form an isomorphism of principal bundles over M.
Further, let $P'(M', G', \pi')$ be another principal G'-bundle, $\Omega' \rightrightarrows M'$ be the corresponding gauge groupoid and $F(f, \varphi) : P'(M, G) \longrightarrow P(M', G')$ be a morphism of principal bundles.

Let $\widetilde{F}:\Omega\longrightarrow\Omega'$ *be the induced morphism of Lie groupoids. Let* $F(u)=u'$ *and* $\pi'(u')=x'$. *Define*

$$\Psi'_x : P'\longrightarrow\Omega'_x \qquad \psi'_x : G'\longrightarrow(\Omega')^x_x$$

as above. Then the diagram

$$\begin{array}{ccc} P & \xrightarrow{\quad F \quad} & P' \\ \downarrow{\scriptstyle \Psi_x} & & \downarrow{\scriptstyle \Psi'_{x'}} \\ \Omega_x & \xrightarrow{\quad \widetilde{F}_x \quad} & \Omega'_{x'} \end{array}$$

is commutative.

Proof: First note that Ψ_x is the composition $P\longrightarrow P\times P\stackrel{p}{\longrightarrow}\Omega_x$ given by $v\mapsto(v,u)\mapsto\langle v,u\rangle$. Hence Ψ_x is smooth. Now consider the commutative diagram

$$\begin{array}{ccc} P\times\pi^{-1}(x) & & \\ \downarrow{\scriptstyle p} & \searrow{\scriptstyle \phi} & \\ \Omega_x & \xrightarrow{\quad \theta \quad} & P \end{array}$$

where $\theta\langle v,ug\rangle=vg^{-1}$ and $\phi(v,ug)=vg^{-1}$. Now the map $\delta:P\times_\pi P\longrightarrow G$ defined by $\delta(vg,v)=g$ is smooth and ϕ is the composition

$$P\times\pi^{-1}(x)\longrightarrow P\times\pi^{-1}(x)\times\{u\}\longrightarrow P\times G\longrightarrow P\times G\longrightarrow P$$

defined by

$$(v,ug)\mapsto(v,ug,u)\mapsto(v,\delta(ug,u))=(v,g)\mapsto(v,g^{-1})=vg^{-1}.$$

So ϕ is smooth. Since p is smooth, it follows that θ is a smooth map.
Also, for any $v\in P$, $g\in G$, we have

$$\theta\Psi_x(v)=\theta\langle v,u\rangle=v \qquad \Psi_x\theta\langle v,ug\rangle=\Psi_x(vg^{-1})=\langle vg^{-1},u\rangle=\langle v,ug\rangle.$$

Hence $\theta=\Psi_x^{-1}$. So, Ψ_x is a diffeomorphism.
Next note that $\psi_x:G\longrightarrow\Omega^x_x$ is clearly a group isomorphism.
Also ψ_x is the composition $G\longrightarrow P\times\{u\}\longrightarrow\Omega^x_x$ defined by $g\mapsto(ug,u)\mapsto\langle ug,u\rangle$ and hence ψ_x is smooth. Also the following is a commutative diagram:

$$\begin{array}{ccc} \pi^{-1}(x)\times\{u\} & & \\ \cong\downarrow{\scriptstyle p} & \searrow{\scriptstyle \delta} & \\ \Omega^x_x & \xrightarrow{\quad \psi_x^{-1} \quad} & G \end{array}$$

So, ψ_x^{-1} is smooth. Hence ψ_x is a Lie group isomorphism.

Again $\beta_x \circ \Psi_x = \pi$ and Ψ_x is equivariant. Therefore (Ψ_x, ψ_x) is an isomorphism of principal bundles over M.
Finally the induced map $\widetilde{F} : \Omega \longrightarrow \Omega'$ is defined by $\langle v, w\rangle \mapsto \langle F(v), F(w)\rangle$. Hence the second part follows. □

Proposition 1.16. *Let $G \rightrightarrows M$ be a locally trivial groupoid. Let $x \in M$. Let $\Omega \rightrightarrows M$ be the gauge groupoid of the principal G_x^x-bundle $G_x(M, G_x^x, \beta_x)$. Define*

$$\chi_x : \Omega \longrightarrow G \qquad \langle h, g\rangle \mapsto hg^{-1} \qquad h, g \in G_x.$$

Then χ_x is a base preserving isomorphism of groupoids.
Further, let (F, f) be a morphism from $G \rightrightarrows M$ to a locally trivial groupoid $G' \rightrightarrows M'$. Let $f(x) = x'$. Let $\Omega' \rightrightarrows M'$ be the gauge groupoid of the principal $(G')_x^x$-bundle $G'_x(M, (G')_x^x, \beta'_x)$ and $\chi'_x : \Omega' \longrightarrow G'$ be the corresponding isomorphism of groupoids. Let $(\widetilde{F}_x, f)$ be the associated groupoid morphism from $\Omega \rightrightarrows M$ to $\Omega' \rightrightarrows M'$. Then the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\widetilde{F}_x} & \Omega' \\ \downarrow{\scriptstyle \chi_x} & & \downarrow{\scriptstyle \chi'_{x'}} \\ G & \xrightarrow{F} & G' \end{array}$$

is commutative.

Proof: First note that χ_x is a base preserving isomorphism of set groupoids.
Also the following is a commutative diagram:

$$\begin{array}{ccc} G_x \times G_x & & \\ \downarrow{\scriptstyle q} & \searrow{\scriptstyle \delta_x} & \\ \Omega & \xrightarrow[\chi_x]{} & G \end{array}$$

where $\delta_x(h, g) = hg^{-1}$ is a surjective submersion and q is the quotient map and hence a surjective submersion. Therefore, χ_x is a surjective submersion. Since χ_x is bijective, it follows that χ_x is a diffeomorphism.
The map $\widetilde{F}_x : \Omega \longrightarrow \Omega'$ is defined by $\langle h, g\rangle \mapsto \langle F(h), F(g)\rangle$. Hence the second part follows. □

References

[1] Mackenzie K. C. H. *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press