Poisson - IV

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1. Duality between Lie algebroids and Poisson vector bundles

The duality between finite dimensional Lie algebras and linear Poisson structures on their duals has a manifestation in a larger class. We shall see that there is a duality between Lie algebroids structures on a vector bundle A and linear Poisson structures on the duals A^*.

Let $q : E \to M$ be a smooth vector bundle and let $q_* : E^* \to M$ be its dual vector bundle. We shall define two types of functions on E.

- Let α be a section of $q_* : E^* \to M$. Then it defines a function on $\ell_\alpha : E \to \mathbb{R}$ by $\ell_X(\alpha) = \alpha(X)$ for all $\alpha \in E^*$. It may be easily seen that ℓ_α is linear on fibres of E. Such functions ℓ_α, $\alpha \in \Gamma(E^*)$ are called linear functions on E.
- On the other hand, a function $f : M \to \mathbb{R}$ defines a function $f \circ q$ on E. We shall refer to this function as a pull-back function.

Definition 1.1. A Poisson structure on E is said to be a linear if the following conditions are satisfied:

(1) The bracket of a two linear functions on E is linear. This means given a pair of sections α, β of E^* there is a section γ of E^* such that $\{\ell_\alpha, \ell_\beta\} = \ell_\gamma$.
(2) The bracket of a linear function and a pull-back function is a pull-back function. This means that given a section α of E^* and a function f on M there exists a function g such that $\{\ell_\alpha, f \circ q\} = g \circ q$.

A vector bundle E with a linear Poisson structure is called a Poisson vector bundle.

Remark: A 1-form on A^* can be decomposed into two components: For any $(x, \alpha) \in A^*$ $T_{(x,\alpha)}A^* = T_xM \oplus A^*_x$ since A^*_x is a linear space. Therefore, $T^*_{(x,\alpha)}A^* = T^*_xM \oplus A_x$. Thus

$$T^*A^* = (q_*)^*A \oplus (q_*)^*T^*M.$$

Let X be a section of A. Then the 1-form $\delta\ell_X$ on A^* corresponds to the section $(q_*)^*X$ of $(q_*)^*A$ under the above identification. On the other hand 1-forms that are sections of $(q_*)^*T^*M$ are of the form $q_*^*\omega$ where ω is a 1-form on M. Thus it is enough to define Poisson brackets of functions which are either linear or pull-back. We define the Poisson bracket as in the statement.

Theorem 1.2. *Let (E, q, M) be a vector bundle with a linear Poisson structure π. Then the bracket on ΓE^* defined by $\ell_{[\alpha,\beta]} = \{\ell_\alpha, \ell_\beta\}$, $\alpha, \beta \in \Gamma E^*$, defines a Lie algebroid structure on E^*.*

Proof. Let E be a Poisson vector bundle. Given $\alpha, \beta \in \Gamma E^*$ define

$$[\alpha, \beta] = \gamma \quad \text{where } \{\ell_\alpha, \ell_\beta\} = \ell_\gamma.$$

For any section $\alpha \in \Gamma E^*$, $\{\ell_\alpha, \ \}$ is a derivation on E such that for any function $f \in C^\infty(M)$ there is a $g \in C^\infty(M)$ satisfying $\{\ell_\alpha, f \circ q\} = g \circ q$. Hence there corresponds a vector field $a(\alpha)$ on M such that

$$\{\ell_\alpha, f \circ q\} = a(\alpha)f \circ q.$$

[1]Lecture delivered at the workshop on Lie Groupoids and Lie Algebroid held at Stat.-Math. Unit, Indian Statistical Institute, Kolkata during December 10-21, 2012.

In order to show that a is defined by a vector bundle morphism we have prove that a is $C^\infty(M)$-linear.

Let $h \in C^\infty(M)$ and $\alpha \in \Gamma E^*$. Then for a section X of E we have $\ell_{h\alpha}(X) = h(q(X))\alpha(X)$ so that

$$\ell_{h\alpha} = (h \circ q)\ell_\alpha.$$

$$\begin{aligned} \{\ell_{h\alpha}, f \circ q\} &= \{(h \circ q)\ell_\alpha, f \circ q\} \\ &= \{h \circ q, f \circ q\}\ell_\alpha + (h \circ q)\{\ell_\alpha, f \circ q\} \\ &= (h \circ q)a(\alpha)f \circ q \\ &= (ha(\alpha)f) \circ q \end{aligned}$$

Therefore, $a(h\alpha) = ha(\alpha)$. It is easy to check that $a(\alpha + \beta) = a(\alpha) + a(\beta)$. Thus a is induced by a vector bundle morphism $E^* \to TM$ which will also be denoted by a.

We shall now verify that the bracket $[\ ,\ ]$ defines a Lie algebra structure on ΓE^*. Since the Poisson bracket is skew-symmetric, so is the bracket $[\ ,\ ]$. For $\alpha, \beta \in \Gamma E^*$ and a scalar λ, we have $\ell_{\lambda\alpha+\beta} = \lambda\ell_\alpha + \ell_\beta$. Therefore

$$\begin{aligned} [\alpha + \lambda\beta, \gamma] &= \{\ell_{\lambda\alpha+\beta}, \ell_\gamma\} \\ &= \lambda\{\ell_\alpha, \ell_\gamma\} + \{\ell_\beta, \ell_\gamma\} \\ &= \lambda[\alpha, \gamma] + [\beta, \gamma], \end{aligned}$$

since the Poisson bracket is bilinear.

To prove that a is a Lie algebra homomorphism we need to show $a([\alpha, \beta]) = [a(\alpha), a(\beta)]$, where on the right hand side we have Lie bracket of vector fields. For any smooth function $f \in C^\infty(M)$,

$$\begin{aligned} a([\alpha, \beta])f \circ q &= \{\ell_{[\alpha,\beta]}, f \circ q\} \\ &= \{\{\ell_\alpha, \ell_\beta\}, f \circ q\} \\ &= -\{\{\ell_\beta, f \circ q\}, \ell_\alpha\} - \{\{f \circ q, \ell_\alpha\}, \ell_\beta\} \\ &= -\{a(\beta)f \circ q, \ell_\alpha\} + \{a(\alpha)f \circ q, \ell_\beta\} \\ &= a(\alpha)(a(\beta)f) \circ q - a(\beta)(a(\alpha)f) \circ q \\ &= [a(\alpha), a(\beta)]f \circ q \end{aligned}$$

Since q is surjective and f is arbitrary, this implies that $a([\alpha, \beta]) = [a(\alpha), a(\beta)]$. Finally we show that $[\alpha, f\beta] = f[\alpha, \beta] + a(\alpha)f\beta$.

$$\begin{aligned} \{\ell_\alpha, \ell_{f\beta}\} &= \{\ell_\alpha, (f \circ q)\ell_\beta\} \\ &= (f \circ q)\{\ell_\alpha, \ell_\beta\} + \{\ell_\alpha, f \circ q\}\ell_\beta \\ &= (f \circ q)\ell_{[\alpha,\beta]} + (a(\alpha)f \circ q)\ell_\beta \\ &= \ell_{f[\alpha,\beta]} + \ell_{a(\alpha)f\beta} \end{aligned}$$

□

Similarly we can prove the following result.

Theorem 1.3. *Let $q : A \to M$ be a Lie algebroid with anchor map ρ. Then the dual vector bundle can be given a linear Poisson bracket having the following properties:*

(1) $\{\ell_X, \ell_Y\} = \ell_{[X,Y]}$ *and*
(2) $\{\ell_X, f \circ q_*\} = \rho(\alpha)f \circ q_*$ *for all sections X, Y of A and functions $f \in C^\infty(M)$.*

Thus there is a one to one correspondence between the Lie algebroids and Poisson vector bundles.

Example 1.4. Consider the tangent Lie Algebroid TM of any manifold M. The dual Poisson structure on the vector bundle $q_* : T^*M \to M$ corresponds to the canonical symplectic form on T^*M.

Let $x_1, \dots, x_n$ be a local coordinate system on some open subset U of M and $y_1, y_2, \dots, y_n$ denote the coordinate functions on the fibre of T^*M. Then the local description of the canonical symplectic form Ω on T^*M is $\sum dx_i \wedge dy_i$. Further, a vector field X can be expressed as $\sum X_i \frac{\partial}{\partial x_i}$, where X_i are smooth functions on U and a local 1-form is given as $\alpha = \sum \alpha_i \, dx_i$. Note that the function ℓ_X on T^*M takes the form $\sum y_i X_i(x)$ in local coordinates, which implies $d\ell_X = \sum X_i \, dy_i + \sum \frac{\partial X_i}{\partial x_i} y_i \, dx_i$. Hence,

$$X_{\ell_X} = -\sum_{i,j} \frac{\partial X_i}{\partial x_j} y_i \frac{\partial}{\partial y_j} + \sum X_i \frac{\partial}{\partial x_i}.$$

The linear Poisson bracket is given by

$$\{\ell_X, \ell_Y\} = \ell_{[X,Y]} \quad \{\ell_X, f \circ q_*\} = Xf \circ q_*, \quad \{f \circ q_*, g \circ q_*\} = 0,$$

where $X, Y \in \Gamma(TM)$ and $f, g \in C^\infty(M)$.

On the other hand Ω defines a Poisson structure π on T^*M which is given by

$$\pi(df, dg) = \Omega(X_f, X_g), \quad f, g \in C^\infty(T^*M)$$

Let $X, Y \in \Gamma(TM)$ and $\alpha = \sum \alpha_i \, dx_i|_x \; \ell_X(\alpha) = \alpha(X)$

$$\begin{aligned} \pi(d\ell_X, d\ell_Y) &= \Omega(X_{\ell_X}, X_{\ell_Y}) \\ &= \textstyle\sum_{i,j} \left(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right) y_i \\ &= \ell_{[X,Y]} \end{aligned}$$

noting that $[X, Y] = \sum_{i.j} [X_i \partial_i, Y_j \partial_j] = \sum \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_j \frac{\partial X_i}{\partial x_j} \right) \partial_i$, where ∂_i stands for $\frac{\partial}{\partial x_i}$. On the other hand for $f \in C^\infty(M)$ we have $d(f \circ q_*) = \sum_i \frac{\partial f}{\partial x_i} dx_i$. Therefore, for any $X \in \Gamma(TM)$

$$\begin{aligned} \pi(d\ell_X, d(f \circ q_*)) &= \Omega(X_{\ell_X}, X_{f \circ q_*}) \\ &= \textstyle\sum X_i \frac{\partial f}{\partial x_i} \\ &= (Xf) \circ q_*. \end{aligned}$$

Finally,

$$\pi(d(f \circ q_*), d(g \circ q_*)) = 0, \quad \text{for } f, g \in C^\infty(M).$$

Example 1.5. Let (M, π) be a Poisson manifold. Then T^*M is a Lie algebroid. The linear Poisson structure on TM satisfies the following properties: For any C^∞ function f on M, The Hamiltonian vector field $X_{\ell_{df}}$ on TM is the complete lift of the vector field X_f and $X_{f \circ q}$ is the vertical lift of X_f.

Definition 1.6. A submanifold C of a Poisson manifold (M, π) is said to be a coisotropic submanifold of M if it satisfies any of the following two equivalent conditions:

(1) The map $\bar{\pi} : T^*M \to TM$ maps $Ann(TC)$ into TC;
(2) If $f, g \in C^\infty(M)$ and f, g vanish on C then $\{f, g\}$ vanishes on C, where $\{\ ,\ \}$ is denotes the Poisson bracket associated to π.

Proposition 1.7. *Let $q : A \to M$ be a Lie algebroid and B be a vector sub-bundle of A. Then B is a Lie subalgebroid of A if and only if the annilhilator B^0 of B is a coisotropic submanifold of A^* with linear Poisson structure.*

Proof. We first assume that $B \to N$ is a Lie subalgebroids of A over an embedded submanifold N. The basic functions on A^* are of the form ℓ_X and $f \circ q_*$, where $X \in \Gamma(A)$ and $f \in C^\infty(M)$. Observe that

- ℓ_X vanishes on B^0 if and only if $X|_N \in \Gamma B$ and
- $f \circ q_*$ vanishes on B^0 if and only if f vanishes on N.

Let $\ell_X, \ell_Y, f \circ q_*$ vanish on B^0. Then $X|_N, Y|_N \in \Gamma B$ and f vanishes on N. Since B is a Lie subalgebroid of A, $[X|_N, Y|_N] = [X, Y]|_N$ and $a(X)|_N \in \Gamma B$. Hence, $\{\ell_X, \ell_Y\}|_{B^0} = \ell_{[X,Y]}|_{B^0} = 0$ and $\{\ell_X, f \circ q_*\}|_{B^0} = a(X)f \circ q_*|_{B^0} = 0$. This proves that B^0 is a coisotropic submanifold of A^*. The converse can also be proved in a similar fashion. □

Proposition 1.8. *Let A and B be two Lie algebroids over the same base M. A vector bundle morphism $F : A \to B$ is a Lie algebroid morphism if and only if $F^* : B^* \to A^*$ is a Poisson map, where B^* and A^* have dual Poisson structure on them.*

Proof. Let $F : A \to B$ be a Lie algebroid morphism. Then $\rho_B \circ F = \rho_A$. Let $F^* : B^* \to A^*$ denote the dual map. For $X, Y \in \Gamma A$, $\ell_X, \ell_Y : A^* \to \mathbb{R}$,

$$\begin{aligned}\{\ell_X \circ F^*, \ell_Y \circ F^*\} &= \{\ell_{FX}, \ell_{FY}\} \\ &= \ell_{F([X,Y])} = \ell_{[X,Y]} \circ F^*\end{aligned}$$

since $\ell_X \circ F^* = \ell_{FX}$. Secondly, for $f \in C^\infty(M)$ and $X \in \Gamma A$,

$$\begin{aligned}\{\ell_X \circ F^*, (f \circ q_A^*) \circ F^*\} &= \{\ell_X \circ F^*, f \circ q_B^*\} \\ &= \{\ell_{FX}, f \circ q_B^*\} \\ &= \rho_B(FX) f \circ q_B^* \\ &= (\rho_A(X) f \circ q_A^*) \circ F^* \\ &= \{\ell_X, f \circ q_A^*\} \circ F^*\end{aligned}$$

Finally, $\{(f \circ q_A^*) \circ F^*, (g \circ q_A^*) \circ F^*\} = \{f \circ q_B^*, g \circ q_B^*\} = 0$ and also $\{f \circ q_A^*, g \circ q_A^*\} \circ F^* = 0$. Thus F^* is a Poisson morphism. The converse also follows along the same line.

□

References

[1] Mackenzie K. C. H. *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press