

Symplectic Geometry

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1 Linear symplectic space

Let V be a vector space over \mathbb{R} . Then a **bilinear form** $\omega : V \times V \rightarrow \mathbb{R}$ is a bilinear map. If (e_1, \dots, e_n) is a basis of V , then a bilinear form ω gives rise to a matrix $A = (a_{ij})$, where $a_{ij} = \omega(e_i, e_j)$. The matrix determines ω uniquely. For, if $u = \sum_i u_i e_i$ and $v = \sum_j v_j e_j$, $u_i, v_j \in \mathbb{R}$, then $\omega(u, v) = \sum_{i,j} a_{ij} u_i v_j$.

Let V^* denote the dual vector space $\text{Hom}(V, \mathbb{R})$ of linear maps $V \rightarrow \mathbb{R}$. Let (f_1, \dots, f_n) be the basis of V^* dual to the basis (e_1, \dots, e_n) of V so that $f_i(e_j) = \delta_{ij}$ (Kronecker delta). Then $f_i(u) = \sum_j f_i(u_j e_j) = \sum_j u_j f_i(e_j) = u_i$, and $f_j(v) = \sum_i f_j(v_i e_i) = \sum_i v_i f_j(e_i) = v_j$, and we may write $\omega = \sum_{i,j} a_{ij} f_i \otimes f_j$ so that $\omega(u, v) = \sum_{i,j} a_{ij} f_i(u) \cdot f_j(v) = \sum_{i,j} a_{ij} u_i v_j$.

The **kernel** of ω is defined by

$$\ker \omega = \{u \in V : \omega(u, v) = 0 \text{ for all } v \in V\}.$$

The bilinear form ω is called **non-degenerate** if $\ker \omega = 0$, that is, if $\omega(u, v) = 0$ for all $v \in V$ implies that $u = 0$, in other words, the map $\bar{\omega} : V \rightarrow V^*$, defined by $\bar{\omega}(u)(v) = \omega(u, v)$, $u, v \in V$, is a linear isomorphism.

It may be seen that the matrix of the linear map $\bar{\omega}$ is the same as the matrix A of ω . Indeed, if $\bar{\omega}(e_i) = \lambda_{i1} f_1 + \dots + \lambda_{in} f_n$, $1 \leq i \leq n$, then the matrix of $\bar{\omega}$ is the coefficient matrix (λ_{ij}) , where $\lambda_{ij} = \bar{\omega}(e_i)(e_j)$, but $\bar{\omega}(e_i)(e_j) = \omega(e_i, e_j) = a_{ij}$, therefore $\lambda_{ij} = a_{ij}$.

Therefore ω is non-degenerate if and only if its matrix A is non-singular.

A bilinear form ω is called **alternating** or **skew-symmetric** if $\omega(u, v) = -\omega(v, u)$ for all $u, v \in V$, that is, if $\omega(e_i, e_j) = -\omega(e_j, e_i)$ for all i, j , or $a_{ij} = -a_{ji}$. Therefore ω is skew-symmetric if and only if its matrix A is skew-symmetric.

A bilinear form ω is called **symplectic form** if it is non-degenerate and skew-symmetric. A **symplectic vector space** (V, ω) is a vector space V with

a symplectic form ω on it. Since the determinate of a skew-symmetric matrix of odd order is zero, the dimension of a symplectic vector space must be even.

A **symplectomorphism** of a symplectic vector space (V, ω) is a linear isomorphism $\phi : V \rightarrow V$ which preserves the symplectic structure in the sense that $\phi^*\omega = \omega$ where $\phi^*\omega$ is the induced symplectic form on the domain of ϕ defined by $\phi^*\omega(u, v) = \omega(\phi(u), \phi(v))$. The symplectomorphisms of (V, ω) form a group $\text{Sp}(V, \omega)$, called **symplectic group**.

Subspaces of symplectic vector space

The **symplectic complement** of a linear subspace W of a symplectic vector space (V, ω) is defined as the subspace

$$W^\omega = \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}$$

Lemma 1.1. *For any subspace W of V , we have*

- (i) $(W^\omega)^\omega = W$,
- (ii) $\dim W + \dim W^\omega = \dim V$.

Proof. (i) By definition, $v \in (W^\omega)^\omega$ if and only if $\omega(v, w) = 0$ for all $w \in W^\omega$, that is, if and only if $v \in W$.

(ii) If $W \subset V$, then we have an exact sequence $0 \rightarrow W \rightarrow V \rightarrow V/W \rightarrow 0$, which gives rise to an exact sequence $0 \rightarrow (V/W)^* \rightarrow V^* \rightarrow W^* \rightarrow 0$. The space $(V/W)^*$ is isomorphic to those linear maps $f : V \rightarrow \mathbb{R}$ in V^* which vanish on W . This subspace of V^* is called the **annihilator** of W in V^* , and is denoted by $\text{Ann } W$. Now the isomorphism $\bar{\omega} : V \rightarrow V^*$ identifies the subspace W^ω of V with $\text{Ann } W$, and therefore $\dim W^\omega = \dim \text{Ann } W$. Since for any subspace W of V , we have $W \oplus \text{Ann } W = V$. Therefore $\dim W + \dim W^\omega = \dim V$. \square

Note that, unlike the annihilator, $W^\omega \cap W$ need not be $\{0\}$.

Exercise 1.2. Show that if W and X are subspaces of (V, ω) , then $W \subseteq X$ if and only if $X^\omega \subseteq W^\omega$.

A subspace W of (V, ω) is called

symplectic if $W^\omega \cap W = \{0\}$. This is true if and only if ω restricts to a non-degenerate form on W .

isotropic if $W \subset W^\omega$. This is true if and only if for any two vectors $w, w' \in W$ $\omega(w, w') = 0$, that is, ω restricts to 0 on W . Thus any subspace of dimension one is isotropic.

coisotropic if $W^\omega \subset W$. Equivalently, W is coisotropic if and only if W^ω is isotropic. This implies in view of Lemma 1.1 that any subspace of codimension one is coisotropic.

Lagrangian if $W = W^\omega$, that is, if W is both isotropic and coisotropic.

Exercise 1.3. Show that if W is a subspace of (V, ω) , then

$$W \text{ is symplectic} \Leftrightarrow W \cap W^\omega = \{0\} \Leftrightarrow V = W \oplus W^\omega.$$

Exercise 1.4. Show that if W is an isotropic subspace of (V, ω) , then $\dim W \leq \frac{1}{2} \dim V$. In particular, if W is a Lagrangian subspace, then $\dim W = \frac{1}{2} \dim V$.

Exercise 1.5. Let (V, ω) be a symplectic vector space and $\phi : V \rightarrow V$ is a linear map. Show that ϕ is a symplectomorphism if and only if its graph $\Gamma = \{(v, \phi(v)) : v \in V\}$ is a Lagrangian subspace of $V \times V$ with symplectic form $(-\omega) \times \omega$.

Symplectic reduction

A coisotropic subspace $W \subset V$ gives rise to symplectic vector space W/W^ω . This construction is called **symplectic reduction**.

Theorem 1.6. *If (V, ω) is a symplectic vector space, and $W \subset V$ a coisotropic subspace, then the vector space $V' = W/W^\omega$ has a natural symplectic structure ω' .*

Proof. The quotient space V' consists of classes $[w] = w + W^\omega$ for $w \in W$. Since subspace W^ω is isotropic, ω vanishes on W^ω . Therefore, for $w, w' \in W$

$$\omega(w + W^\omega, w' + W^\omega) = \omega(w, w').$$

Thus ω induces a bilinear form ω' on W/W^ω , where $\omega'([w], [w']) = \omega(w, w')$. Now, if $v \in W$ and $\omega(v, w) = 0$ for all $w \in W$, then $v \in W^\omega$, and hence ω' is non-degenerate. This completes the proof. \square

Symplectic structure on \mathbb{R}^{2n}

We recall some fact about the Euclidean space \mathbb{R}^{2n} . A point $p \in \mathbb{R}^{2n}$ is given by $2n$ coordinate functions $x_1, \dots, x_n, y_1, \dots, y_n : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, where x_i maps p onto its i -th coordinate, and y_i maps p onto its $(n+i)$ -th coordinate, for $i = 1, \dots, n$. The tangent space $T(\mathbb{R}^{2n})_p$ at $p \in \mathbb{R}^{2n}$ is the vector space with basis $\partial/\partial x_1(p), \dots, \partial/\partial x_n(p), \partial/\partial y_1(p), \dots, \partial/\partial y_n(p)$. The dual space of $T(\mathbb{R}^{2n})_p$ is the cotangent space of $T^*(\mathbb{R}^{2n})_p$, and this has the dual basis $dx_1, \dots, dx_n, dy_1, \dots, dy_n$ so that $dx_i(\partial/\partial x_j) = \delta_{ij} = dy_i(\partial/\partial y_j)$, $dx_i(\partial/\partial y_j) = 0 = dy_i(\partial/\partial x_j)$. A vector of the space $T^*(\mathbb{R}^{2n})_p$ is of the form $\sum_i a_i dx_i + \sum_i b_i dy_i$, $a_i, b_i \in \mathbb{R}$. These vectors are called the **differential 1-forms** on \mathbb{R}^{2n} .

We shall identify the spaces $T(\mathbb{R}^{2n})_p$ and $T^*(\mathbb{R}^{2n})_p$ with \mathbb{R}^{2n} .

The vector space $\Omega^2(\mathbb{R}^{2n})$ of skew symmetric forms on \mathbb{R}^{2n} is generated by the forms $dx_i \wedge dy_j \in \Omega^2(\mathbb{R}^{2n})$ defined by

$$dx_i \wedge dy_j(\partial/\partial x_r, \partial/\partial y_s) = dx_i(\partial/\partial x_r) \cdot dy_j(\partial/\partial y_s) = \delta_{ir} \cdot \delta_{js}.$$

The vectors of the space $\Omega^2(\mathbb{R}^{2n})$ are called the **differential 2-forms** on \mathbb{R}^{2n} .

The 2-form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ is called the **canonical symplectic form** on \mathbb{R}^{2n} . Its matrix is given by

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix.

In general, we have the following which asserts that all symplectic vector spaces of the same dimension are linearly symplectomorphic.

Theorem 1.7. *If (V, ω) is a symplectic vector space of dimension $2n$, then there is a basis $u_1, \dots, u_n, v_1, \dots, v_n$ of V such that*

$$\omega(u_i, u_j) = \omega(v_i, v_j) = 0, \quad \omega(u_i, v_j) = \delta_{ij}.$$

(Such a basis is called a **symplectic basis**.)

Moreover, there is a linear isomorphism $\phi : \mathbb{R}^{2n} \rightarrow V$ such that $\phi^*\omega = \omega_0$.

Proof. The proof is by induction on n .

Since ω is non-degenerate, there exists vectors $u_1, v_1 \in V$ such that $\omega(u_1, v_1) = 1$. Therefore the subspace generated by u_1 and v_1 is symplectic. Let W be the symplectic complement of this subspace. Then (W, ω) is a symplectic vector space of dimension $2n-2$. By induction, there is a symplectic basis $u_2, \dots, u_n, v_2, \dots, v_n$ of W . Then the vectors $u_1, \dots, u_n, v_1, \dots, v_n$ is a symplectic basis of V .

Finally, define the linear map $\phi : \mathbb{R}^{2n} \rightarrow V$ by

$$\phi(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{i=1}^n x_i u_i + \sum_{i=1}^n y_i v_i.$$

This satisfies $\phi^*\omega = \omega_0$. □

Exercise 1.8. Show that if W is a Lagrangian subspace of (V, ω) , then any basis u_1, \dots, u_n of W can be extended to a symplectic basis $u_1, \dots, u_n, v_1, \dots, v_n$ of V .

Hint. Choose $v_1 \in W^\omega$, where W is the linear span of $\{u_2, \dots, u_n\}$.

2 Symplectic manifold

Let M be a smooth manifold, and $\Omega^2(M)$ denote the space of smooth 2-forms on M . A 2-form $\omega \in \Omega^2(M)$ assigns to each point $p \in M$, a skew-symmetric

bilinear form $\omega_p : T_p(M) \times T_p(M) \rightarrow \mathbb{R}$ on the tangent space $T_p(M)$ of M at p such that ω_p varies smoothly with p . A form $\omega \in \Omega^2(M)$ is called **symplectic** if each tangent space $(T_p M, \omega_p)$ is a symplectic vector space, and ω is closed, that is, its exterior derivative $d\omega$ is zero.

A **symplectic manifold** is a pair (M, ω) , where M is a smooth manifold and ω is a symplectic form on M .

We shall describe how to construct a symplectic form on M . But, first note that a symplectic manifold (M, ω) must be even dimensional. Moreover, if $\dim M = 2n$, then the n -th exterior power $\omega^n = \omega \wedge \cdots \wedge \omega$ is non-vanishing, that is, ω^n is a volume form. Therefore any symplectic manifold is canonically oriented.

Cotangent bundle

The manifold structure on M is given by local coordinate charts (U, x_1, \dots, x_n) with coordinate functions $x_i : U \rightarrow \mathbb{R}$. Then at a point $x \in U$, the differentials $(dx_1)_x, \dots, (dx_n)_x$ form a basis of the cotangent space $T_x^* M$ of M at $x \in M$ so that if $\xi \in T_x^* M$, then $\xi = \sum_{i=1}^n \xi_i (dx_i)_x$, $\xi_i \in \mathbb{R}$. Then if $T^*U = \{(x, \xi) : x \in U, \xi \in T_x^* M\}$, that is, if $T^*U = \cup_{x \in U} T_x^* M$, we have a map $T^*U \rightarrow \mathbb{R}^{2n}$ given by $(x, \xi) \rightarrow (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$. This gives a coordinate chart $(T^*U, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ for the cotangent bundle $T^*M = \cup_{x \in M} T_x^* M$. The coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ are the **cotangent coordinates** associated to the coordinates x_1, \dots, x_n on U . The transition functions on the overlaps of coordinate charts are smooth. For, given two charts $(U, x_1, \dots, x_n), (U', x'_1, \dots, x'_n)$, a point $x \in U \cap U'$, and a vector $\xi \in T_x^* M$, we have

$$\xi = \sum_{i=1}^n \xi_i (dx_i)_x = \sum_{i,j} \xi_i \left(\frac{\partial x_i}{\partial x'_j} \right) (dx'_j)_x = \sum_{j=1}^n \xi'_j (dx'_j)_x,$$

where each $\xi'_j = \sum_i \xi_i \left(\frac{\partial x_i}{\partial x'_j} \right)$ is smooth. Therefore the tangent bundle T^*M is a smooth manifold of dimension $2n$.

Canonical symplectic form

Let (U, x_1, \dots, x_n) be a coordinate chart on M , with associated coordinate chart $(T^*U, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ on T^*M . Consider the 1-form on T^*U

$$\alpha = \sum_{i=1}^n \xi_i dx_i.$$

Define a 2 form ω on T^*U by $\omega = -d\alpha$. Then

$$\omega = \sum_{i=1}^n dx_i \wedge d\xi_i.$$

Lemma 2.1. *The form α , and hence the 2-form ω , is well-defined on T^*M .*

Proof. Let (U, x_1, \dots, x_n) and (U', x'_1, \dots, x'_n) be two cotangent coordinate charts. On $U \cap U'$, the two sets of coordinates are related by

$$\xi'_j = \sum_i \xi_i \left(\frac{\partial x_i}{\partial x'_j} \right). \text{ Since } dx'_j = \sum_i \left(\frac{\partial x'_j}{\partial x_i} \right) dx_i, \text{ we have}$$

$$\alpha = \sum_i \xi_i dx_i = \sum_j \xi'_j dx'_j = \alpha'.$$

and therefore $\omega = \omega'$ on $U \cap U'$. This completes the proof. \square

Note that the form ω is closed, since its exterior derivative

$$d\omega = -d(d\alpha) = 0.$$

The 1-form α on T^*M is called the **Liouville 1-form**, and the 2-form ω on T^*M is called the **canonical symplectic form**.

Submanifolds of symplectic manifold

A submanifold N of a symplectic manifold (M, ω) is called **symplectic** (or **isotropic**, **coisotropic**, **Lagrangian**) if for every point $p \in N$ the subspace $T_p N$ of the symplectic vector space $(T_p M, \omega_p)$ is symplectic, (or isotropic, coisotropic, Lagrangian).

Any 1-dimensional submanifold of a symplectic surface is Lagrangian, because a 1-dimensional subspace of a symplectic vector space is always isotropic.

Exercise 2.2. If N is a submanifold of a symplectic manifold (M, ω) with inclusion map $i : N \hookrightarrow M$, then N is a Lagrangian submanifold of M if and only if $i^*\omega = 0$ and $\dim N = \frac{1}{2} \dim M$.

Exercise 2.3. (i) Let V be a vector space and W a subspace of V . Define a canonical symplectic form σ on $V \oplus V^*$ by $\sigma(v \oplus v^*, w \oplus w^*) = v^*(w) - w^*(v)$ where $v, w \in V$ and $v^*, w^* \in V^*$. Show that if $\text{Ann } W$ is the annihilator of W in V^* , then σ vanishes on $W \oplus \text{Ann } W$, and conclude that $W \oplus \text{Ann } W$ is a Lagrangian subspace of the symplectic vector space $(V \oplus V^*, \sigma)$.

(ii) Use the ideas in (i) to define a canonical symplectic structure λ on the cotangent bundle T^*M , and show that if N is a submanifold of M , then the annihilator of TN in T^*M is a Lagrangian submanifold of the symplectic manifold (T^*M, λ) .

Action of smooth maps

Recall that if $f : M \rightarrow N$ is a smooth map between smooth manifolds, then its differential df induces for each $p \in M$ a linear map $df_p : T_p M \rightarrow T_{f(p)} N$ whose matrix is the Jacobian matrix $(\partial f_i / \partial x_j(p))$, which is an $n \times m$ matrix if $\dim M = m$ and $\dim N = n$. The dual f_p^* of df gives a linear map $f_p^* =$

$(df_p)^* : T_{f(p)}^*N \rightarrow T_p^*M$ which carries $\theta \in T_{f(p)}^*N$ to $\theta \circ df_p \in T_p^*M$. If θ is a 2-form on N , then $f_p^*(\theta_p)(v_1, v_2) = \theta_p(df_p(v_1), df_p(v_2))$, where $v_1, v_2 \in T_pM$.

Darboux's theorem

A **symplectomorphism** of a symplectic manifold (M, ω) is a diffeomorphism $\phi : M \rightarrow M$ which preserves the symplectic form : $\phi^*\omega = \omega$. The group of symplectomorphisms of (M, ω) is denoted by $\text{Symp}(M, \omega)$.

Two symplectic manifolds (M, ω) and (M', ω') are called **equivalent**, if there is a diffeomorphism $\phi : M \rightarrow M'$ such that $\phi^*\omega' = \omega$. In general, it is not known when two given symplectic manifolds of the same dimension are equivalent. However, Darboux's theorem asserts that locally two symplectic manifolds of the same dimension are equivalent.

Theorem 2.4 (Darboux). *At each point p of a symplectic manifold (M, ω) of dimension $2n$ without boundary there is a coordinate chart $\phi : U \rightarrow \mathbb{R}^{2n}$ such that $\phi(p) = 0$ and $\phi^*\omega_0 = \omega|_U$, where ω_0 is the canonical symplectic form on \mathbb{R}^{2n} .*

The classical proof of Darboux's theorem is by induction on n in the spirit of Theorem 1.7 (see Arnold [1], and also Sternberg [5]). Here we indicate a proof of the theorem which was suggested by Moser in [3]. The proof uses Moser's stability theorem.

Theorem 2.5 (Moser's Stability Theorem). *Let M be a closed manifold, and ω_t , $1 \leq t \leq 1$, be a smooth family of cohomologous symplectic forms on M . Then there is a family of diffeomorphisms ψ_t of M such that*

$$\psi_0 = \text{Id} \quad \text{and} \quad \psi_t^*\omega_t = \omega_0.$$

Moreover, if $\omega_t(q) = \omega_0(q)$ for all points q on a compact submanifold N of M , then we may assume that $\psi_t = \text{Id}$ on N .

Here a closed manifold is a compact manifold without boundary, and family of cohomologous forms ω_t means that each $\omega_t - \omega_0$ is exact, that is $\omega_t - \omega_0 = d\sigma_t$ for some 1-forms σ_t , in other words the de Rham cohomology class $[\omega_t]$ is independent of t .

Actually, we need the following relative version of Moser's theorem.

Theorem 2.6 (Moser's Theorem – Relative Version). *Suppose that ω_0 and ω_1 are symplectic forms on a manifold M , and N is a compact submanifold of M so that $\omega_0(p) = \omega_1(p)$ at all points $p \in N$. Then there exist neighbourhoods U_0 and U_1 of N in M , and a diffeomorphism $\phi : U_0 \rightarrow U_1$ such that $\phi^*\omega_1 = \omega_0$, and ϕ restricted to N is the Id map.*

A proof of this theorem which is due to Weinstein may be found in [4], Theorem 3.6.6 (Darboux-Weinstein), p. 111. The proof of Darboux's theorem is obtained by applying this theorem to the case when N is a single point.

Hamiltonian flows

Let (M, ω) be a Symplectic manifold with tangent bundle TM . A vector field on M is a smooth function $X : M \rightarrow TM$ which assigns to each point $p \in M$ a tangent vector $X_p \in T_pM$. Let $\chi(M)$ denote the space of vector fields on M . Then any $X \in \chi(M)$ defines a 1-form on M by contraction $\iota(X)\omega : TM \rightarrow \mathbb{R}$ given by $\iota(X)\omega(v) = \omega(X_p, v)$, $v \in T_pM$. Since ω is non-degenerate, the homomorphism $T_pM \rightarrow T_p^*M$ given by $X_p \mapsto \iota(X_p)\omega = \bar{\omega}(X_p)$ is bijective. Thus there is a one-to-one correspondence $\chi(M) \rightarrow \Omega^1(M) : X \mapsto \iota(X)\omega$.

A vector field X is called **symplectic** if the 1-form $\iota(X)\omega$ is closed. Let $\chi(M, \omega)$ denote the space of symplectic vector fields. A classical result says that if M is closed, then $\chi(M, \omega)$ is the Lie algebra of the group $\text{Sym}(M, \omega)$ of symplectomorphisms of M (McDuff-Salamon [2], Proposition 3.2).

Let $H : M \rightarrow \mathbb{R}$ be a smooth function. We shall call H a **Hamiltonian function**. The function H determines a vector field $X_H : M \rightarrow TM$ by the identity

$$\iota(X_H)\omega = dH.$$

The vector field X_H is called the **Hamiltonian vector field** associated to the Hamiltonian function H . If M is a closed manifold, the vector field X_H generates a smooth 1-parameter group of diffeomorphisms $\phi_H^t \in \text{Diff}(M)$ satisfying

$$\frac{d}{dt}\phi_H^t = X_H \circ \phi_H^t, \quad \phi_H^0 = \text{Id}.$$

This is called the **Hamiltonian flow** associated to H . The identity

$$dH(X_H) = (\iota(X_H)\omega)(X_H) = \omega(X_H, X_H) = 0$$

shows that the vector field X_H is tangent to the level sets $\{H = \text{constants}\}$ of H .

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