Atiyah classes and homotopy algebras

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- Atiyah (1957): obstruction to existence of holomorphic connections
- Rozansky-Witten theory (1997)
  - invariants of 3-dim topological manifolds
  - explanation using Atiyah class (Kontsevich, **Kapranov**)
- Deformation quantization
  - Kontsevich
  - Dolgushev, Tamarkin, Tsygan
  - Calaque, Van den Bergh
  - Markarian, Căldăraru, Ramadoss
- HERE: new algebraic structures from Lie algebroid/algebra pairs via Atiyah classes

Atiyah class of a holomorphic vector bundle

- $E \to X$ holomorphic vector bundle
- $\mathscr{J}^1 E$ space of 1-jets of holomorphic sections of $E \to X$
- short exact sequence of holomorphic vector bundles over X

$$\boxed{0 \to T_X^* \otimes E \to \mathscr{J}^1 E \to E \to 0}$$

- Its extension class

$$\alpha_E \in \operatorname{Ext}_X^1(E, T_X^* \otimes E) \cong H^1(X; T_X^* \otimes \operatorname{End} E)$$

  is the Atiyah class of $E \to X$.
- Constitutes the obstruction to the existence of a holomorphic connection on $E \to X$.

Theorem (Kapranov): Let X be a complex manifold and E a holomorphic vector bundle over X. Then

- $T_X[-1]$ is a Lie algebra object in the derived category $D^b(X)$ of coherent sheaves of $\mathcal{O}_X$-modules,
- and $E[-1]$ is a module object over the Lie algebra object $T_X[-1]$ in $D^b(X)$.

Consequently, for any commutative $\mathcal{O}_X$-algebra $\mathcal{C}$, $\bigoplus_{i\geq 1} H^{i-1}(X; T_X \otimes \mathcal{C})$ is a graded Lie algebra of which $\bigoplus_{i\geq 1} H^{i-1}(X; E \otimes \mathcal{C})$ is a module.

A *Lie algebra object* in $D^b(X)$ is an object Λ of $D^b(X)$ together with a morphism $\lambda \in \operatorname{Hom}_{D^b(X)}(\Lambda \otimes \Lambda, \Lambda)$ such that

$$\lambda \circ \tau = -\lambda \quad \text{(skew-symmetry)},$$

$$\lambda \circ (\mathrm{id} \otimes \lambda) = \lambda \circ (\lambda \otimes \mathrm{id}) + \lambda \circ (\mathrm{id} \otimes \lambda) \circ (\tau \otimes \mathrm{id}) \quad \text{(Jacobi identity)},$$

where $\tau : \Lambda \otimes \Lambda \to \Lambda \otimes \Lambda$ is the transposition map.

Theorem (Kapranov): Suppose X is a Kähler manifold. Write T for $T_X^{1,0}$ and $\Omega^{0.k}(T)$ for the T-valued forms of type $(0,k)$ on X. The Dolbeault complex $\Omega^{0,\bullet}(T)$ is an $L_\infty[1]$ algebra.

An $L_\infty[1]$ *algebra* is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ endowed with a sequence $(\lambda_k)_{k=1}^{\infty}$ of multilinear maps $\lambda_k : \otimes^k V \to V$ of degree 1 sastifying the identity

$$\sum_{k=1}^{n} \sum_{\sigma\in\mathfrak{S}_k^{n-k}} \varepsilon(\sigma; v_1, \cdots, v_n) \\ \lambda_{1+n-k}\big(\lambda_k(v_{\sigma(1)}, \cdots, v_{\sigma(k)}), v_{\sigma(k+1)}, \cdots, v_{\sigma(n)}\big) = 0$$

for each $n \in \mathbb{N}$ and for any homogeneous vectors $v_1, v_2, \ldots, v_n \in V$.

V is an $L_\infty[1]$ algebra $\iff$ $V[-1]$ is an L_∞ algebra

Here $\mathfrak{S}_k^{n-k}$ denotes the set of $(k, n-k)$-shuffles, and $\varepsilon(\sigma; v_1, \cdots, v_n)$ denotes the Koszul sign of the permutation σ of the (homogeneous) vectors $v_1, v_2, \ldots, v_n$.

- A $(k, n-k)$-shuffle is a permutation σ of the set $\{1, 2, \cdots, n\}$ such that $\sigma(1) \leqslant \sigma(2) \leqslant \cdots \leqslant \sigma(k)$ and $\sigma(k+1) \leqslant \sigma(k+2) \leqslant \cdots \leqslant \sigma(n)$.
- The Koszul sign of a permutation σ of the (homogeneous) vectors $v_1, v_2, \ldots, v_n$ is determined by the relation $v_{\sigma(1)} \odot v_{\sigma(2)} \odot \cdots \odot v_{\sigma(n)} = \varepsilon(\sigma; v_1, \cdots, v_n) \cdot v_1 \odot v_2 \odot \cdots \odot v_n$.

Remark: If all λ_k are zero except for λ_1, (V, λ_1) is simply a cochain complex. If $\lambda_k = 0$ for $k \geq 3$, then $(V[-1], [-,-], d)$ is a differential graded Lie algebra, where $[x, y] = (-1)^{|x|+1}\lambda_2(x, y)$, and $d = \lambda_1$.

Theorem (Kapranov): Suppose X is a Kähler manifold. Write T for $T_X^{1,0}$ and $\Omega^{0.k}(T)$ for the T-valued forms of type $(0,k)$ on X. The Dolbeault complex $\Omega^{0,\bullet}(T)$ is an $L_\infty[1]$ algebra.

Outline of construction

Let $\nabla^{\mathbb{C}}$ be the complexification of its Levi-Cività connection.

Consider the curvature
$R_2 = R^{\nabla^{\mathbb{C}}} \in \Omega^{1,1}(\mathrm{End}\, T) \cong \Omega^{0,1}\big(\mathrm{Hom}(T\otimes T, T)\big)$
and its higher covariant derivatives
$R_n = \partial^{\nabla} R_{n-1} \in \Omega^{0,1}\big(\mathrm{Hom}(T^{\otimes n}, T)\big)$.

The n-th multibracket is given by

$$\begin{tikzcd}
\Omega^{0,j_1}(T)\otimes\cdots\otimes\Omega^{0,j_n}(T) \arrow[d, "\wedge"'] \arrow[drr, "\lambda_n"] & & \\
\Omega^{0,j_1+\cdots+j_n}(\otimes^n T) \arrow[rr, "R_n\in\Omega^{0,1}(\mathrm{Hom}(\otimes^n T, T))"'] & & \Omega^{0,j_1+\cdots+j_n+1}(T)
\end{tikzcd}$$

Kapranov's results can be extended to a much wider class of situations: *vector bundles over Lie pairs*.

Outline:

1. Definition and examples of vector bundles over Lie pairs
2. Atiyah class of a Lie pair
3. Generalizations of Kapranov's results

Definition: A Lie algebroid is

- a vector bundle A over a smooth (complex) manifold M
- together with a bundle map $A \xrightarrow{\rho} T_M (\otimes \mathbb{C})$
- such that $\Gamma(A)$ is a Lie algebra
- and $\Gamma(A) \xrightarrow{\rho} \mathfrak{X}(M)(\otimes \mathbb{C})$ is a morphism of Lie algebras.

Examples:

- $M = \{*\}, \quad A = \mathfrak{g}, \quad \rho = 0$
- $M =$ real manifold, $\quad A = T_M, \quad \rho = \mathrm{id}$
- $X =$ complex manifold, $\quad A = T_X^{0,1}, \quad \rho : T_X^{0,1} \hookrightarrow T_X \otimes \mathbb{C}$
- $M =$ Poisson manifold, $\quad A = T_M^*, \quad \rho = \pi^\sharp$

Lie algebroid $A \to M$ with anchor $A \xrightarrow{\rho} T_M$
Vector bundle $E \xrightarrow{\pi} M$

Two equivalent descriptions of an A-connection on E:

- **covariant derivative** $\Gamma(A) \times \Gamma(E) \xrightarrow{\nabla} \Gamma(E)$
  such that $\nabla_{fa} e = f(\nabla_a e)$
  and $\nabla_a(fe) = \rho(a)f \cdot e + f(\nabla_a e)$
- **infinitesimal action / horizontal lift**

$$\begin{tikzcd}
& \mathfrak{X}_\pi(E) \arrow[d, "\pi_*"] \\
\Gamma(A) \arrow[ur, "h"] \arrow[r, "\rho"'] & \mathfrak{X}(M)
\end{tikzcd}$$

$$\begin{tikzcd}
& T_{e_x} E \arrow[d, "\pi_*"] \\
A_x \arrow[ur, "{h(-,e_x)}"] \arrow[r, "\rho"'] & T_x M
\end{tikzcd}$$

commutative diagram of $C^\infty(M)$-modules

If ∇ is flat, or equivalently h is a morphism of Lie algebras, we say that E is an A-module.

Keep in mind:

- complex vector bundle E over complex manifold X
- $L = T_X \otimes \mathbb{C}$ and $A = T_X^{0,1}$

$$\boxed{E \text{ is an } A\text{-}\textit{module}} \iff \boxed{E \text{ is a } \textit{holomorphic} \text{ vector bundle}}$$

Definition: **Vector bundle over a Lie pair**

- Smooth manifold M
- Lie algebroid pair (L, A): a Lie algebroid L (over M) together with a Lie subalgebroid A
- Vector bundle $E \xrightarrow{\pi} M$ which is a module over the Lie algebroid A

Lemma (Bott connection): The quotient L/A of a Lie pair (L, A) is naturally an A-module.

$$0 \to A \to L \xrightarrow{\pi} L/A \to 0 \qquad \nabla_a\big(\pi(l)\big) = \pi\big([a, l]\big)$$

Remark: A matched pair of Lie algebroids $L = A \bowtie B$ can be seen as a Lie pair (L, A) together with a splitting $j : B \to L$ of the short exact sequence $0 \to A \to L \to B \to 0$, whose image $j(B)$ happens to be a Lie subalgebroid of L.

Examples:

1. Lie subalgebras in a Lie algebra
2. Lie bialgebras
3. Foliation $\mathcal{F}$ of a manifold M: $L = T_M$ and $A = T_{\mathcal{F}}$.
   $\alpha_{N_{\mathcal{F}}}$, Atiyah-Molino class
4. Complex manifold X: $L = T_X \otimes \mathbb{C}$ and $A = T_X^{0,1}$,
   E is any holomorphic vector bundle over X.
5. Infinitesimal Poisson action of a Lie algebra $\mathfrak{g}$ on a Poisson manifold M: matched pair of Lie algebroids $\left(T_M^*, M \rtimes \mathfrak{g}\right)$ [Lu].

- An L-jet (of order 1) on E extending the infinitesimal A-action (h) is a linear map $L_{\pi(e_x)} \xrightarrow{\phi} T_{e_x}E$ such that the diagram

$$\begin{tikzcd}
 & A_{\pi(e_x)} \arrow[dl] \arrow[dr, "h(-,e_x)"] & \\
L_{\pi(e_x)} \arrow[rr, "\phi"] \arrow[dr, "\rho"'] & & T_{e_x}E \arrow[dl, "\pi_{*e_x}"] \\
 & T_{\pi(e_x)}M &
\end{tikzcd}$$

  commutes.
- The "jet space" $\mathscr{J}^1_{L/A}E$ is the manifold whose points are h-extending L-jets on E.
- It is a vector bundle over M: the projection $\mathscr{J}^1_{L/A}E \to M$ maps $L_{\pi(e_x)} \xrightarrow{\phi} T_{e_x}E$ to $\pi(e_x)$.
- The smooth sections of $\mathscr{J}^1_{L/A}E \to E$ are the L-connections on E extending the infinitesimal A-action on E.

Theorem (Chen,S,Xu):

- The jet bundle $\mathscr{J}^1_{L/A}E$ is an A-module.
- It fits into a short exact sequence of A-modules

$$\boxed{0 \to A^\perp \otimes E \to \mathscr{J}^1_{L/A}E \to E \to 0}$$

$(A^\perp = (L/A)^*$ is the annihilator of A in $L^*)$.

Atiyah class: extension class

$$\alpha_E \in \operatorname{Ext}^1_{\mathcal{A}}(E, A^\perp \otimes E) \cong H^1_{\mathrm{CE}}(A; A^\perp \otimes \operatorname{End} E)$$

of this short exact sequence.

Geometrically, the obstruction to the existence of a *compatible* L-connection on E, which extends the given infinitesimal A-action.

Theorem (Bordemann):

- $\mathfrak{d} \supset \mathfrak{g}$ Lie algebras
- $D \supset G$ corresponding Lie groups

$$\boxed{\alpha_{\mathfrak{d}/\mathfrak{g}} = 0} \Longleftrightarrow \boxed{\exists\ G\text{-invariant connection on } D/G}$$

$\boxed{D^b(\mathcal{A})}$

- Lie algebroid A over manifold M
- $\mathcal{U}(A)$, its universal enveloping algebra
- The category $\mathcal{A}$ of left $\mathcal{U}(A)$-modules is an abelian category which contains all vector bundles over M endowed with an A-action, i.e. A-modules.
- $D^b(\mathcal{A})$, its derived category

Theorem (Chen,S,Xu): Let (L, A) be a Lie algebroid pair with quotient L/A and let E be an A-module. Then

- $L/A[-1]$ is a Lie algebra object in the derived category $D^b(\mathcal{A})$,
- and $E[-1]$ is a module object over the Lie algebra object $L/A[-1]$ in $D^b(\mathcal{A})$.

Consequently, for any commutative A-algebra $\mathcal{C}$, $\bigoplus_{i\geq 1} H^{i-1}_{\mathrm{CE}}(A; L/A \otimes \mathcal{C})$ is a graded Lie algebra of which $\bigoplus_{i\geq 1} H^{i-1}_{\mathrm{CE}}(A; E \otimes \mathcal{C})$ is a module.

The Atiyah class of L/A makes L/A into a Lie algebra object in the derived category $D^b(\mathcal{A})$.

Indeed,

$$\begin{aligned}\alpha_{L/A} &\in \operatorname{Ext}^1_{\mathcal{A}}(L/A, A^\perp \otimes L/A) \\ &\cong \operatorname{Ext}^1_{\mathcal{A}}(L/A \otimes L/A, L/A) \\ &\cong \operatorname{Hom}_{D^b(\mathcal{A})}(L/A[-1] \otimes L/A[-1], L/A[-1])\end{aligned}$$

defines a "Lie bracket" on $L/A[-1]$.

Moreover, if E is an A-module, its Atiyah class

$$\begin{aligned}\alpha_E &\in \operatorname{Ext}^1_{\mathcal{A}}(E, A^\perp \otimes E) \\ &\cong \operatorname{Ext}^1_{\mathcal{A}}(L/A \otimes E, E) \\ &\cong \operatorname{Hom}_{D^b(\mathcal{A})}(L/A[-1] \otimes E[-1], E[-1])\end{aligned}$$

defines a "representation" on $E[-1]$ of the "Lie algebra" $L/A[-1]$.

Any homotopy algebra behind the scene?

To understand the algebraic structure of $L/A[-1] \in D^b(\mathcal{A})$ at the cochain complex level, it is necessary to get hold of a cocycle representing the Atiyah class.

Atiyah cocycle associated to a connection

- Lie pair (L, A), A-module E
- L-connection ∇ on E extending given infinitesimal A-action
- $R^{\nabla} : \wedge^2 L \to \operatorname{End} E$, curvature of ∇
- Since E is an A-module, the restriction of R^{∇} to $\wedge^2 A$ vanishes. Hence the curvature induces a section R_E of $\Gamma(A^* \otimes A^{\perp} \otimes \operatorname{End} E)$, which is a 1-cocycle for the Lie algebroid A with values in the A-module $A^{\perp} \otimes \operatorname{End} E$.
- Its cohomology class is the Atiyah class $\alpha_E \in H^1_{\mathrm{CE}}(A; A^{\perp} \otimes \operatorname{End} E)$.
- The L-connection ∇ is *compatible* with the A-module structure of E if and only if $R_E = 0$.

Example:

- Let $\mathfrak{g}$ be a Lie subalgebra of a Lie algebra $\mathfrak{d}$.
- Given an $\mathfrak{g}$-module E (ie. a Lie algebra morphism $\boldsymbol{A} : \mathfrak{g} \to \operatorname{End} E$), and a $\mathfrak{d}$-connection on E extending it (i.e. a linear map $\boldsymbol{L} : \mathfrak{d} \to \operatorname{End} E$ whose restriction to $\mathfrak{g}$ is $\boldsymbol{A}$),
- The Atiyah class is the class of $\partial^{\mathfrak{g}} \boldsymbol{L}$ in the Chevalley-Eilenberg cohomology group $H^1_{\mathrm{CE}}(\mathfrak{g}; \mathfrak{g}^\perp \otimes \operatorname{End}(E))$.
- Note that $\boldsymbol{L} \in \mathfrak{d}^* \otimes \operatorname{End}(E)$ but, in general, $\boldsymbol{L} \notin \mathfrak{g}^\perp \otimes \operatorname{End}(E)$.

In particular:

- $\mathfrak{sl}_2(\mathbb{C})$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$[e,f] = h, \qquad [h,e] = 2e, \qquad [h,f] = -2f$$

- $\mathfrak{g}$, Lie subalgebra spanned by h and e
- quotient $\mathfrak{sl}_2(\mathbb{C})/\mathfrak{g}$ identified to nilpotent Lie subalgebra $\mathfrak{n}$ spanned by f
- $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{g} \oplus \mathfrak{n}$ is thus a matched pair of Lie algebras
- the $\mathfrak{g}$-module $\mathfrak{g}^\perp \otimes \mathrm{End}(\mathfrak{n}) \cong \mathrm{Hom}(\mathfrak{n} \otimes \mathfrak{n}, \mathfrak{n})$ is 1-dimensional
- $H^1_{\mathrm{CE}}(\mathfrak{g}; \mathfrak{g}^\perp \otimes \mathrm{End}(\mathfrak{n}))$ is a 1-dimensional vector space generated by the Atiyah class $\alpha_{\mathfrak{n}}$ of the $\mathfrak{g}$-module $\mathfrak{n}$

- Choose a splitting of $0 \to A \to L \to B \to 0$.
  It determines a 'connection' $\Gamma(B) \times \Gamma(A) \xrightarrow{\Delta} \Gamma(A)$.
- Choose an extension of the A-action on E to an L-connection ∇ on E and an extension of the A-action on B to an L-connection ∇ on B.
- The Atiyah cocycle $R_{L/A} \in \Gamma(A^* \otimes (L/A)^* \otimes \mathrm{End}(L/A))$, may be regarded as a bundle map $R_2 : L/A \otimes L/A \to A^* \otimes L/A$.
- A sequence $\{R_n\}_{n=2}^{\infty}$ of bundle maps

$$R_n : \otimes^n L/A \to \mathrm{Hom}(A, L/A)$$

is defined inductively by the relation

$$R_{n+1} = \partial^{\nabla} R_n, \ \text{for } n \geq 2.$$

Chosen splitting, Δ, and ∇ involved in definition of "covariant differential" ∂^{∇}.

Consider the sequence of k-ary operations $\lambda_k : \otimes^k V \to V$ $(k \in \mathbb{N})$ on the graded vector space $V = \bigoplus_{n=0}^{\infty} \Gamma(\wedge^n A^* \otimes B)$ defined by $\lambda_1 = \partial^A$ and, for $k \geq 2$,

$$\lambda_k(\xi_1 \otimes b_1, \cdots, \xi_k \otimes b_k) = (-1)^{|\xi_1|+\cdots+|\xi_k|} \xi_1 \wedge \cdots \wedge \xi_k \wedge R_k(b_1, \cdots, b_k),$$

where $b_1, \ldots, b_k \in \Gamma(B)$ and $\xi_1, \ldots, \xi_k$ are homogeneous elements of $\Gamma(\wedge^\bullet A^*)$.

Theorem (Chen,S,Xu): Let (L, A) be a Lie pair (with quotient $B := L/A$) and let E be an A-module.

- When endowed with the sequence of multibrackets $(\lambda_k)_{k \in \mathbb{N}}$ defined above, the graded vector space $V = \bigoplus_{n=0}^{\infty} \Gamma(\wedge^n A^* \otimes B)$ becomes a Leibniz$_\infty$[1] algebra.
- A sequence of multibrackets $(\mu_k)_{k \in \mathbb{N}}$ can be defined that endows the graded vector space $W = \bigoplus_{n=0}^{\infty} \Gamma(\wedge^n A^* \otimes E)$ with a structure of Leibniz$_\infty$[1] module over the Leibniz$_\infty$[1] algebra $(V, (\lambda_k)_{k \in \mathbb{N}})$.

Leibniz algebra = Lie algebra without skew-symmetry

Recall that a *graded Leibniz algebra* is a $\mathbb{Z}$-graded vector space

$$V = \bigoplus_{k \in \mathbb{Z}} V_k$$

equipped with a bilinear bracket

$$V \otimes V \xrightarrow{[-,-]} V$$

satisfying the graded Leibniz rule

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]],$$

for all homogeneous elements $x, y, z \in V$.

Leibniz$_\infty$ algebra $\quad = \quad$ L_∞ algebra without skew-symmetry

A **Leibniz$_\infty$[1] algebra** is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ endowed with a sequence $(\lambda_k)_{k=1}^{\infty}$ of multilinear maps $\lambda_k : \otimes^k V \to V$ of degree 1 sastifying the identity

$$0 = \sum_{1\leqslant j\leqslant k\leqslant n} \sum_{\sigma\in\mathfrak{S}_{k-j}^{j-1}} \varepsilon(\sigma; v_1, \cdots, v_{k-1})(-1)^{|v_{\sigma(1)}|+|v_{\sigma(2)}|+\cdots+|v_{\sigma(k-j)}|}$$

$$\lambda_{n-j+1}\Big(v_{\sigma(1)}, \cdots, v_{\sigma(k-j)}, \lambda_j\big(v_{\sigma(k+1-j)}, \cdots, v_{\sigma(k-1)}, v_k\big), v_{k+1}, \cdots, v_n\Big)$$

for each $n \in \mathbb{N}$ and for any homogeneous vectors $v_1, v_2, \ldots, v_n \in V$.

V is a Leibniz$_\infty$[1] algebra $\quad\Leftrightarrow\quad$ $V[-1]$ is a Leibniz$_\infty$ algebra

Kapranov's example

Suppose X is a Kähler manifold.
Lie pair: $L = T_X \otimes \mathbb{C}$ and $A = T_X^{0,1}$.
Let $\nabla^{\mathbb{C}}$ be the complexification of its Levi-Cività connection.
Write T for $L/A = T_X^{1,0}$.

In this context, the tensors $R_n \in \Omega^{0,1}\big(\mathrm{Hom}(T^{\otimes n}, T)\big)$ are the curvature $R_2 = R^{\nabla^{\mathbb{C}}} \in \Omega^{1,1}(\mathrm{End}\, T)$ and its higher covariant derivatives: $R_{i+1} = \partial^{\nabla} R_i$.

The n-th multibracket is given by

$$\begin{tikzcd}
\Omega^{0,j_1}(T) \otimes \cdots \otimes \Omega^{0,j_n}(T) \arrow[d, "\wedge"'] \arrow[drr, "\lambda_n"] & & \\
\Omega^{0,j_1+\cdots+j_n}(\otimes^n T) \arrow[rr, "R_n \in \Omega^{0,1}(\mathrm{Hom}(\otimes^n T, T))"'] & & \Omega^{0,j_1+\cdots+j_n+1}(T)
\end{tikzcd}$$

Theorem (Kapranov): The Dolbeault complex $\Omega^{0,\bullet}(T)$ of a Kähler manifold is an $L_\infty[1]$ algebra.

Example

- $\mathfrak{l} = \mathfrak{a} \bowtie \mathfrak{b}$, matched pair of Lie algebras
- trivial $\mathfrak{b}$-connection on $\mathfrak{b}$
- Atiyah cocycle: $R_B(a, b)b' = \nabla_{\nabla_b a} b'$
- $R_n \in \mathrm{Hom}(\otimes^n \mathfrak{b}, \mathfrak{a}^* \otimes \mathfrak{b})$ given by

$$R_n(b_1, b_2, b_3, \cdots, b_n) = \nabla_{\nabla_{b_{n-1}} \nabla_{b_{n-2}} \cdots \nabla_{b_1}(-)} b_n$$

- $\bigoplus_{i \geq 0}(\wedge^i \mathfrak{a}^* \otimes \mathfrak{b})[-1]$ becomes a Leibniz$_\infty[1]$ algebra

In particular, if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, then both $\bigoplus_{i \geq 0}(\wedge^i \mathfrak{g}^* \otimes \mathfrak{g}^*)[-1]$ and $\bigoplus_{i \geq 0}(\wedge^i \mathfrak{g} \otimes \mathfrak{g})[-1]$ are Leibniz$_\infty[1]$ algebras.

Theorem (Chen,S,Xu): IF (L, A) is a Lie algebroid pair s.t.

- the quotient L/A of the pair can be embedded in L as a Lie subalgebroid transversal to A
- and there exists a torsion-free flat L/A-connection ∇ on L/A,

THEN the multibrackets λ_n are skew-symmetric so that $\big(\bigoplus_n \Gamma(\wedge^n A^* \otimes L/A)\big)[-1]$ is an L_∞ algebra.

Example

- Lie algebra pair $(\mathfrak{gl}_n(\mathbb{C}), \mathfrak{u}(n))$
- $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}(n) \oplus b(n)$, where $b(n)$ denotes the Lie algebra of upper triangular matrices with real diagonal coefficients
  $(\mathfrak{u}(n), b(n))$ is a Lie bialgebra
- Torsion-free flat $b(n)$-connection on $b(n)$:

$$\nabla_X Y = XY, \quad \forall X, Y \in b(n)$$

- Theorem $\Rightarrow$ $\bigoplus(\wedge^{\bullet-1} b(n)) \otimes b(n)$ is an L_∞ algebra

Theorem (Laurent-Gengoux,S,Xu): The definition of the multibrackets λ_n (actually R_n) can be modified in such a way that 'Leibniz$_\infty$' becomes 'L_∞'. Choices involved in the construction yield different homotopy algebras, which are however canonically isomorphic.

Proof of [Chen,S,Xu] is computational.

Proof of [Laurent-Gengoux,S,Xu] is geometric (via groupoids).