Poisson Lie 2-groups and Lie 2-bialgebras

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DEFINITION: A **Poisson Lie group** is a Lie group G endowed with a multiplicative Poisson structure, i.e. a multiplicative field of bivectors π which is Poisson.

DEFINITION: A **Lie bialgebra** is a vector space $\mathfrak{g}$ endowed with a compatible structures of Lie algebra ($\wedge^2\mathfrak{g} \to \mathfrak{g}$) and Lie coalgebra ($\mathfrak{g} \to \wedge^2\mathfrak{g}$).

THEOREM (DRINFELD): The category of all Lie bialgebras is isomorphic to the category of all connected, simply-connected Poisson Lie groups.

QUESTION: What about 2-groups?

strict 2-groupoid:

$$\begin{tikzcd}
\Gamma_2 \arrow[d, shift left] \arrow[d, shift right] \arrow[dd, bend left=40] \arrow[dd, bend right=40] & \ni \alpha \\
\Gamma_1 \arrow[d, shift left] \arrow[d, shift right] & \ni f, g \\
\Gamma_0 & \ni x, y
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, bend left=40, "f"{name=U}] \arrow[r, bend right=40, "g"'{name=D}] & y
\arrow[Rightarrow, from=U, to=D, "\alpha"]
\end{tikzcd}$$

Horizontal and vertical multiplications:

$$\begin{tikzcd}
x \arrow[r, bend left=60, "f_1"{name=F1}] \arrow[r, "g_1"{name=G1}] \arrow[r, bend right=60, "h_1"'{name=H1}] & y \arrow[r, bend left=60, "f_2"{name=F2}] \arrow[r, "g_2"{name=G2}] \arrow[r, bend right=60, "h_2"'{name=H2}] & z
\arrow[Rightarrow, from=F1, to=G1, "\alpha_1"]
\arrow[Rightarrow, from=G1, to=H1, "\beta_1"]
\arrow[Rightarrow, from=F2, to=G2, "\alpha_2"]
\arrow[Rightarrow, from=G2, to=H2, "\beta_2"]
\end{tikzcd}$$

+ many natural conditions.

strict 2-group: particular case where $\Gamma_0 = *$, $\Gamma_1 = G$, and $\Gamma_2 = D$

Other characterization of a **strict 2-group**:

$$\begin{tikzcd} D \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & G \arrow[d, shift left] \arrow[d, shift right] \\ * \arrow[r, shift left] \arrow[r, shift right] & * \end{tikzcd}$$

is a double groupoid in the sense of Mackenzie

DEFINITION: A **strict Lie 2-group**

- is a Lie groupoid $D \rightrightarrows G$
- where D and G are Lie groups
- and $$\begin{tikzcd} D \times D \arrow[r, "m"] \arrow[d, shift left] \arrow[d, shift right] & D \arrow[d, shift left] \arrow[d, shift right] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$ is a morphism of Lie groupoids.

crossed module of groups:

$\Theta \xrightarrow{\Phi} G$ homomorphism of groups

together with action of G on Θ by automorphisms: $\alpha \mapsto {}^{g}\alpha$

satisfying

$$\Phi({}^{g}\beta) = g \cdot \Phi(\beta) \cdot g^{-1}$$
$${}^{\Phi(\alpha)}\beta = \alpha \cdot \beta \cdot \alpha^{-1}$$

Examples:

1. $G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(G)$
2. $1 \to G$
3. $Z(G) \to 1$

equivalence between strict 2-groups and crossed modules of groups:

THEOREM (R. BROWN):

crossed modules $\Theta \xrightarrow{\Phi} G$ $\underset{\text{bijection}}{\longleftrightarrow}$ strict 2-groups $D \rightrightarrows G \rightrightarrows *$

- $D = G \ltimes \Theta$ semi-direct product of **groups**

$$\boxed{(g,\alpha) \bullet (h,\beta) = (gh, {}^{h^{-1}}(\alpha)\beta)}$$

- $D = G \times \Theta \rightrightarrows G$ transformation **groupoid** for the Θ-action on G given by $g \cdot \alpha = g\Phi(\alpha)$

$$\boxed{\begin{array}{ll} s(g,\alpha) = g & t(g,\alpha) = g\Phi(\alpha) \\ (g,\alpha) \circ (h,\beta) = (g,\alpha\beta) & \text{if } h = g\Phi(\alpha) \end{array}}$$

strict Lie 2-algebra:

- A **crossed module of Lie algebras** consists of a pair θ, $\mathfrak{g}$ of Lie algebras, a homomorphism $\phi : \theta \to \mathfrak{g}$ and a $\mathfrak{g}$-action on θ by derivations satisfying

  $$\phi({}^{x}u) = [x, \phi(u)] \qquad \text{and} \qquad {}^{\Phi(u)}v = [u, v]$$

  for all $x \in \mathfrak{g}$ and $u, v \in \theta$.

- strict Lie 2-algebra = crossed module of Lie algebras

multiplicative vector fields:

- $\boxed{G}$ = Lie group

  $\Pi \in \mathfrak{X}^k(G)$ is multiplicative if

  $$\Pi_{gh} = L_{g*}\Pi_h + R_{h*}\Pi_g$$

- $\boxed{\Gamma \rightrightarrows M}$ = Lie groupoid $\boxed{\mathcal{A} \to M}$ = its Lie algebroid

  $\Pi \in \mathfrak{X}^k(\Gamma)$ is multiplicative if

  $$\Pi \in C^\infty(T^*\Gamma \times_\Gamma \overset{(k)}{\cdots} \times_\Gamma T^*\Gamma)$$

  is a 1-cocycle w.r.t. groupoid

  $$T^*\Gamma \times_\Gamma \underset{(k)}{\cdots} \times_\Gamma T^*\Gamma \rightrightarrows \mathcal{A}^* \times_M \underset{(k)}{\cdots} \times_M \mathcal{A}^*$$

DEFINITION: A **strict Poisson Lie 2-group** is a strict Lie 2-group $D \rightrightarrows G$ where D is endowed with a Poisson structure π, which is multiplicative w.r.t. both the group and the groupoid structures.

DEFINITION: A **strict Lie 2-bialgebra** (or crossed module of Lie bialgebras) is a pair of crossed modules of Lie algebras in duality

$$\theta \xrightarrow{\phi} \mathfrak{g} \qquad \text{and} \qquad \mathfrak{g}^* \xrightarrow{-\phi^*} \theta^*$$

such that the pair $(\mathfrak{g} \ltimes \theta, \theta^* \ltimes \mathfrak{g}^*)$ is a Lie bialgebra.

$\mathfrak{g} \ltimes \theta$ = semi-direct product of Lie algebras

EXAMPLE:

Given a crossed module $\theta \xrightarrow{\phi} \mathfrak{g}$
and $r \in \wedge^2\theta$ such that $x \triangleright [r, r] = 0, \forall x \in \mathfrak{g}$,

we have $[[r, r], u] = 0, \forall u \in \theta$
and $[[\phi(r), \phi(r)], x] = 0, \forall x \in \mathfrak{g}$.

Hence (θ, θ^*) is a Lie bialgebra with r-matrix r
and $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra with r-matrix $\phi(r)$.

THEOREM: The category of connected, simply-connected, strict Poisson Lie 2-groups is isomorphic to the category of strict Lie 2-bialgebras.

DEFINITION (BAEZ,CRANS): A **(weak) Lie 2-algebra** is a 2-term L_∞-algebra, i.e. a sequence $(l_k)_{k\in\mathbb{Z}}$ of homomorphisms $l_k : \wedge^k V \to V$ of degree $k-2$ (where $V = V_0 \oplus V_1$ is a 2-term graded vector space) satisfying a certain compatibility condition.

Given a (weak) Lie 2-algebra $V = V_0 \oplus V_1$ with $l_3 = 0$, set $\mathfrak{g} := V_0$ and $\theta := V_1$.

$$l_1 : \theta \xrightarrow{\phi} \mathfrak{g} \qquad l_2 : \begin{cases} \mathfrak{g} \wedge \mathfrak{g} \xrightarrow{[,]} \mathfrak{g} \\ \theta \otimes \mathfrak{g} \xrightarrow{\cdot} \theta \end{cases} \qquad l_3 : \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \xrightarrow{0} \theta$$

- $\mathfrak{g}$ is a Lie algebra: $[x, y] = l_2(x, y)$
- θ is a Lie algebra: $[u, v] = l_2(u, l_1(v))$

(weak) Lie 2-algebra with $l_3 = 0$ $\Longleftrightarrow$ strict Lie 2-algebra

Lie bialgebras and the big bracket:

[Lecomte, Roger; Kosmann-Schwarzbach]

Take a vector space $\mathfrak{g}$.

Endow the graded vector space $\bigoplus_k \wedge^k(\mathfrak{g} \oplus \mathfrak{g}^*)$ with the graded Poisson bracket of degree -2 characterized by

$$\{x, \xi\} = \langle \xi | x \rangle \qquad \{x_1, x_2\} = 0 \qquad \{\xi_1, \xi_2\} = 0$$

for all $x, x_1, x_2 \in \mathfrak{g}$ and $\xi, \xi_1, \xi_2 \in \mathfrak{g}^*$.

$(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra with bracket $b_2 \in \wedge^2\mathfrak{g} \otimes \mathfrak{g}^*$ and cobracket $c_2 \in \wedge^2\mathfrak{g}^* \otimes \mathfrak{g}$

$\Longleftrightarrow$

$s := b_2 + c_2 \in \wedge^3(\mathfrak{g} \oplus \mathfrak{g}^*)$ satisfies $\{s, s\} = 0$

Odd big bracket:

Take a graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V_k$.

Endow the graded symmetric algebra

$$\mathcal{E}^\bullet(V) := S^\bullet(V^*[1] \oplus V[2])$$

with the graded Poisson bracket of degree -3 characterized by

$$\begin{aligned}
\{x_i, \xi_j\} &= \delta_{ij}\langle \xi_j | x_i\rangle && \forall x_i \in V_i, \xi_j \in (V_j)^*;\\
\{x, y\} &= 0 && \forall x, y \in V;\\
\{\xi, \eta\} &= 0 && \forall \xi, \eta \in V^*.
\end{aligned}$$

- $S^\bullet(V^*[1] \oplus V[2]) \cong \Gamma(\wedge^\bullet T(V^*))$ as vector spaces
- $\{,\}$ is the Schouten bracket of polyvector fields

Let $V = V_0 \oplus V_1$ with $V_0 = \mathfrak{g}$ and $V_1 = \theta$.

DEFINITION: A **(weak) Lie 2-bialgebra** on $V = V_0 \oplus V_1$ consists in an element $\varepsilon \in \mathcal{E}(V)_{-4}$ such that $\{\varepsilon, \varepsilon\} = 0$.

$$\varepsilon = \phi + l_2^1 + l_2^2 + l_3 + c_2^1 + c_2^2 + c_3 \in \mathcal{E}(V)_{-4}$$

with

$$\phi \in \theta^* \otimes \mathfrak{g}$$

$$\begin{aligned} l_2^1 &\in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g} & c_2^1 &\in \theta^* \otimes \theta \otimes \theta \\ l_2^2 &\in \mathfrak{g}^* \otimes \theta^* \otimes \theta & c_2^2 &\in \mathfrak{g}^* \otimes \mathfrak{g} \otimes \theta \\ l_3 &\in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \theta & c_3 &\in \mathfrak{g}^* \otimes \theta \otimes \theta \otimes \theta \end{aligned}$$

$\theta \xrightarrow{\phi} \mathfrak{g}$ is a Lie 2-algebra

$\theta \xrightarrow{\phi} \mathfrak{g}$ is a Lie 2-coalgebra (i.e. $\mathfrak{g}^* \xrightarrow{-\phi^*} \theta^*$ is a Lie 2-algebra)

together with compatibility relations

PROBLEM: Integrate (weak) Lie 2-bialgebras.

DEFINITION: A **quasi-Poisson Lie 2-group** is a triple consisting of

- a Lie 2-group $G \ltimes \Theta \rightrightarrows G$;
- a multiplicative bivector field $\pi \in \mathfrak{X}^2(G \ltimes \Theta)$;
- and a 1-cocycle $\eta : G \to \wedge^3\theta \quad \in \Gamma(\mathrm{Lie}(G \ltimes \theta \rightrightarrows G))$

satisfying $\frac{1}{2}[\pi,\pi] = \overleftarrow{\eta} - \overrightarrow{\eta}$ and $[\pi, \overleftarrow{\eta}] = 0$.

THEOREM: $\boxed{\text{connected simply connected quasi-Poisson 2-groups}} \quad \overset{\longleftrightarrow}{\text{bijection}} \quad \boxed{\text{(weak) Lie 2-bialgebras with } l_3 = 0}$

REMARK: The Lie 2-groups integrating weak Lie 2-algebras are simplicial manifolds satisfying Kan conditions [Getzler, Henriques]

$\boxed{G}$ = Lie group

$\Pi \in \mathfrak{X}^k(G)$ is multiplicative if $\Pi_{gh} = L_{g*}\Pi_h + R_{h*}\Pi_g$

When endowed with the Schouten bracket, the multiplicative polyvector fields on G form a graded Lie algebra $\mathscr{X}^\bullet(G)$.

$\mathscr{D}^k(\mathfrak{g}) := \{$deg k-1 derivations of Gerstenhaber algebra $\wedge^\bullet\mathfrak{g}\}$

$\delta \in \mathscr{D}^k(\mathfrak{g}) \quad \Longleftrightarrow \quad \delta|_{\mathfrak{g}} : \mathfrak{g} \to \wedge^k\mathfrak{g}$ is a 1-cocycle

When endowed with the graded commutator, $\mathscr{D}^\bullet(\mathfrak{g})$ becomes a graded Lie algebra.

THEOREM (DRINFELD): $\mathscr{X}^\bullet(G) \cong \mathscr{D}^\bullet(\mathfrak{g})$ (as graded Lie algebras)

$\boxed{\Gamma \rightrightarrows M}$ = Lie groupoid $\boxed{\mathcal{A} \to M}$ = its Lie algebroid

$\Pi \in \mathfrak{X}^k(\Gamma)$ is multiplicative if $\Pi \in C^\infty(T^*\Gamma \times_\Gamma \overset{(k)}{\cdots} \times_\Gamma T^*\Gamma)$ is a 1-cocycle w.r.t. groupoid $T^*\Gamma \times_\Gamma \underset{(k)}{\cdots} \times_\Gamma T^*\Gamma \rightrightarrows \mathcal{A}^* \times_M \underset{(k)}{\cdots} \times_M \mathcal{A}^*$

$\mathscr{X}^k(\Gamma)$ = space of multiplicative k-vector fields on Γ

$\mathscr{D}^k(\mathcal{A})$ = space of degree k-1 derivations of the Gerstenhaber algebra $\wedge^\bullet \mathcal{A}$

THEOREM (IGLESIAS PONTE, LAURENT-GENGOUX, XU):
$\mathscr{X}^\bullet(\Gamma) \cong \mathscr{D}^\bullet(\mathcal{A})$ (as graded Lie algebras)

$\boxed{G \ltimes \Theta \rightrightarrows G}$ = 2-group associated to crossed module $\Theta \to G$

$\mathscr{X}^\bullet(G \ltimes \Theta)$ = graded Lie algebra of polyvector fields on $G \times \Theta$ which are multiplicative w.r.t. both the group and the groupoid structures

$\mathscr{D}^\bullet(\mathfrak{g} \ltimes \theta)$ = some graded Lie algebra whose elements of degree k are couples of Chevalley-Eilenberg 1-cocycles

$$\omega : \theta \to \wedge^k \theta \qquad\qquad \delta : \mathfrak{g} \to \mathfrak{g} \otimes \wedge^{k-1}\theta$$

compatible in a certain sense

THEOREM (CHEN, STIÉNON, X): $\mathscr{X}^\bullet(G \ltimes \Theta) \cong \mathscr{D}^\bullet(\mathfrak{g} \ltimes \theta)$
(as graded Lie algebras)

$\boxed{G \ltimes \Theta \rightrightarrows G}$ = 2-group associated to crossed module $\Theta \to G$

$\boxed{\mathcal{A} = G \ltimes \theta \to G}$ = its Lie algebroid

1. $\Pi \in \mathfrak{X}^k(G \ltimes \Theta)$ multiplicative w.r.t. groupoid structure
2. Iglesias, Laurent, Xu:
   $\hat{\delta} : \Gamma(\wedge^0 A) \to \Gamma(\wedge^{k-1} A)$ s.t. $\hat{\delta}(fg) = f\hat{\delta}(g) + \hat{\delta}(f)g$
3. $\hat{\delta} \in \Gamma(T_G \otimes \wedge^{k-1} A) \cong C^\infty(G, \mathfrak{g} \otimes \wedge^{k-1}\theta)$
4. $\hat{\delta} : G \to \mathfrak{g} \otimes \wedge^{k-1}\theta$ is in fact a 1-cocycle for Lie group G
5. $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \wedge^{k-1}\theta$ is a 1-cocycle for Lie algebra $\mathfrak{g}$

1. $\Theta = \Theta \times \{1\}$ is a Lie subgroup of $\Theta \times G$
2. $\Pi \in \mathfrak{X}^k(G \ltimes \Theta)$ multiplicative w.r.t. group structure
3. [1] + [2] $\Longrightarrow \Pi_\Theta \in \mathscr{X}^k(\Theta)$
4. Drinfeld: $\omega : \theta \to \wedge^k \theta$ is a 1-cocycle

How the Universal Lifting Theorem is used to integrate Lie 2-bialgebras to quasi-Poisson 2-groups:

$\mathscr{X}^2(G \ltimes \Theta) \ni \pi$	$\mathscr{D}^2(\mathfrak{g} \ltimes \theta) \ni$ $\begin{cases} \omega : \theta \to \wedge^2\theta & \dashleftarrow c_2^1 \\ \delta : \mathfrak{g} \to \mathfrak{g} \otimes \theta & \dashleftarrow c_2^2 \end{cases}$
$\mathscr{X}^3(G \ltimes \Theta) \ni \overleftarrow{\eta} - \overrightarrow{\eta}$ coming from $\eta : G \to \wedge^3\theta$	$\mathscr{D}^3(\mathfrak{g} \ltimes \theta) \ni \begin{cases} \omega_\eta : \theta \to \wedge^3\theta \\ \delta_\eta : \mathfrak{g} \to \mathfrak{g} \otimes \wedge^2\theta \end{cases}$ obtained from $\eta : \mathfrak{g} \to \wedge^3\theta \quad \dashleftarrow c_3$ using $\phi : \theta \to \mathfrak{g}$
$\frac{1}{2}[\pi, \pi] = \overleftarrow{\eta} - \overrightarrow{\eta}$ and $[\pi, \overleftarrow{\eta}] = 0$	$\varepsilon = \phi + l_2^1 + l_2^2 + \underset{\substack{\Vert \\ 0}}{l_3} + c_2^1 + c_2^2 + c_3$ satisfies $\{\varepsilon, \varepsilon\} = 0$

— Thank You —