Geometry of Maurer-Cartan Elements on Complex Manifolds

PING XU

PENNSTATE
1855

Workshop on Lie groupoids and Lie algebroids
Kolkata, December 2012

JOINT WITH ZHUO CHEN AND MATHIEU STIÉNON

	Quantum	Classical
Real	$*$-product	Poisson manifolds
Complex	stacks of algebroids (Kashiwara, Kontsevich)	Maurer-Cartan elements

A **complex Lie algebroid** consists of

- a complex vector bundle $A \to M$,
- a bundle map $a : A \to T_{\mathbb{C}}M$ called **anchor**
- and a Lie algebra **bracket** $[\cdot, \cdot]$ on the space of sections $\Gamma(A)$

such that

- a gives a Lie algebra homomorphism from $\Gamma(A)$ to $\mathfrak{X}_{\mathbb{C}}(M)$
- and the Leibniz rule

$$[u, fv] = \big(a(u)f\big)v + f[u, v]$$

holds $\forall\ f \in C^{\infty}(M, \mathbb{C})$ and $u, v \in \Gamma(A)$.

A — vector bundle over M

FACT:

$$\boxed{\begin{array}{c} A \text{ is a Lie algebroid} \\ \Updownarrow \\ (\Gamma(\wedge^{\bullet} A), \wedge, [,]) \text{ is a Gerstenhaber algebra} \\ \Updownarrow \\ (\Gamma(\wedge^{\bullet} A^*), \wedge, d) \text{ is a differential graded algebra} \end{array}}$$

Here d is a degree 1 derivation of the graded commutative algebra $(\Gamma(\wedge^{\bullet} A^*), \wedge)$ such that $d^2 = 0$:

$$\begin{aligned}(d\alpha)(u_0, u_1, \cdots, u_n) &= \sum_{i=0}^{n} (-1)^i a(u_i)\alpha(u_0, \cdots, \widehat{u_i}, \cdots, u_n) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([u_i, u_j], u_0, \cdots, \widehat{u_i}, \cdots, \widehat{u_j}, \cdots, u_n).\end{aligned}$$

$A \to M$ — complex vector bundle.

$(\Gamma(\wedge^\bullet A), \wedge, [,], d_*)$ is a differential Gerstenhaber algebra

$\Updownarrow$

Both A and A^* are Lie algebroids + compatibility conditions.

$\underbrace{\hphantom{xxxxxxxxxxxx}}$

LIE BIALGEBROID

Deformations of Lie bialgebroids and the Maurer-Cartan equation

- (A, A^*) — complex Lie bialgebroid
- $H \in \Gamma(\wedge^2 A)$

Twist the differential $d_* : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+1} A)$ by H:

$$d_*^H : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+1} A), \qquad d_*^H u = d_* u + [H, u].$$

THEOREM (LIU-WEINSTEIN-XU, 1997): If H satisfies the **Maurer-Cartan equation**

$$d_* H + \frac{1}{2}[H, H] = 0, \tag{1}$$

then $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot]), d_*^H)$ is again a differential Gerstenhaber algebra. Thus (A, A_H^*) is a Lie bialgebroid.

Any solution $H \in \Gamma(\wedge^2 A)$ to (1) is called a **Hamiltonian operator.**

The Lie algebroid structure on A^*_H can be described explicitly:

$$a^H_* = a_* + a \circ H^\sharp$$

and

$$[\alpha, \beta]^H_* = [\alpha, \beta]_* + [\alpha, \beta]_H.$$

Here

$$[\alpha, \beta]_H = L_{H^\sharp(\alpha)}\beta - L_{H^\sharp(\beta)}\alpha - d_*\langle H^\sharp(\alpha)|\beta\rangle,$$

for all $\alpha, \beta \in \Gamma(A^*)$.

MAURER-CARTAN ELEMENTS ON COMPLEX MANIFOLDS
EXTENDED POISSON STRUCTURES

$\boldsymbol{X}$ — complex manifold of complex dimension n with almost complex structure J

$T_{\mathbb{C}}\boldsymbol{X} = T\boldsymbol{X} \otimes \mathbb{C}$
$T^*_{\mathbb{C}}\boldsymbol{X} = T^*\boldsymbol{X} \otimes \mathbb{C}$

$\mathbb{J} : T_{\mathbb{C}}\boldsymbol{X} \to T_{\mathbb{C}}\boldsymbol{X}$ — $\mathbb{C}$-linear extension of J
$T^{1,0}\boldsymbol{X}$ and $T^{0,1}\boldsymbol{X}$ — its eigenbundles ($\lambda = \pm\sqrt{-1}$)

$T^{p,q}\boldsymbol{X} = \wedge^p T^{1,0}\boldsymbol{X} \otimes \wedge^q T^{0,1}\boldsymbol{X}$
$(T^{p,q}\boldsymbol{X})^* = \wedge^p (T^{1,0}\boldsymbol{X})^* \otimes \wedge^q (T^{0,1}\boldsymbol{X})^*$

Consider

$$A = T^{1,0}\boldsymbol{X} \oplus (T^{0,1}\boldsymbol{X})^*, \qquad A^* = T^{0,1}\boldsymbol{X} \oplus (T^{1,0}\boldsymbol{X})^*.$$

Endow A with a complex Lie algebroid structure as follows:

1. anchor is Pr_1
2. bracket fully determined by
   - $[\theta, \omega] = 0, \quad \forall \theta, \omega \in \Omega^{0,1}$
   - $[X, Y]$ defined in the usual way, $\quad \forall X, Y \in \Gamma(T^{1,0}\boldsymbol{X})$
   - $[X, \theta] = 0, \quad \forall X \in \Gamma(T^{1,0}\boldsymbol{X})$ and $\theta \in \Omega^{0,1}$ s.t. $\bar{\partial} X = 0$ and $\partial\theta = 0$.

Similarly, A^* may also be endowed with a complex Lie algebroid structure.

We have

$$\wedge^k A = \bigoplus_{i+j=k} T^{i,0}\boldsymbol{X} \otimes (T^{0,j}\boldsymbol{X})^*,$$
$$\wedge^k A^* = \bigoplus_{i+j=k} T^{0,i}\boldsymbol{X} \otimes (T^{j,0}\boldsymbol{X})^*.$$

The Lie algebroid differentials d_* and d induced by the Lie algebroid structures on A^* and A are the usual $\bar{\partial}$- and ∂-operators, respectively:

$$\bar{\partial}: \ \Omega^{0,j}(T^{i,0}\boldsymbol{X}) \to \Omega^{0,j+1}(T^{i,0}\boldsymbol{X}),$$
$$\partial: \ \Omega^{j,0}(T^{0,i}\boldsymbol{X}) \to \Omega^{j+1,0}(T^{0,i}\boldsymbol{X}).$$

$$A = T^{1,0}\boldsymbol{X} \oplus (T^{0,1}\boldsymbol{X})^* \qquad A^* = T^{0,1}\boldsymbol{X} \oplus (T^{1,0}\boldsymbol{X})^*$$

FACT: (A, A^*) is a Lie bialgebroid

DEFINITION: An **extended Poisson manifold** $(\boldsymbol{X}, H)$ is a complex manifold $\boldsymbol{X}$ equipped with an $H \in \Gamma(\wedge^2 A)$ which is a Hamiltonian operator with respect to (A, A^*), i.e.

$$\bar{\partial} H + \frac{1}{2}[H, H] = 0. \tag{2}$$

H is called an **extended Poisson structure.**

Write

$$H = \pi + \theta + \omega,$$

where

- $\pi \in \Gamma(T^{2,0}\boldsymbol{X})$;
- $\theta \in \Gamma(T^{1,0}\boldsymbol{X} \otimes (T^{0,1}\boldsymbol{X})^*)$;
- $\omega \in \Gamma((T^{0,2}\boldsymbol{X})^*)$.

Then

$$\boxed{H = \pi + \theta + \omega \text{ is extended Poisson}} \iff \begin{cases} [\omega, \omega] = 0 \\ \bar{\partial}\omega + [\omega, \theta] = 0 \\ \bar{\partial}\theta + \frac{1}{2}[\theta, \theta] + [\omega, \pi] = 0 \\ \bar{\partial}\pi + [\theta, \pi] = 0 \\ [\pi, \pi] = 0 \end{cases}$$

REMARK: If $\pi + \theta + \omega$ is a solution of the MC equation, then so is $\lambda\pi + \theta + \lambda^{-1}\omega$ for every $\lambda \in \mathbb{C}^\times$.

EXAMPLES:

1. $H = \pi$ is an extended Poisson $\Leftrightarrow$ π is a holomorphic Poisson bivector field.
2. $H = \theta$ is an extended Poisson $\Leftrightarrow \bar{\partial}\theta + \frac{1}{2}[\theta, \theta] = 0$.
   Moreover, if $\overline{\theta}^\flat \circ \theta^\flat - \mathrm{I}$ is invertible, θ is equivalent to a deformed complex structure.
3. $H = \omega$ is an extended Poisson $\Leftrightarrow \bar{\partial}\omega = 0$.

MORPHISMS OF EXTENDED POISSON STRUCTURES (VIA COISOTROPIC SUBMANIFOLDS)

Coisotropic submanifolds

$\boldsymbol{Y} \subset \boldsymbol{X}$ — complex submanifold

$$(T^{1,0}\boldsymbol{Y})^{\perp} = \left\{ \xi \in (T^{1,0}\boldsymbol{X})^{*}|_{\boldsymbol{Y}} \text{ s.t. } \langle \xi | v \rangle = 0, \ \forall v \in T^{1,0}\boldsymbol{Y} \right\}$$

DEFINITION: A complex submanifold $\boldsymbol{Y}$ of an extended Poisson manifold $(\boldsymbol{X}, H)$ is called **coisotropic** if

$$H(v_1, v_2) = 0, \quad \forall v_1, v_2 \in T^{0,1}\boldsymbol{Y} \oplus (T^{1,0}\boldsymbol{Y})^{\perp}.$$

1. IF $H = \pi$, THEN $\boldsymbol{Y}$ is coisotropic $\Leftrightarrow$ it is coisotropic in the usual sense, i.e. $\pi^\sharp((T^{1,0}\boldsymbol{Y})^\perp) \subseteq T^{1,0}\boldsymbol{Y}$ or $\pi(\xi_1, \xi_2) = 0$, $\forall \xi_1, \xi_2 \in (T^{1,0}\boldsymbol{Y})^\perp$.
2. IF $H = \omega$, THEN $\boldsymbol{Y}$ is coisotropic $\Leftrightarrow$ $\iota^*\omega = 0$, where $\iota : \boldsymbol{Y} \to \boldsymbol{X}$ is the embedding map.
3. IF $H = \theta$, THEN $\boldsymbol{Y}$ is coisotropic $\Leftrightarrow$ $\theta^\flat(T^{0,1}\boldsymbol{Y}) \subseteq T^{1,0}\boldsymbol{Y}$.

Morphisms of extended Poisson structures

Let $(\boldsymbol{X}_1, H_1)$ and $(\boldsymbol{X}_2, H_2)$ be two extended Poisson manifolds. A holomorphic map $f: \boldsymbol{X}_1 \to \boldsymbol{X}_2$ is called an **extended Poisson map** if its graph

$$G_f = \{(x, f(x)) \mid x \in \boldsymbol{X}_1\}$$

is a coisotropic submanifold of $\boldsymbol{X}_1 \times \boldsymbol{X}_2^\dagger$, where $\boldsymbol{X}_2^\dagger$ means $(\boldsymbol{X}_2, H_2^\dagger = -\pi + \theta - \omega)$.

FACTS:

1. Let $f : \boldsymbol{X}_1 \to \boldsymbol{X}_2$ be a holomorpic map between two extended Poisson manifolds $(\boldsymbol{X}_1, H_1)$ and $(\boldsymbol{X}_2, H_2)$ with $H_i = \pi_i + \theta_i + \omega_i$, $i = 1, 2$.

$$\boxed{f \text{ is an extended Poisson map}} \iff \begin{cases} f_*\pi_1 = \pi_2 \\ f^*\omega_2 = \omega_1 \\ f_* \circ \theta_1^\flat = \theta_2^\flat \circ f_* \end{cases}$$

2. The composition of two extended Poisson maps is again an extended Poisson map.

Homology and cohomology of extended Poisson manifolds

Lichnerowicz-Poisson cohomology

$H^\bullet(\boldsymbol{X}, H)$ = cohomology of the Lie algebroid A_H^*

(aka tangent cohomology)

It is the cohomology of the cochain complex

$$\cdots \xrightarrow{\bar{\partial}^H} \Gamma(\wedge^k A) \xrightarrow{\bar{\partial}^H} \Gamma(\wedge^{k+1} A) \xrightarrow{\bar{\partial}^H} \cdots ,$$

where

$$\Gamma(\wedge^k A) = \bigoplus_{i+j=k} \Omega^{0,j}(T^{i,0}\boldsymbol{X})$$

and

$$\bar{\partial}^H = \bar{\partial} + [H, \cdot].$$

PARTICULAR CASE I

$H = \pi$ — holomorphic Poisson bivector field

$A_H^* = T^{0,1}\boldsymbol{X} \bowtie (T^{1,0}\boldsymbol{X})_\pi^*$ “matched pair”

$H^\bullet(\boldsymbol{X}, H) =$ total cohomology of the double complex

$$
\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\uparrow d_\pi & & \uparrow d_\pi & & \uparrow d_\pi & & \\
\Omega^{0,0}(X, T^{2,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,1}(X, T^{2,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,2}(X, T^{2,0}X) & \xrightarrow{\bar\partial} & \cdots \\
\uparrow d_\pi & & \uparrow d_\pi & & \uparrow d_\pi & & \\
\Omega^{0,0}(X, T^{1,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,1}(X, T^{1,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,2}(X, T^{1,0}X) & \xrightarrow{\bar\partial} & \cdots \\
\uparrow d_\pi & & \uparrow d_\pi & & \uparrow d_\pi & & \\
\Omega^{0,0}(X, T^{0,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,1}(X, T^{0,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,2}(X, T^{0,0}X) & \xrightarrow{\bar\partial} & \cdots
\end{array}
$$

EXAMPLE

Consider $X = \mathbb{CP}^1 \times \mathbb{CP}^1$

- $\dim H^0(X, \wedge^2 TX) = 9$
- In the affine chart

$$U = \{([1 : x], [1 : w])\},$$

any holomorphic bivector field reads as

$$\pi = (a_1 + a_2 x + a_3 x^2 + a_4 w + a_5 xw + a_6 x^2 w + a_7 w^2 + a_8 xw^2 + a_9 x^2 w^2) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w},$$

where a_i, $i = 1, 2, ...9$ are constant numbers.

THEOREM (HONG & XU): The Poisson cohomology of $(\boldsymbol{X} = \mathbb{CP}^1 \times \mathbb{CP}^1, \pi)$ is

$$\begin{aligned} \dim H^0_\pi(X) &= 1 \\ \dim H^1_\pi(X) &= 6 - rank(M) \\ \dim H^2_\pi(X) &= 9 - rank(M) \\ \dim H^i_\pi(X) &= 0 \quad (i \geq 3) \end{aligned}$$

where M is the matrix

$$M = \begin{pmatrix} -a_2 & a_1 & 0 & -a_4 & a_1 & 0 \\ -2a_3 & 0 & 2a_1 & -a_5 & a_2 & 0 \\ 0 & -a_3 & a_2 & -a_6 & a_3 & 0 \\ -a_5 & a_4 & 0 & -2a_7 & 0 & 2a_1 \\ -2a_6 & 0 & 2a_4 & -2a_8 & 0 & 2a_2 \\ 0 & -a_6 & a_5 & -2a_9 & 0 & 2a_3 \\ -a_8 & a_7 & 0 & 0 & -a_7 & a_4 \\ -2a_9 & 0 & 2a_7 & 0 & -a_8 & a_5 \\ 0 & -a_9 & a_8 & 0 & -a_9 & a_6 \end{pmatrix}$$

Particular case II

$H = \theta \in \Omega^{0,1}(T^{1,0}\boldsymbol{X})$ — a Maurer-Cartan element such that $\overline{\theta}^\flat \circ \theta^\flat - \mathrm{I}$ is invertible

Then

$$H^k(\boldsymbol{X}, H) \cong \bigoplus_{i+j=k} H^i(\boldsymbol{X}, \wedge^j T_\theta \boldsymbol{X}).$$

$T_\theta \boldsymbol{X}$ — the holomorphic tangent bundle of the deformed complex manifold $\boldsymbol{X}$

In particular, if $H = 0$, then

$$H^k(\boldsymbol{X}, H) \cong \bigoplus_{i+j=k} H^i(\boldsymbol{X}, \wedge^j T\boldsymbol{X}).$$

Koszul-Brylinski homology

$H_\bullet(\boldsymbol{X}, H)$ = cohomology of the Lie algbd A_H^* with coeff in $\mathscr{L}$

$\mathscr{L} \otimes \mathscr{L} = Q_{A_H^*}$ — Evens-Lu-Weinstein module

$B \to \boldsymbol{X}$ — complex Lie algebroid with anchor map a

EVENS-LU-WEINSTEIN MODULE

$$\boxed{Q_B = \wedge^{\mathrm{top}} B \otimes \wedge^{\mathrm{top}} T^*_{\mathbb{C}} \boldsymbol{X}}$$

The representation of B on Q_B is given by

$$\nabla_\alpha(\alpha_1 \wedge \cdots \wedge \alpha_n \otimes \mu) = \sum_{i=1}^{n} \left(\alpha_1 \wedge \cdots \wedge [\alpha, \alpha_i] \wedge \cdots \wedge \alpha_n \otimes \mu\right) + \alpha_1 \wedge \cdots \wedge \alpha_n \otimes L_{a(\alpha)} \mu,$$

where $\alpha,\ \alpha_1, \cdots, \alpha_n \in \Gamma(B)$, $\mu \in \Gamma(\wedge^{\mathrm{top}} T^*_{\mathbb{C}} \boldsymbol{X})$.

Now

$$A_H^* = T^{0,1}\boldsymbol{X} \oplus (T^{1,0}\boldsymbol{X})^*,$$

$$Q_{A_H^*} = \wedge^n (T^{1,0}\boldsymbol{X})^* \otimes \wedge^n (T^{1,0}\boldsymbol{X})^*.$$

Accordingly,

$$\mathscr{L} = Q_{A_H^*}^{\frac{1}{2}} \cong \wedge^n (T^{1,0}\boldsymbol{X})^*.$$

Associated differential operator:

$$\breve{d}_*^H : \Gamma(\wedge^k A \otimes \mathscr{L}) \to \Gamma(\wedge^{k+1} A \otimes \mathscr{L}).$$

LEMMA: The diagram

$$\begin{array}{ccc} \Gamma(\wedge^k A \otimes \mathscr{L}) & \xrightarrow{\breve{d}_*^H} & \Gamma(\wedge^{k+1} A \otimes \mathscr{L}) \\ \tau \downarrow \cong & & \cong \downarrow \tau \\ \bigoplus_{i-j=n-k} \Omega^{i,j} & \xrightarrow[\bar{\partial}+\lfloor \partial, \iota_H \rfloor]{} & \bigoplus_{i-j=n-k-1} \Omega^{i,j} \end{array}$$

commutes.

$$\iota_H \lambda = -H \cdot \lambda$$

"$\cdot$" — Clifford action

PROPOSITION: The Koszul-Brylinski homology $H_k(\boldsymbol{X}, H)$ of the extended Poisson manifold $(\boldsymbol{X}, H)$ is isomorphic to the cohomology of the cochain complex

$$\Big(\bigoplus_{i-j=n-k} \Omega^{i,j} \, , \; \bar{\partial} + \lfloor \partial, \iota_H \rfloor \Big).$$

Example I (S, arXiv:0903.5065)

$H = \pi$ — holomorphic Poisson bivector field

Then $H_\bullet(\boldsymbol{X}, H)$ is isomorphic to the total cohomology of the double complex

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Omega^{n-k+1,0} & \xrightarrow{(-1)^k \partial_\pi} & \Omega^{n-k,0} & \xrightarrow{(-1)^{k+1} \partial_\pi} & \Omega^{n-k-1,0} & \longrightarrow & \cdots \\
 & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial & & \\
\cdots & \longrightarrow & \Omega^{n-k+1,1} & \xrightarrow{(-1)^{1+k} \partial_\pi} & \Omega^{n-k,1} & \xrightarrow{(-1)^{1+k+1} \partial_\pi} & \Omega^{n-k-1,1} & \longrightarrow & \cdots \\
 & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial & & \\
\cdots & \longrightarrow & \Omega^{n-k+1,2} & \xrightarrow{(-1)^{2+k} \partial_\pi} & \Omega^{n-k,2} & \xrightarrow{(-1)^{2+k+1} \partial_\pi} & \Omega^{n-k-1,2} & \longrightarrow & \cdots \\
 & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

$$\partial_\pi = [\partial, i_\pi]$$

THEOREM (S): For any holomorphic Poisson bivector field π on $X = \mathbb{CP}^1 \times \mathbb{CP}^1$, we have

$$H_0(X, \pi) = 0$$
$$H_1(X, \pi) = 0$$
$$H_2(X, \pi) \cong \mathbb{C}^4$$
$$H_3(X, \pi) = 0$$
$$H_4(X, \pi) = 0.$$

Example II

$H = \omega \in \Omega^{0,2}(\boldsymbol{X})$ with $\bar{\partial}\omega = 0$

Then

$$H_\bullet(\boldsymbol{X}, H) \cong H^\bullet\Big(\bigoplus_{i-j=n-k} \Omega^{i,j}, \bar{\partial} + \partial\omega\wedge\Big)$$

— the twisted Dolbeault cohomology

Example III

$H = \theta \in \Omega^{0,1}(T^{1,0}\boldsymbol{X})$ — a Maurer-Cartan element such that $\overline{\theta}^{\flat} \circ \theta^{\flat} - \mathrm{I}$ is invertible

Then

$$H_k(\boldsymbol{X}, \theta) \cong \bigoplus_{j-i=n-k} H_\theta^{i,j}(\boldsymbol{X}),$$

where $H_\theta^{i,j}(\boldsymbol{X})$ is the Dolbeault cohomology of the deformed complex structure.

Evens-Lu-Weinstein duality

- $\boldsymbol{X}$ — compact complex manifold
- B — a complex Lie algebroid over $\boldsymbol{X}$ with $\mathrm{rank}_{\mathbb{C}} B = r$

There is a natural map

$$\phi : \Gamma(\wedge^k B^* \times Q_B^{\frac{1}{2}}) \otimes \Gamma(\wedge^{\mathrm{top}-k} B^* \times Q_B^{\frac{1}{2}}) \to \Gamma(\wedge^{\mathrm{top}} B^* \otimes Q_B) \cong \Gamma(\wedge^{\mathrm{top}} T_{\mathbb{C}}^* \boldsymbol{X})$$

Integrating, one gets the pairing

$$\begin{aligned} \Gamma(\wedge^k B^* \times Q_B^{\frac{1}{2}}) \otimes \Gamma(\wedge^{\mathrm{top}-k} B^* \times Q_B^{\frac{1}{2}}) &\to \mathbb{C}, \\ \xi \otimes \eta &\mapsto \int_{\boldsymbol{X}} \phi(\xi \otimes \eta). \end{aligned} \tag{3}$$

THEOREM (EVENS-LU-WEINSTEIN 1997, BLOCK 2005): For a complex Lie algebroid B over a compact manifold $\boldsymbol{X}$, map (3) induces a pairing

$$H^k(B, Q_B^{\frac{1}{2}}) \otimes H^{\mathrm{top}-k}(B, Q_B^{\frac{1}{2}}) \to \mathbb{C}.$$

Moreover,

$\boxed{B \text{ is an elliptic Lie algebroid}} \quad \Rightarrow \quad \boxed{\text{pairing is non-degenerate}}$

A complex Lie algebroid B is called **elliptic**,
if $\mathrm{Re} \circ a_B : \; B \to T\boldsymbol{X}$ is surjective.

$$a_B : B \to T_{\mathbb{C}}\boldsymbol{X} \quad \text{— anchor map}$$
$$\mathrm{Re} : T_{\mathbb{C}}\boldsymbol{X} \to T\boldsymbol{X} \quad \text{— taking real part}$$

MAIN THEOREM: Assume that $(\boldsymbol{X}, H)$ is a compact extended Poisson manifold of complex dimension n.

1. Then the map

$$\Omega^{i,j} \otimes \Omega^{k,l} \to \mathbb{C} : \zeta \otimes \eta \mapsto \int_{\boldsymbol{X}} (\zeta \wedge \eta)^{\text{top}}$$

   induces a pairing on the Koszul-Brylinski Poisson homology:

$$H_k(\boldsymbol{X}, H) \otimes H_{2n-k}(\boldsymbol{X}, H) \to \mathbb{C} \tag{4}$$

2. And if

$$F = (\text{conjugation} + \theta^{\flat}) \oplus \pi^{\sharp} \ : \ T^{0,1}\boldsymbol{X} \oplus (T^{1,0}\boldsymbol{X})^* \to T^{1,0}\boldsymbol{X},$$

   where $H = \pi + \theta + \omega$, is surjective, then
   - $H^{\bullet}(\boldsymbol{X}, H)$ and $H_{\bullet}(\boldsymbol{X}, H)$ are all finite-dimensional;
   - the pairing (4) is non-degenerate.

NOTE:

1. If the θ-term is missing, i.e. $H = \pi + \omega$, then F is automatically surjective.
2. If $H = 0$, we recover the Serre duality pairing on

$$H_k(\boldsymbol{X}, H) = \bigoplus_{j-i=n-k} H^{i,j}(\boldsymbol{X}).$$

QUESTION: What is $\sum_k (-1)^k \dim H_k(\boldsymbol{X}, H)$?

For a compact holomorphic Poisson manifold (X, π), we have

$$\sum_k (-1)^k \dim H_k(X, \pi) = (-1)^n \chi(X),$$

where $\chi(X)$ denotes the standard Euler characteristic of X. (S, `arXiv:0903.5065`)

Modular classes

modular class of a complex Lie algebroid B	=	obstruction class for the Evens-Lu-Weinstein module Q_B to be trivial

- B — complex Lie algebroid over $\boldsymbol{X}$ with $\operatorname{rank}_{\mathbb{C}} B = r$ and $\dim \boldsymbol{X} = m$
- $Q_B = \wedge^r B \otimes \wedge^m T^*_{\mathbb{C}} \boldsymbol{X}$ — ELW module of B

Consider the complex of sheaves over $\boldsymbol{X}$

$$\hat{\mathcal{S}}^0 \xrightarrow{\hat{d}_B} \mathcal{S}^1 \xrightarrow{d_B} \mathcal{S}^2 \cdots \xrightarrow{d_B} \mathcal{S}^r \tag{5}$$

where

- $\hat{\mathcal{S}}^0$ — sheaf of nowhere vanishing smooth complex valued functions on $\boldsymbol{X}$
- $\mathcal{S}^\bullet$ — sheaf of sections of $\wedge^\bullet B^*$ (soft sheaf)
- $\hat{d}_B f = d_B \log f = \frac{d_B f}{f}, \quad \forall f \in C^\infty(U, \mathbb{C}^\times), \quad U \subset \boldsymbol{X}$
- d_B — usual Lie algebroid cohomology differential
- $\hat{H}^\bullet(B, \mathbb{C}^\times)$ — hypercohomology of this complex of sheaves

$\hat{H}^\bullet(B, \mathbb{C}^\times)$ can be computed by the total cohomology of the Čech double complex

$$\begin{array}{ccccccc}
\check{C}^0(\mathscr{U}; \hat{\mathcal{S}}^0) & \xrightarrow{\hat{d}_B} & \check{C}^0(\mathscr{U}; \mathcal{S}^1) & \xrightarrow{d_B} & \check{C}^0(\mathscr{U}; \mathcal{S}^2) & \xrightarrow{d_B} & \cdots \\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
\check{C}^1(\mathscr{U}; \hat{\mathcal{S}}^0) & \xrightarrow{\hat{d}_B} & \check{C}^1(\mathscr{U}; \mathcal{S}^1) & \xrightarrow{d_B} & \check{C}^1(\mathscr{U}; \mathcal{S}^2) & \xrightarrow{d_B} & \cdots \\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
\check{C}^2(\mathscr{U}; \hat{\mathcal{S}}^0) & \xrightarrow{\hat{d}_B} & \check{C}^2(\mathscr{U}; \mathcal{S}^1) & \xrightarrow{d_B} & \check{C}^2(\mathscr{U}; \mathcal{S}^2) & \xrightarrow{d_B} & \cdots \\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \\
\cdots & & \cdots & & \cdots & &
\end{array} \tag{6}$$

where

- $\mathscr{U} = \{U_i\}_{i \in I}$ — good covering of $\boldsymbol{X}$
- δ — usual Čech coboundary operator

MODULAR CLASS OF B

1. ω_i — nowhere vanishing sections of Q_B over U_i
2. $\exists!\ f_{ij} \in C^\infty(U_{ij}, \mathbb{C}^\times)$ such that $\omega_i = f_{ij}\omega_j$
3. $\xi_i \in \Gamma(B^*|_{U_i})$ s.t. $\nabla_X \omega_i = \langle \xi_i | X \rangle \omega_i$, $\forall X \in \Gamma(B|_{U_i})$ — modular 1-form on U_i corresponding to ω_i

Hence

$$\xi_i = \xi_j + \frac{d_B f_{ij}}{f_{ij}} = \xi_j + \hat{d}_B f_{ij}$$

$$\Downarrow$$

(ξ_i, f_{ij}) is a 1-cocycle of the double complex (6)

$$\Downarrow$$

defines a class $\boxed{\text{mod}\ (B) = [(\xi_i, f_{ij})] \in \hat{H}^1(B, \mathbb{C}^\times)}$

A complex Lie algebroid B is said to be **unimodular** if its modular class vanishes.

Because the sheaves $\mathcal{S}^\bullet$ are soft, the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \to 0$$

induces the long exact sequence of cohomology groups

$$\cdots \to H^1(B, \mathbb{C}) \to \hat{H}^1(B, \mathbb{C}^\times) \xrightarrow{\tau} H^2(M, \mathbb{Z}) \to \cdots .$$

We have

$$\tau(\mathrm{mod}(B)) = c_1(Q_B).$$

B is unimodular

$$\Updownarrow$$

Q_B is isomorphic to the trivial module $\mathbb{C}$

$$\Updownarrow$$

$c_1(Q_B) = 0$

and $\exists$ a nowhere vanishing $\omega \in \Gamma(Q_B)$

s.t. $\nabla_X \omega = 0, \forall X \in \Gamma(B)$

DEFINITION: An extended Poisson manifold $(\boldsymbol{X}, H)$ is unimodular if its corresponding complex Lie algebroid A_H^* is unimodular.

PROPOSITION: Let $(\boldsymbol{X}, H)$ be an extended Poisson manifold.

$(\boldsymbol{X}, H)$ is unimodular $\iff$ $\exists$ a nowhere vanishing $\omega \in \Omega^{(n,0)}$ s.t. $\bar{\partial}\omega + \partial\iota_H\omega = 0$

Generalized Calabi-Yau condition
(usual Calabi-Yau for $H = 0$)

COROLLARY: For any unimodular extended Poisson manifold $(\boldsymbol{X}, H)$ of complex dimension n, we have

$$H_k(\boldsymbol{X}, H) \cong H^{2n-k}(\boldsymbol{X}, H).$$

Proof:

$(\boldsymbol{X}, H)$ is unimodular	$\Longleftrightarrow$	$\exists$ a nowhere vanishing $\omega \in \Omega^{(n,0)}$ s.t. $\bar{\partial}\omega + \partial\iota_H\omega = 0$

$\Longrightarrow$ the cohomologies of

$$\wedge^{\bullet} A_H \otimes \mathscr{L} \xrightarrow{\breve{d}_*^H} \wedge^{\bullet+1} A_H \otimes \mathscr{L}$$

$$\text{where } \mathscr{L} = Q_{A_H^*}^{\frac{1}{2}} = \wedge^n (T_X^{1,0})^*$$

and

$$\wedge^{\bullet} A_H \xrightarrow{d_*^H} \wedge^{\bullet+1} A_H$$

are isomorphic

Thus, if $(\boldsymbol{X}, H)$ is unimodular, then $H_k(\boldsymbol{X}, H) \cong H^k(\boldsymbol{X}, H)$ and the result follows from ELW duality on KB homology.

— Thank You —