

## LIE GROUPOIDS-I

Neeta Pandey<sup>1</sup>

Brahmananda Keshab Chandra College, 111/2; B.T.Road, Kolkata, 700 108

### 1. BASIC CONCEPTS

Groupoids are a certain generalization of groups. We begin by recalling a concept associated in general with any topological space, that of its fundamental groupoid.

Let  $M$  be a smooth manifold. Let  $\mathcal{P}(M)$  be the set of smooth paths  $\gamma : [0, 1] \rightarrow M$  on  $M$  with the compact open topology. Define an equivalence relation  $\sim$  on  $\mathcal{P}(M)$  by  $\sigma \sim \tau$  if  $\sigma$  and  $\tau$  are homotopic relative to end points. Denote the equivalence class of a path  $\gamma$  by  $\langle \gamma \rangle$ . The *fundamental groupoid* of  $M$  is the quotient space  $\Pi(M) := \mathcal{P}(M)/\sim$ .

For any  $x \in M$  let  $c_x$  denote the constant path at  $x$ . Then the map  $\epsilon : M \rightarrow \Pi(M)$  defined by  $\epsilon(x) = \langle c_x \rangle$  is injective.

For any  $\langle \gamma_1 \rangle$  and  $\langle \gamma_2 \rangle$  in  $\Pi(M)$  such that  $\gamma_1(0) = \gamma_2(1)$ , the multiplication of  $\langle \gamma_1 \rangle$  and  $\langle \gamma_2 \rangle$  is defined by  $\langle \gamma_1 \rangle \langle \gamma_2 \rangle = \langle \gamma_1 \gamma_2 \rangle$  where  $\gamma_1 \gamma_2$  is the concatenation of paths:

$$\gamma_1 \gamma_2(t) = \begin{cases} \gamma_2(2t); & 0 \leq t \leq \frac{1}{2} \\ \gamma_1(2t - 1); & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Note that  $\gamma_1 \gamma_2(0) = \gamma_2(0)$  and  $\gamma_1 \gamma_2(1) = \gamma_2(1)$ . Also any  $\langle \gamma \rangle$ , has an inverse given by  $\langle \gamma \rangle^{-1} = \langle \gamma^{-1} \rangle$  where  $\gamma^{-1}(t) = \gamma(1 - t)$  for  $t \in [0, 1]$ .

For any  $x \in M$ , the set of homotopy classes (relative to end points) of smooth paths  $\langle \gamma \rangle$  such that  $\gamma(0) = \gamma(1) = x$  form a group  $\pi_1(M, x)$  under the above multiplication called the *fundamental group of  $M$  based at  $x$* .

We can define surjections  $\alpha : \Pi(M) \rightarrow M$  by  $\alpha(\langle \gamma \rangle) = \gamma(0)$  and  $\beta : \Pi(M) \rightarrow M$  by  $\beta(\langle \gamma \rangle) = \gamma(1)$ . The map  $(\alpha, \beta) : \Pi(M) \rightarrow M$  is a covering map and hence the smooth structure of  $M$  can be lifted to  $\Pi(M)$ . This structure is compatible with the quotient topology. Thus  $\Pi(M)$  is a smooth manifold. It can be shown that  $\alpha, \beta$  are surjective submersions and  $m, \epsilon$  are smooth maps.

The above structures recur in various situations. The basic notion can be formulated as follows:

**Definition 1.1.** A *set groupoid* consists of two sets  $G$  and  $M$  called the *groupoid* and *base* respectively, together with maps:

$$\begin{aligned} \alpha : G &\longrightarrow M && \text{(source projection),} \\ \beta : G &\longrightarrow M && \text{(target projection),} \\ \epsilon : M &\longrightarrow G && \text{(object inclusion)} \end{aligned}$$

(where, for any  $x \in M$ ,  $\epsilon(x)$  will also be denoted by  $1_x$ ) and a *partial multiplication*  $m : G \times_M G \rightarrow G$  defined on the subset

$$G \times_M G = \{(h, g) \in G \times G : \alpha(h) = \beta(g)\},$$

of  $G \times G$ ,  $m(h, g)$  being denoted by  $hg$ , subject to the following conditions:

- (i)  $\alpha(hg) = \alpha(g)$  and  $\beta(hg) = \beta(h)$  for all  $(h, g) \in G \times_M G$

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- (ii)  $k(hg) = (kh)g$ , for all  $(k, h), (h, g) \in G \times_M G$
- (iii)  $\alpha(1_x) = \beta(1_x) = x$  for all  $x \in M$
- (iv)  $g1_{\alpha(g)} = g$  and  $1_{\beta(g)}g = g$  for all  $g \in G$ .
- (v) each  $g \in G$  has a two sided *inverse*  $\iota(g) = g^{-1}$  such that

$$\alpha(g^{-1}) = \beta(g) \quad \beta(g^{-1}) = \alpha(g) \quad g^{-1}g = 1_{\alpha(g)} \quad gg^{-1} = 1_{\beta(g)}$$

A set groupoid is denoted by  $(G \rightrightarrows M, \alpha, \beta, \epsilon, m, \iota)$  or simply  $G \rightrightarrows M$ . The maps  $\alpha, \beta, \epsilon, m$  and  $\iota$  are called the *structure maps* of the set groupoid.

**Proposition 1.2.** *Let  $G \rightrightarrows M$  be a set groupoid. Let  $g \in G$ . Then*

- (i) *If  $(h, g) \in G \times_M G$  and  $hg = g$ , then  $h = 1_{\beta(g)}$ .*
- (ii) *If  $(g, k) \in G \times_M G$  and  $gk = g$ , then  $k = 1_{\alpha(g)}$ .*
- (iii) *If  $(h, g) \in G \times_M G$  and  $hg = 1_{\alpha(g)}$ , then  $h = g^{-1}$ .*
- (iv) *If  $(g, k) \in G \times_M G$  and  $gk = 1_{\beta(g)}$ , then  $k = g^{-1}$ .*

**Remark 1.3.** A set groupoid  $G \rightrightarrows M$  can be thought of as a small category, with elements of  $M$  the objects of this category and elements of  $G$ , the morphisms between these objects. Hence elements of  $G$  are referred as *arrows* and any  $g \in G$  is written as  $\beta(g) \curvearrowright_{\alpha(g)}$ .

For any  $x \in M$ ,  $1_x$  is called the *unity* or *identity* corresponding to  $x$ . Any  $1_x, x \in M$  is written as  $\overset{x}{\circlearrowleft}$ .

**Notation 1.4.** Let  $G \rightrightarrows M$  be a set groupoid. Let  $x \in M$ . The  $\alpha$ -fibre over  $x$  is denoted by  $G_x := \alpha^{-1}(x)$  and the  $\beta$ -fibre over  $x$  is denoted by  $G^x := \beta^{-1}(x)$ . For any  $x, y \in M$ ,  $G_x^y = G_x \cap G^y$ . Similarly for any  $U, V \subseteq M$ ,  $G_U := \alpha^{-1}(U)$ ,  $G^V := \beta^{-1}(V)$  and  $G_U^V = G_U \cap G^V$ . The set  $\{1_x : x \in M\}$  is denoted by  $1_M$  or  $\epsilon(M)$ .

**Remark 1.5.** Let  $G \rightrightarrows M$  be a set groupoid. Let  $x \in M$ . The set  $G_x^x$  is a group under the restriction of partial multiplication and is called the *vertex group* at  $x$ .

**Definition 1.6.** A set groupoid  $G \rightrightarrows M$  is a *Lie groupoid* if  $G$  and  $M$  are smooth manifolds,  $\alpha, \beta$  are surjective submersions and the maps  $\epsilon$  and  $m$  are smooth maps.

**Remark 1.7.** Let  $G \rightrightarrows M$  be a Lie groupoid. Let  $\Delta(M) = \{(x, x) : x \in M\}$ . Then  $\Delta(M)$  is a submanifold of  $M \times M$  of codimension  $\dim M$ . Also  $\alpha \times \beta : G \times G \rightarrow M \times M$  is a surjective submersion. Hence  $(\alpha \times \beta)^{-1}(\Delta(M)) = G \times_M G$  is a submanifold of  $G \times G$  and  $\text{codim}(G \times_M G)$  in  $G \times G$  is  $\dim M$ .

**Remark 1.8.** Let  $G \rightrightarrows M$  be a Lie groupoid. Let  $x \in M$ . Then  $G_x$  and  $G^x$  are closed embedded submanifolds of  $G$ .

**Remark 1.9.** Let  $G \rightrightarrows M$  be a Lie groupoid. Then  $\alpha \circ \epsilon = \text{id}_M$ . Hence  $\epsilon$  is an immersion. Therefore  $1_M$  is a closed embedded submanifold of  $G$ .

**Definition 1.10.** Let  $G \rightrightarrows M$  be a Lie groupoid. Let  $x \in M$ . The *orbit* of  $G \rightrightarrows M$  through  $x$  is defined to be  $\beta(\alpha^{-1}(x))$ .

The orbit of  $G \rightrightarrows M$  through  $x$  can also be expressed as  $\alpha(\beta^{-1}(x))$ .

## 2. SOME EXAMPLES OF LIE GROUPOIDS

- (1) Let
- $M$
- be a smooth manifold. Define
- $\alpha = \beta = \epsilon = \text{id}_M$
- . Then

$$M \times_M M = \{(x, y) \in M \times M : x = y\} = \Delta(M).$$

Define the partial multiplication  $m : M \times_M M \rightarrow M$  by  $(x, x) \mapsto x$ . Then clearly  $M \rightrightarrows M$  is a Lie groupoid. This is a *base groupoid* i.e. a groupoid in which every element is a unity.

- (2) Let
- $M$
- be a smooth manifold. Let
- $G$
- be a Lie group. Let
- $e$
- be the identity element of
- $G$
- . Define a Lie groupoid
- $M \times G \times M \rightrightarrows M$
- as follows: for any
- $x, y \in M$
- ,
- $g \in G$
- , let

$$\alpha : (x, g, y) \mapsto y \quad \beta : (x, g, y) \mapsto x \quad \epsilon : x \mapsto (x, e, x)$$

To define the partial multiplication  $m$ , let  $\alpha(x, g, y) = \beta(z, h, w)$ . Then  $y = z$ . Define

$$m((x, g, y), (y, h, w)) = (x, gh, w).$$

For any  $(x, g, y) \in M \times G \times M$ , we get  $(x, g, y)^{-1} = (y, g^{-1}, x)$ .

The Lie groupoid  $M \times G \times M \rightrightarrows M$  is called the *trivial groupoid on  $M$  with group  $G$* .

- (3) Take
- $M$
- to be a point in the above example. This shows that any Lie group can be considered as a Lie groupoid.

- (4) Let
- $M$
- be a smooth manifold. Let
- $G$
- be the trivial group. The corresponding trivial groupoid
- $M \times M \rightrightarrows M$
- is called the
- pair groupoid*
- on
- $M$
- .

- (5) Let
- $M$
- be a smooth manifold. The fundamental groupoid
- $\Pi(M) \rightrightarrows M$
- is a Lie groupoid.

- (6) Let
- $G \rightrightarrows M$
- be a Lie groupoid. Applying the tangent functor to each of the structure maps defining
- $G$
- gives the Lie groupoid
- $TG \rightrightarrows TM$
- which is called the
- tangent prolongation groupoid*
- or simply
- tangent groupoid*
- .

- (7) Let
- $G$
- be a Lie group. Let
- $M$
- be a smooth
- $G$
- manifold where
- $G$
- has a left action on
- $M$
- . Then
- $G \times M \rightrightarrows M$
- is a Lie groupoid described as follows: for any
- $x \in M$
- ,
- $g \in G$
- and
- $e$
- identity element of
- $G$
- ,

$$\alpha : (g, x) \mapsto x \quad \beta : (g, x) \mapsto g \cdot x \quad \epsilon : x \mapsto (e, x)$$

To define the partial multiplication  $m$ , let  $\alpha(g, x) = \beta(h, y)$ . Then  $x = h \cdot y$ . Define

$$m((g, h \cdot y), (h, y)) = (gh, y).$$

For any  $(g, x) \in G \times M$ , we get  $\iota((g, x)) = (g, x)^{-1} = (g^{-1}, gx)$ . Clearly,  $\alpha, \beta, \epsilon$  and  $\iota$  are smooth maps. Since  $\iota^{-1} = \iota$  this implies that  $\iota$  is a diffeomorphism. Also  $\alpha$  is projection onto the second factor and hence is a surjective submersion. Again  $\beta = \alpha \circ \iota$  and hence  $\beta$  is a surjective submersion. The partial multiplication  $m$  can be identified with the map

$$G \times G \times M \rightarrow G \times M \quad \text{given by} \quad (g, h, y) \mapsto (gh, y).$$

Hence  $m$  is smooth.

This Lie groupoid is called the *action groupoid* of  $G \times M \rightarrow M$  and is denoted by  $G \ltimes M$ .

- (8) Let
- $G$
- be a Lie group and
- $\mathfrak{g}$
- be its Lie algebra. We describe a Lie groupoid structure
- $T^*G \rightrightarrows \mathfrak{g}^*$
- as follows: Let
- $e$
- be the identity element of
- $G$
- . Let
- $g \in G$
- . Let us denote the left

translation corresponding to  $g$  by  $L_g : G \rightarrow G$  and the right translation corresponding to  $g$  by  $R_g : G \rightarrow G$ .

For any  $\omega \in T_g^*G$ , define

$$\alpha : (g, \omega) \mapsto L_g^*(\omega) \qquad \beta : (g, \omega) \mapsto R_g^*\omega$$

We can identify  $T^*G$  with  $G \times \mathfrak{g}^*$  by the diffeomorphism

$$\Phi : T^*G \longrightarrow G \times \mathfrak{g}^* \qquad (g, \omega) \mapsto (g, L_g^*(\omega))$$

Thus  $\alpha = \text{pr}_2$  and hence is a surjective submersion. Similarly  $\beta$  is a surjective submersion. Define  $\epsilon : \mathfrak{g}^* \rightarrow T^*G$  by  $v^* \mapsto (e, v^*)$  for  $v^* \in \mathfrak{g}^*$ . Then  $\epsilon$  is smooth.

The partial multiplication  $m$  on  $T^*G$  is defined as follows: for any  $(g, \omega), (h, \theta) \in T^*G$  such that  $\alpha(g, \omega) = \beta(h, \theta)$ , define

$$(g, \omega) \cdot (h, \theta) = \left( gh, \frac{1}{2}[R_{h^{-1}}^*(\omega) + L_{g^{-1}}^*(\theta)] \right).$$

Clearly  $m$  is smooth. Next define

$$\iota(g, \omega) = (g^{-1}, L_g^*R_g^*(\omega))$$

Then  $\iota$  is a diffeomorphism.

So  $T^*G \rightrightarrows \mathfrak{g}^*$  is a Lie groupoid called the cotangent groupoid of the Lie group  $G$ .

Recall that any  $g \in G$  defines an inner automorphism

$$I_g : G \longrightarrow G \qquad h \mapsto ghg^{-1}.$$

Then  $G$  acts on itself from the left by

$$I : G \times G \longrightarrow G \qquad (g, h) \mapsto ghg^{-1}$$

and  $e$  is a fixed point of this action. Let

$$\text{Ad } g = T(I_g)_e : \mathfrak{g} \longrightarrow \mathfrak{g}$$

Then

$$\text{Ad} : G \times \mathfrak{g} \longrightarrow \mathfrak{g} \qquad (g, v) \mapsto \text{Ad } g(v),$$

is a left action and hence defines a representation of  $G$  into  $\text{Aut}(\mathfrak{g})$ . This is called the *adjoint representation* of  $G$ .

For any  $g \in G$ , let

$$\text{coad } g := (\text{Ad } g^{-1})^* : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*.$$

This defines a left action

$$\text{coad} : G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \qquad (g, v^*) \mapsto \text{coad } g(v^*).$$

We now describe the orbits of  $T^*G \rightrightarrows \mathfrak{g}^*$ . Let  $v^* \in \mathfrak{g}^*$ . Then

$$\begin{aligned} \beta\alpha^{-1}(v^*) &= \{\beta(g, \omega) : g \in G, L_g^*(\omega) = v^*\} = \{R_g^*(\omega) : g \in G, L_{g^{-1}}^*(v^*) = \omega\} \\ &= \{R_g^*L_{g^{-1}}^*(v^*) : g \in G\} = \{(\text{Ad } g^{-1})^*(v^*) : g \in G\} \\ &= \{(\text{coad } g)(v^*) : g \in G\}. \end{aligned}$$

So, the orbit of the groupoid  $T^*G \rightrightarrows \mathfrak{g}^*$  is the orbit of the coadjoint action of  $G$  on  $\mathfrak{g}^*$ .

3. TRANSLATION MAPS IN  $G \rightrightarrows M$ 

**Definition 3.1.** Let  $G \rightrightarrows M$  be a Lie groupoid and let  $g \in G$ . Define

$$L_g : G^{\alpha(g)} \longrightarrow G^{\beta(g)} \quad \text{by} \quad h \mapsto gh \quad (\text{left translation corresponding to } g)$$

and

$$R_g : G_{\beta(g)} \longrightarrow G_{\alpha(g)} \quad \text{by} \quad h \mapsto hg \quad (\text{right translation corresponding to } g)$$

**Proposition 3.2.** Let  $G \rightrightarrows M$  be a Lie groupoid and let  $g \in G$ . Then  $L_g$  and  $R_g$  are diffeomorphisms.

**Proposition 3.3.** Let  $G \rightrightarrows M$  be a Lie groupoid. The inversion map  $\iota : G \longrightarrow G$  is a diffeomorphism.

*Proof:* Let  $g \in G$ . Then  $gg^{-1} = 1_{\beta(g)} = 1_{\alpha(g^{-1})}$ . Hence  $g = (g^{-1})^{-1}$ . Thus  $\iota^{-1} = \iota$  by Proposition 1.2. Hence to prove that  $\iota$  is a diffeomorphism it is enough to prove that  $\iota$  is a smooth map.

Let  $G \times_{\beta} G = \{(k, l) : \beta(k) = \beta(l)\}$ . Then  $G \times_{\beta} G = (\beta \times \beta)^{-1}(\Delta(M))$  and hence is a submanifold of  $G \times G$  of codimension  $\dim M$ . Define

$$\theta : G \times_M G \longrightarrow G \times_{\beta} G \quad \text{by} \quad (k, l) \mapsto (k, kl).$$

Then  $\theta = \text{pr}_1 \times m$  and hence  $\theta$  is a smooth map. Also  $\theta$  is a bijection.

Note that  $\iota$  can be expressed as the composition

$$G \xrightarrow{id \times \epsilon_{\beta}} G \times_{\beta} G \xrightarrow{\theta^{-1}} G \times_M G \xrightarrow{\text{pr}_2} G$$

i.e. for any  $g \in G$ ,  $g \mapsto (g, 1_{\beta(g)}) \mapsto (g, g^{-1}1_{\beta(g)}) = (g, g^{-1}) \mapsto g^{-1}$ .

Hence to prove that  $\iota$  is smooth, it is enough to show that  $\theta$  is a diffeomorphism. Note that

$$\dim(G \times_M G) = \dim(G \times_{\beta} G) = \dim(G \times G) - \dim M.$$

Hence to show that  $\theta$  is a diffeomorphism it is enough to show that  $\theta$  is an immersion.

We have

$$\begin{aligned} T(G \times_M G) &= T[(\alpha \times \beta)^{-1}(\Delta(M))] &&= (T(\alpha) \times T(\beta))^{-1}(T\Delta(M)) \\ &= \{(X, Y) \in TG \times TG : T(\alpha)(X) = T(\beta)(Y)\} &&= TG \times_{TM} TG. \end{aligned}$$

Let  $m$  be the partial multiplication on  $G$ . Let  $(g, h) \in G \times_M G$  and let  $\alpha(g) = \beta(h) = x$ . We have the restriction of  $m$ ,

$$m : G_x \times G^x \longrightarrow G.$$

So,

$$T(m) : T_g G_x \times T_h G^x \longrightarrow T_{gh} G.$$

Now for any  $X \in T_g G_x$  and  $Y \in T_h G^x$ ,

$$T(m)_{(g,h)}(X, Y) = T(m)_{(g,h)}(X, 0) + T(m)_{(g,h)}(0, Y).$$

Also  $m : G_x \times \{h\} \longrightarrow G$  is just  $R_h : G_x \longrightarrow G_{\alpha(h)}$ . So,

$$T(m)_{(g,h)}(X, 0) = T(R_h)_g(X).$$

Similarly,  $m : \{g\} \times G^x \longrightarrow G$  is just  $L_g : G^x \longrightarrow G^{\beta(g)}$ . So,

$$T(m)_{(g,h)}(0, Y) = T(L_g)_h(Y).$$

Hence

$$T(m)_{(g,h)}(X, Y) = T(R_h)_g(X) + T(L_g)_h(Y).$$

Let  $(X, Y) \in T_{(g,h)}(G \times_M G)$ . Suppose that  $T(\theta)_{(g,h)}(X, Y) = (0, 0)$ . Now  $\text{pr}_1 \circ \theta = \text{pr}_1$ . So

$$T(\text{pr}_1 \circ \theta)_{(g,h)}(X, Y) = T(\text{pr}_1)_{(g,h)}(X, Y) = X.$$

Also

$$T(\text{pr}_1 \circ \theta)_{(g,h)}(X, Y) = T(\text{pr}_1)_{(g,gh)} \circ T(\theta)_{(g,h)}(X, Y) = T(\text{pr}_1)_{(g,gh)}(0, 0) = 0$$

Hence  $X = 0$ . Next note that  $\text{pr}_2 \circ \theta = m$ . So

$$\begin{aligned} T(\text{pr}_2)_{(g,gh)} \circ T(\theta)_{(g,h)}(X, Y) &= T(\text{pr}_2 \circ \theta)_{(g,h)}(X, Y) &= T(m)_{(g,h)}(X, Y) \\ &= T(R_h)_g(X) + T(L_g)_h(Y) &= T(L_g)_h(Y). \end{aligned}$$

Also

$$T(\text{pr}_2)_{(g,gh)} \circ T(\theta)_{(g,h)}(X, Y) = T(\text{pr}_2)_{(g,gh)}(0, 0) = 0.$$

Hence  $T(L_g)_h(Y) = 0$  which implies that  $Y = 0$ , since  $L_g$  is a diffeomorphism.

Thus  $\theta$  is an immersion. □

#### REFERENCES

- [1] Mackenzie K. C. H. *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press