Outline

PCP reduction

Given: $\langle M, w\rangle$ in encoded form

To construct: an instance (A, B) of MPCP such that M accepts w if and only if (A, B) has a solution.

Preliminaries:

- Assume w.l.o.g. that
  - M never prints a blank;
  - M never moves left from the first tape square (i.e. never hangs).
- Starting configuration: $(q_0,\ \underline{\#}w)$
- Representation of configuration: $\alpha\mathbf{q}\beta$
  (If head is on a blank, i.e. at the end of the written portion of the tape, $\beta = \varepsilon$.)

PCP instance

	List A	List B	
Group I	$\vdash$	$\vdash q_o w \vdash$	
Group II	a	a	for any $a \in \Sigma$
	$\vdash$	$\vdash$	
Group IIIA	qa	pb	if $b \in \Sigma$
when $\delta(q,a) = (p,b)$	qa	ap	if $b = R$
	cqa	pca	if $b = L$
Group IIIB	$q \vdash$	$pb \vdash$	if $b \in \Sigma$
when $\delta(q,\#) = (p,b)$	$q \vdash$	$\#' p \vdash$	if $b = R$
	$cq \vdash$	$pc \vdash$	if $b = L$

PCP instance

	List A	List B
Group IV	ahb	h
	ah	h
	hb	h
Group V	$h \vdash\vdash$	$\vdash$

PCP is undecidable

Lemma: M accepts w if and only if the MPCP instance (A, B) above has a solution.

Proof sketch: $(\Rightarrow)$

1. Start with pair from group I.
2. Use pairs from group III to simulate one move in the region around the head position;
   use pairs from group II to copy rest of the tape contents.
3. If M halts, use pairs from group IV to progressively erase the "tape" in the B string.
   A string: $\ldots \vdash a_1 \ldots a_i h a_{i+1} \ldots \vdash$
   B string: $\ldots \vdash a_1 \ldots a_i h a_{i+1} \ldots \vdash a_1 \ldots a_{i-1} h a_{i+2} \ldots \vdash$
4. Terminate with pair from group V.

PCP is undecidable

Proof sketch (contd.): $(\Leftarrow)$

Solution must begin with: A string = $\vdash$ B string = $\vdash q_o w \vdash$.

Partial solutions have the form: A string = x B string = xy

where

$x =$ sequence of configurations separated by $\vdash$, last one (w_C, say) may be incomplete

$y =$ computation upto and including w_C followed by prefix of a configuration that follows w_C

Partial solution can be completed only if M halts; otherwise, B string is always longer.

Outline

Ambiguity of CFGs

Given: $A = w_1, w_2, \ldots, w_k, \quad B = x_1, x_2, \ldots, x_k.$

Reduction:

Let $G_A \ : \ A \rightarrow w_1 A a_1 \ | \ \ldots \ | \ w_k A a_k \ \ | \ \ w_1 a_1 \ | \ \ldots \ | \ w_k a_k$

and $G_B \ : \ B \rightarrow x_1 B a_1 \ | \ \ldots \ | \ x_k B a_k \ \ | \ \ x_1 a_1 \ | \ \ldots \ | \ x_k a_k$

Let $G_{AB} \ : \ S \rightarrow A \ | \ B \ldots$

Then G_{AB} is ambiguous if and only if (A, B) has a solution.

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List languages

$L(G_A)$ and $L(G_B)$ are called *list languages*.

List languages

Lemma: If L_A is a list language, then $\overline{L_A}$ is a CFL.

Proof: Let P be a (D)PDA that works as follows.

1. On symbols in Σ, P pushes the input symbol on stack.
2. On symbols $a_1, \ldots, a_k$, it pops the stack to see if the top of the stack contains w_i^R.
3. If input and stack are both empty, reject.
4. In all other cases, accept.

CFG related problems

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- $L(G_1) \subseteq L(G_2)$? Let $L(G_2) = \overline{L_A} \cup \overline{L_B}$, $L(G_1) = (\Sigma \cup I)^*$.

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- $L(G_1) \subseteq L(G_2)$? — Let $L(G_2) = \overline{L_A} \cup \overline{L_B}$, $L(G_1) = (\Sigma \cup I)^*$.
- $L(R) \subseteq L(G_1)$?