Projective modules over smooth real affine varieties

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1 Introduction

Let $X = \mathrm{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over a field k and P be a projective A-module of rank n.

A result of Murthy ([Mu], Theorem 3.8) says that if k is an algebraically closed field and if the top Chern class $C_n(P)$ in $CH_0(X)$ is zero, then P splits off a free summand of rank 1 (i.e. $P \simeq A \oplus Q$). We note that over any base field, the vanishing of the top Chern class is a *necessary* condition for P to split off a free summand of rank 1. However, as is shown by the example of the tangent bundle of an even dimensional real sphere, this condition is not sufficient. In view of these examples, it is of interest to explore whether one can classify examples of projective modules over smooth real varieties

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which have the property that the top Chern class of the projective module vanishes but the projective module does not split off a free summand of rank one. More precisely we ask the following:

Question. *Let $X = \mathrm{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over $\mathbb{R}$ (the field of real numbers) and P a projective A-module of rank n. Under what further restrictions, does $C_n(P) = 0$ imply that $P \simeq A \oplus Q$?*

The investigation of this question was initiated in [B-RS 1] and it was shown that if n is odd and $K_A \simeq A \simeq \wedge^n(P)$ then the vanishing of the top Chern class $C_n(P)$ is sufficient to conclude that $P \simeq A \oplus Q$, where $K_A = \wedge^n(\Omega_{A/\mathbb{R}})$ denotes the canonical module of A.

In this paper (which should be regarded as a sequel to [B-RS 1]) we carried out this investigation further and settle the above question in complete generality. For instance we prove the following result (see (4.31) for general statement):

Theorem *Let X be as above. Let $\mathbb{R}(X)$ denote the localization A_S of A with respect to the multiplicatively closed subset S of A consisting of all elements which do not belong to any real maximal ideal. Assume that the manifold $X(\mathbb{R})$ of real points of X is connected. Let P be a projective A-module of rank n such that its top Chern class $C_n(P) \in CH_0(X)$ is zero. Then $P \simeq A \oplus Q$ in the following cases:*

1. *$X(\mathbb{R})$ is not compact.*

2. *n is odd.*

3. *n is even, $X(\mathbb{R})$ is compact and $\wedge^n(P) \otimes_A \mathbb{R}(X) \not\simeq K_A \otimes_A \mathbb{R}(X)$.*

Moreover, if n is even and $X(\mathbb{R})$ is compact then there exists a projective A-module P of rank n such that $P \oplus A \simeq K_A \oplus A^{n-1} \oplus A$ (hence $C_n(P) = 0$) but P does not have a free summand of rank 1.

Note that the above theorem says that if $\dim(X)$ is odd then the only obstruction for an algebraic vector bundle of top rank over X to split off a trivial subbundle of rank 1 is algebraic, namely the possible nonvanishing of its top Chern class. However if $\dim(X)$ is even and $X(\mathbb{R})$ is compact and connected, then, apart from the possible nonvanishing of its top Chern class, the only other obstruction for an algebraic vector bundle of top rank over X to split off a trivial subbundle of rank 1 is purely topological viz. the associated topological vector bundle and the manifold $X(\mathbb{R})$ have the same orientation. Moreover, in this case, the theorem assures us that indeed these obstructions genuinely exist.

As an immediate consequence of our theorem we obtain that if $X = \mathrm{Spec}(A)$ is a smooth affine surface over $\mathbb{R}$ such that $X(\mathbb{R})$ is a compact, connected, non-orientable manifold then stably free A-modules are free. This settles Question 6.5 of [B-RS 1] affirmatively (see (4.32)).

The main thrust of our proof in brief consists in showing that if the top Chern class $C_n(P)$ of a projective A-module P of rank n vanishes, then its Euler class (which is an "oriented zero cycle" associated to P) vanishes. We then appeal to a result of ([B-RS 2], Corollary 4.4) to conclude that P splits off a free summand of rank 1.

The layout of this paper is as follows: In Section 2, we first recall the definition of the Euler Class group $E(A, L)$ of a Noetherian commutative ring A with respect to a rank 1 projective A-module L. This group, in the case where A is a smooth affine domain, can be essentially thought of as the group of "oriented zero cycles" modulo a "rational equivalence". We also introduce another group $E_0(A, L)$ which is a quotient of $E(A, L)$ obtained by ignoring "orientations". Then we quote some results which will be used subsequently. In Section 3 we introduce some constructions regarding " thickenings of oriented cycles" which are crucial in deducing our main theorem. As another application of these constructions, we prove some results about torsion elements in $K_0(A)$ which are of independent interest. In section 4, we prove the theorem mentioned in the introduction.

We owe our success in carrying out the investigation of the question to a satisfactory conclusion, mainly to the insight of Madhav Nori. We thank him profusely but inadequately. We thank Albert Sheu for clarifying some results. We also thank the referee for carefully going through the manuscript and suggesting improvements in the exposition.

2 Preliminaries

All rings considered in this paper are commutative Noetherian and contain the field $\mathbb{Q}$ of rational numbers. All modules are assumed to be finitely generated.

We begin this section by stating following two results. The following theorem of Eisenbud-Evans [E-E] can be deduced from [Pl, p. 1420].

Theorem 2.1 *Let A be a ring and P be a projective A-module of rank d. Let $(\alpha, a) \in P^* \oplus A$. Then there exists an element $\beta \in P^*$ such that $\mathrm{ht}(I_a) \geq d$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq d$ then $\mathrm{ht}(I) \geq d$. Further, if $(\alpha(P), a)$ is an ideal of height $\geq d$ and I is a proper ideal of A, then $\mathrm{ht}(I) = d$.*

When A is a geometrically reduced affine ring we have the following version of Bertini's theorem (due to Swan) which is a refinement of (2.1). This version can be deduced from [Sw, Theorems 1.3 and 1.4]; see also [Mu]. In this paper we shall refer to this result as Swan's Bertini theorem.

Theorem 2.2 *Let A be a geometrically reduced affine ring over an infinite field and P be a projective A-module of rank r. Let $(\alpha, a) \in P^* \oplus A$. Then there exists an element $\beta \in P^*$ such that if $I = (\alpha + a\beta)(P)$ then*

1. *Either $I_a = A_a$ or I_a is an ideal of height r such that $(A/I)_a$ is a geometrically reduced ring.*
2. *If $r < \dim A$ and A_a is geometrically integral, then $(A/I)_a$ is also geometrically integral.*
3. *If A_a is smooth, then $(A/I)_a$ is also smooth.*

Rest of the section may be considered as a quick reference guide to the generalities of Euler class group theory. We first accumulate some basic definitions, namely, the definitions of Euler class groups, weak Euler class groups, the Euler class of a projective module and then quote some results which are very much relevant to this paper. A detailed account of these topics can be found in [B-RS 2]. A reader familiar with this topic can safely go to the next section.

We start with the definition of the Euler class group.

Definition of $E(A, L)$ **:** Let A be a ring of dimension $n \geq 2$ and let L be a projective A-module of rank 1. Write $F = L \oplus A^{n-1}$. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Two surjections α, β from F/JF to J/J^2 are said to be related if there exists $\sigma \in SL_{A/J}(F/JF)$ such that $\alpha\sigma = \beta$. Clearly this is an equivalence relation on the set of surjections from F/JF to J/J^2. Let $[\alpha]$ denote the equivalence class of α. Such an equivalence class $[\alpha]$ is called a *local* L*-orientation* of J. By abuse of notation, we shall identify an equivalence class $[\alpha]$ with α. A local L-orientation α is called a *global* L*-orientation* if $\alpha : F/JF \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : F \twoheadrightarrow J$.

Let G be the free abelian group on the set of pairs $(\mathcal{N}, \omega_{\mathcal{N}})$ where $\mathcal{N}$ is an $\mathcal{M}$-primary ideal for some maximal ideal $\mathcal{M}$ of height n such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements and $\omega_{\mathcal{N}}$ is a local L-orientation of $\mathcal{N}$. Now let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements and ω_J be a local L-orientation of J. Let $J = \cap_i \mathcal{N}_i$ be the (irredundant) primary decomposition of J. We associate to the pair (J, ω_J), the element $\sum_i (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ of G where $\omega_{\mathcal{N}_i}$ is the local orientation of $\mathcal{N}_i$ induced by ω_J. By abuse of notation, we denote $\sum_i (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ by (J, ω_J).

Let H be the subgroup of G generated by set of pairs (J, ω_J), where J is an ideal of height n and ω_J is a global L-orientation of J.

The Euler class group of A with respect to L is $E(A, L) \stackrel{\text{def}}{=} G/H$.

Let P be a projective A-module of rank n such that $L \simeq \wedge^n(P)$ and let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism. Let $\varphi : P \twoheadrightarrow J$ be a surjection where J is an ideal of height n. Therefore we obtain an induced surjection $\overline{\varphi} : P/JP \twoheadrightarrow J/J^2$. Let $\overline{\gamma} : L/JL \oplus (A/J)^{n-1} \simeq P/JP$, be an isomorphism such that $\wedge^n(\overline{\gamma}) = \overline{\chi}$. Let ω_J be the local L-orientation of J given by $\overline{\varphi}\,\overline{\gamma} : L/JL \oplus (A/J)^{n-1} \twoheadrightarrow J/J^2$. Let $e(P, \chi)$ be the image in $E(A, L)$ of the element (J, ω_J) of G. The assignment sending the pair (P, χ) to the element $e(P, \chi)$ of $E(A, L)$ is well defined. The *Euler class* of (P, χ) is defined to be $e(P, \chi)$.

Remark 2.3 If A is Cohen-Macaulay and J is an ideal of height n such that J/J^2 is generated by n elements, then J/J^2 is a free A/J-module of rank n and hence a local L-orientation ω_J of J gives rise to a unique isomorphism $L/JL \stackrel{\sim}{\to} \wedge^n(J/J^2)$. Conversely, an isomorphism $L/JL \stackrel{\sim}{\to} \wedge^n(J/J^2)$ gives rise to a local L-orientation of J. Hence we can regard a local L-orientation on J as an isomorphism $L/JL \simeq \wedge^n(J/J^2)$.

Now assume that A is a smooth affine domain of dimension $n \geq 2$. Let $\mathcal{N}$ be an $\mathcal{M}$-primary ideal for some maximal ideal $\mathcal{M}$ such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements. Let $\phi' : L/\mathcal{N}L \oplus (A/\mathcal{N})^{n-1} \simeq \mathcal{N}/\mathcal{N}^2$ be an isomorphism. By Swan's Bertini theorem (2.2), there exists an ideal $J \subset A$ and a surjection $\phi : L \oplus A^{n-1} \twoheadrightarrow \mathcal{N} \cap J$ such that (1) $\mathcal{N} + J = A$, (2) J is a finite intersection of maximal ideals or $J = A$ and (3) $\phi \otimes A/\mathcal{N} = \phi'$. Hence $E(A, L)$ is generated by elements of the type (M, ω_M) where M is a maximal ideal of A and ω_M is a local L-orientation of M. In fact $E(A, L) = G'/H'$ where G' is a free abelian group on the set of pairs (M, ω_M) (M: maximal ideal, ω_M: local L-orientation of M) and H' is a subgroup generated by (J, ω_J) where J is a *reduced* ideal of height n and ω_J is a global L-orientation of J (see [B-RS 2, Remark 4.7] for further details).

Definition of $E_0(A, L)$: Let A, L be as above and let $F = L \oplus A^{n-1}$. Let S be the set of ideals $\mathcal{N}$ of A where $\mathcal{N}$ is an $\mathcal{M}$-primary ideal of height n such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements. Let F be the free abelian group on S. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let $J = \cap_i \mathcal{N}_i$ be the (irredundant) primary decomposition of J. We associate to J, the element $\sum_i \mathcal{N}_i$ of F. By abuse of notation, we denote this element by (J). Let K be the subgroup of F generated by elements of the type (J), where J is an ideal of height n such that there exists a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow J$.

The *weak Euler class group* of A with respect to L is denoted by $E_0(A, L)$ and is defined as $E_0(A, L) \stackrel{\text{def}}{=} F/K$. When $L = A$, we denote $E(A, L)$ by $E(A)$ and $E_0(A, L)$ by $E_0(A)$.

Remark 2.4 It is clear from the above definitions that there is an obvious canonical surjective group homomorphism $\Theta_L : E(A, L) \twoheadrightarrow E_0(A, L)$ which sends an element (J, ω_J) of $E(A, L)$ to (J) in $E_0(A, L)$.

We state some results on Euler class group and weak Euler class group for later use. The following result is proved in [B-RS 2, Corollary 4.4].

Theorem 2.5 *Let A be a ring of dimension $n \geq 2$ and L be a projective A-module of rank 1. Let P be a projective A-module of rank n with $L \simeq \wedge^n(P)$ and let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism. Let $J \subset A$ be an ideal of height n and ω_J be a local L-orientation of J. Then,*

1. *Suppose that the image of (J, ω_J) is zero in $E(A, L)$. Then there exists a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow J$ such that ω_J is induced by α (in other words, ω_J is a global L-orientation).*

2. *$P \simeq Q \oplus A$ for some projective A-module Q of rank $n-1$ if and only if $e(P, \chi) = 0$ in $E(A, L)$.*

The following result from [B-RS 2, Lemma 5.4] is very much relevant to this paper.

Lemma 2.6 *Let A be a ring of dimension $n \geq 2$. Let $J \subset A$ be an ideal of height n and ω_J be a local L-orientation of J. Let $\overline{a} \in A/J$ be a unit. Then $(J, \omega_J) = (J, \overline{a^2}\omega_J)$ in $E(A, L)$.*

Let S be a multiplicatively closed subset of A such that $\dim(A_S) = \dim(A) \geq 2$. Then from definitions of Euler class groups and weak Euler class groups, it is easy to see that there exist (canonical) surjective maps $E(A, L) \twoheadrightarrow E(A_S, L \otimes_A A_S)$ and $E_0(A, L) \twoheadrightarrow E_0(A_S, L \otimes_A A_S)$. Now we state a result which describes kernels of these maps and can be proved by adapting the proof of [B-RS 2, Lemma 5.6].

Lemma 2.7 *Let A be a ring of dimension $n \geq 2$ and let L be a projective A-module of rank 1. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements and let ω_J be a local L-orientation of J. Let $f \in A$. Suppose that $(J, \omega_J) \neq 0$ in $E(A, L)$, but the image of $(J, \omega_J) = 0$ in $E(A_f, L_f)$. Then, there exists an ideal I of height n and a local L-orientation ω_I of I such that $I_f = A_f$ and $(J, \omega_J) = (I, \omega_I)$ in $E(A, L)$.*

Remark 2.8 Let A be a ring of dimension $n \geq 2$ and let L be a projective A-module of rank 1. Let J be an ideal of A of height n such that J/J^2 is generated by n elements. For notational clarity let us denote the element of $E_0(A, L)$ associated to J by $(J)_L$ and the element of $E_0(A)$ associated to J by (J).

Keeping this convention in mind, we quote a result proved in [B-RS 2, Theorem 6.8].

Theorem 2.9 *Let A, L be as above. Then the canonical map $\beta_L : E_0(A, L) \to E_0(A)$ defined by $(J)_L \mapsto (J)$ is an isomorphism of groups.*

Now suppose that $X = \mathrm{Spec}(A)$ is a smooth affine variety of dimension $n \geq 2$ over a field k. Let $CH_0(X)$ denote the group of zero cycles of X modulo rational equivalence. In this set up, it is easy to see that there exists a canonical surjection $\Psi : E_0(A) \twoheadrightarrow CH_0(X)$ such that $\Psi((J)) = [J]$ where $[J]$ denotes the image in $CH_0(X)$ of the zero cycle associated to A/J.

In the case $k = \mathbb{R}$ (the field of real numbers) the following result is proved in [B-RS 1, Theorem 5.5].

Theorem 2.10 *Let $X = \mathrm{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over $\mathbb{R}$. Then the canonical surjection $\Psi : E_0(A) \twoheadrightarrow CH_0(X)$ is an isomorphism.*

3 Useful constructions, torsion in $K_0(A)$ and set-theoretic complete intersection

Let A be a ring (which we always assume to be Noetherian) and let $K_0(A)$ denote the Grothendieck group of the category of finitely generated projective A-modules.

Recall that an ideal I of A is said to be a local complete intersection ideal of height d if for every maximal ideal $\mathcal{M}$ of A containing I, the ideal $IA_{\mathcal{M}}$ is generated by a *regular sequence* of d elements. It is easy to see that if an ideal I is a local complete intersection ideal of height d then the A-module A/I has projective dimension d. Since $K_0(A)$ is canonically isomorphic to the Grothendieck group of the category of finitely generated A-modules of *finite projective dimension*, if I is a local complete intersection ideal of A then the A-module A/I defines an element of $K_0(A)$ which we denote by $[A/I]$.

Now assume that A is a ring of dimension $n \geq 2$. Let I be an ideal of A of height n such that I/I^2 is generated by n elements. Then there exists an ideal $J \subset I$ of height $n-1$ such that J/IJ is generated by $n-1$ elements and $I/(J+I^2)$ is cyclic. Given such a pair (J, I) and a positive integer r, we denote the ideal $J + I^r$ by $I^{(r)}$. The following lemma and subsequent remark say that $I^{(r)}$ (in some sense) is independent of J and hence we may not always specify J.

Lemma 3.1 *Let A, I, J be as above. Let Q be a projective A-module of rank $n-1$. Then there exists an ideal J_1 of height $n-1$ such that $J_1 \subset J$, J_1 is a surjective image of Q, and $I^{(r)} = J + I^r = J_1 + I^r$.*

Proof Since $\dim(A/I) = 0$, $Q/IQ \simeq (A/I)^{n-1}$. Hence there exists a surjection $\theta' : Q/IQ \twoheadrightarrow J/IJ$. Let $\theta : Q \to J$ be a lift of θ' and let $\theta(Q) = J_1$. Then $J_1 + IJ = J$. Since height of $J = n-1$, by repeated application of (2.1), we can modify θ and assume that height of J_1 is $n-1$. Since $J_1 + IJ = J$ it is easy to see that $J_1 + I^r J = J$. Therefore $I^{(r)} = J + I^r = J_1 + I^r J + I^r = J_1 + I^r$. ∎

Remark 3.2 Let $I, J, I^{(r)}$ be as above. Since $I/(J+I^2)$ is cyclic, there exists $a \in I$ such that $I = J + (a) + I^2$. Let L be a projective A-module of rank 1 and let $F = L \oplus A^{n-2}$. Then, by (3.1), we can assume that J is a surjective image of F. Therefore it is easy to see that $(I^{(r)})_L = r(I)_L$ in $E_0(A, L)$.

In simple terms the following lemma means that if A is a ring of dimension $n \geq 2$ and $K \subset A$ is an ideal of height n such that $K = (a_1, \cdots, a_{n-1}, a_n^2)$, then any set of n generators of K/K^2 can be lifted to a set of n generators of K. However, we will need the general version stated below for later parts of the paper.

Lemma 3.3 *Let A be a ring of dimension $n \geq 2$ and let I be an ideal of height n such that I/I^2 is generated by n elements. Let L be a projective A-module of rank 1. Let $F = L \oplus A^{n-2}$ and let $\theta : F \oplus A \twoheadrightarrow I$ be a surjective map. Assume* *that $\theta(F) = J$ is an ideal of height $n-1$ and let $I^{(2)} = J + I^2$. Let ω be a local L-orientation of $I^{(2)}$. Then $(I^{(2)}, \omega) = 0$ in $E(A, L)$.*

Proof Let $\theta(0,1) = a$. Then $I = J + (a)$ and hence $I^{(2)} = J + (a^2)$. Therefore there exists a surjective map $\beta : F \oplus A \twoheadrightarrow I^{(2)}$ such that $\beta(p) = \theta(p); p \in F$ and $\beta(0,1) = a^2$. The surjection β defines a global L-orientation ω_0 on $I^{(2)}$. Hence there exists $u \in A$ such that $(u) + I^{(2)} = A$ and $\overline{u}\omega_0 = \omega$ where $\overline{u}$ denotes the image of u in $A/I^{(2)}$. Since $J + (a^2) + (u) = A$, adding a suitable multiple of a^2 to u if necessary, we assume that $J + (u) = I_1$ is an ideal of height n. Let $\alpha : F \oplus A \twoheadrightarrow I_1$ be the surjection such that $\alpha(p) = \beta(p)$ for $p \in F$ and $\alpha(0,1) = u$. The surjection α defines a global L-orientation ω_1 on I_1.

Let $\widetilde{a^2}$ denotes the image of a^2 in A/I_1. Note that a is a unit modulo I_1. Since $J + (ua^2) = I^{(2)} \cap I_1$, it is easy to see that $(I^{(2)}, \overline{u}\omega_0) + (I_1, \widetilde{a^2}\omega_1) = 0$ in $E(A, L)$. But, by (2.6), $(I_1, \widetilde{a^2}\omega_1) = (I_1, \omega_1)$ and we have $(I_1, \omega_1) = 0$ in $E(A, L)$. Therefore it follows that $(I^{(2)}, \omega) = (I^{(2)}, \overline{u}\omega_0) = 0$ in $E(A, L)$. This proves the lemma. ∎

In the next couple of lemmas we deal with $I^{(2)}$. One interesting property of $I^{(2)}$ is that, any two local L-orientations of $I^{(2)}$ induce the same image in $E(A, L)$, as the following lemma shows.

Lemma 3.4 *Let A be a ring of dimension $n \geq 2$ and L be a projective A-module of rank 1. Let I be an ideal of A of height n such that I/I^2 is generated by n elements and ω_I be a local L-orientation of I. Let $F = L \oplus A^{n-2}$ and let $\alpha : F/IF \oplus A/I \twoheadrightarrow I/I^2$ be a surjection corresponding to ω_I. Let $\theta : F \oplus A \to I$ be a lift of α such that $\theta(F) = J$ is an ideal of height $n-1$ (which exists by (2.1)) and let $I^{(2)} = J + I^2$. Let ω and ω' be two local L-orientations of $I^{(2)}$. Then, $(I^{(2)}, \omega) = (I^{(2)}, \omega')$ in $E(A, L)$.*

Proof Recall that ω_I is given by the equality $I = J + (a) + I^2$. It is easy to see that there exists an ideal I_1 of A of height n such that $I + I_1 = A$ and $J + (a) = I \cap I_1$. If we set ${I_1}^{(2)} = J + {I_1}^2$, then $J + (a^2) = I^{(2)} \cap {I_1}^{(2)}$.

Note that $I^{(2)} + {I_1}^{(2)} = A$. Let us write $I' = J + (a^2)$ and fix a local L-orientation $\widetilde{\omega}$ of ${I_1}^{(2)}$. This $\widetilde{\omega}$, together with the local L-orientation ω of

$I^{(2)}$ will induce a local L-orientation, say, $\omega_{I'}$ of I'. By (3.3), $(I', \omega_{I'}) = 0$ and hence

$$(I^{(2)}, \omega) + ({I_1}^{(2)}, \widetilde{\omega}) = 0$$

in $E(A, L)$. By a similar argument we obtain

$$(I^{(2)}, \omega') + ({I_1}^{(2)}, \widetilde{\omega}) = 0$$

in $E(A, L)$. Therefore, $(I^{(2)}, \omega) = (I^{(2)}, \omega')$ in $E(A, L)$. ■

Remark 3.5 In view of the above lemma, in what follows, we will not specify any local L-orientation of $I^{(2)}$ and denote by $(I^{(2)}, *)$ an element of $E(A, L)$ supported on $I^{(2)}$.

Lemma 3.6 *Let A be a ring of dimension $n \geq 2$ and L be a projective A-module of rank 1. Let $I, \omega_I, J, I^{(2)}$ be as in the above lemma. Then $(I^{(2)}, *) = (I, \omega_I) + (I, -\omega_I)$ in $E(A, L)$. As a consequence, if ω'_I is any other local L-orientation of I, then $(I, \omega_I) + (I, -\omega_I) = (I, \omega'_I) + (I, -\omega'_I)$ in $E(A, L)$.*

Proof Recall that ω_I is given by the equality $I = J + (a) + I^2$, where $a \in I$ and J is an ideal of height $n-1$ such that J is a surjective image of $L \oplus A^{n-2}$.

Let $B = A/J, \overline{I} = I/J$. Note that B is a ring of dimension 1, $\overline{I}$ is an ideal of B of height 1 and $(h) + \overline{I}^2 = \overline{I}$ where h denotes the image of a in B.

Then, since $J + (a) + I^2 = I$, we have $(h) = \overline{I} \cap K_1$ where K_1 is an ideal of B which is comaximal with $\overline{I}$ and of height ≥ 1. If height of $K_1 > 1$ then $K_1 = B$. In that case $I = J + (a)$, $(I, \omega_I) = (I, -\omega_I) = 0$ and $I^{(2)} = J + (a^2)$ and hence we are through. So we assume that K_1 is an ideal of B of height 1. Since $(h) + \overline{I}^2 = \overline{I}$, there exists $e \in \overline{I}^2$ such that $(h, e) = \overline{I}$. It is easy to see that $K_1 + (e) = B$.

Claim: There exists $w \in B$ such that $K_1 + (w) = B$ and $\mathrm{ht}(h + w^2 e^2) = 1$.

Proof of the claim. Note that, since $\mathrm{ht}(\overline{I}) = 1$, if a minimal prime ideal $\mathcal{N}$ of B contains $h + z$ for some $z \in \overline{I}^2$, then $\overline{I} + \mathcal{N} = B$. Since K_1 is a proper ideal of B and $K_1 + (e) = B$, e is not nilpotent and hence can not belong to all minimal prime ideals of B. Let $\{\mathcal{N}_1, \cdots, \mathcal{N}_r\}$ be the set of all minimal prime ideals of B which do not contain e and let $T = \{i | \mathcal{N}_i + K_1 = B\}$.

Then it is easy to see that there exists $w \in \cap_{i\in T}\mathcal{N}_i$ such that $K_1 + (w) = B$. Now we show that $\mathrm{ht}(h + w^2e^2) = 1$.

Let $\mathcal{N}$ be a minimal prime ideal of B. If $w^2e^2 \in \mathcal{N}$ then, since $\mathrm{ht}(h) = 1$, $h + w^2e^2 \notin \mathcal{N}$. If $w^2e^2 \notin \mathcal{N}$ then $\mathcal{N} \in \{\mathcal{N}_1, \cdots, \mathcal{N}_r\}$ and is not comaximal with K_1. Therefore if $h + w^2e^2 \in \mathcal{N}$ then $K_1 + (h + w^2e^2)$ is a proper ideal of B which contains h and hence w^2e^2. But this is a contradiction since $K_1 + (w^2e^2) = B$.

The above argument shows that $h + w^2e^2$ does not belong to any minimal prime ideal of B and hence $\mathrm{ht}(h+w^2e^2) = 1$. Thus the claim is proved.

Let $h' = h + w^2e^2$. Since $\mathrm{ht}(h + w^2e^2) = 1$ we have $(h') = \overline{I} \cap K_2$ where K_2 is an ideal of B of height ≥ 1. If $K_2 = B$ then as before we conclude that $(I, \omega_I) + (I, -\omega_I) = (I^{(2)}, *)$ in $E(A, L)$. So we assume that K_2 is an ideal of height 1. Note that $\overline{I}, K_1, K_2$ are pairwise comaximal and $(hh') = \overline{I}^2 \cap K_1 \cap K_2$.

Let $x, y \in A$ be preimages of w, e respectively and let $b = a + x^2y^2$. Let I_1 and I_2 be inverse images of K_1 and K_2 in A respectively. The equalities

$$(h) = \overline{I} \cap K_1, \quad (h') = \overline{I} \cap K_2, \quad (hh') = \overline{I}^2 \cap K_1 \cap K_2$$

give rise to following equalities

(i) $J + (a) = I \cap I_1$.

(ii) $J + (b) = I \cap I_2$.

(iii) $J + (ab) = I^{(2)} \cap I_1 \cap I_2$.

.

Let $\omega_{I_1}, \omega_{I_2}$ be the local L- orientations of I_1 and I_2 obtained from $J+(a)$ and $J + (b)$ respectively. Since ω_I is obtained from $J + (a)$ and $b = a + x^2y^2$, $xy \in I$, ω_I is also obtained from $J + (b)$. Therefore we have the following equations in $E(A, L)$.

(i) $(I, \omega_I) + (I_1, \omega_{I_1}) = 0$.

(ii) $(I, \omega_I) + (I_2, \omega_{I_2}) = 0$.

(iii) $(I, -\omega_I) + (I_2, -\omega_{I_2}) = 0$.

Since $ab = a^2 + (xy)^2 a = b^2 - (xy)^2 b$, by (2.6), we see that (I_1, ω_{I_1}) and $(I_2, -\omega_{I_2})$ are obtained from $J + (ab)$. Since $J + (ab) = I^{(2)} \cap I_1 \cap I_2$, we have (iv) $(I^{(2)}, *) + (I_1, \omega_{I_1}) + (I_2, -\omega_{I_2}) = 0$ in $E(A, L)$. Therefore, using equations (i), (ii), (iii) and (iv), we get that $(I, \omega_I) + (I, -\omega_I) = (I^{(2)}, *)$ in $E(A, L)$, as desired.

For the last part of the lemma, let ω'_I be another local L-orientation of I. Recall that ω_I is given by the equality $I = J + (a) + I^2$. Therefore, there exists $u \in A$ such that u is a unit modulo I and ω'_I is given by the equality $I = J + (ua) + I^2$.

Now we can apply the same arguments as in the first part of this lemma to obtain that $(I, \omega'_I) + (I, -\omega'_I) = (I^{(2)}, *)$ in $E(A, L)$. Therefore $(I, \omega_I) + (I, -\omega_I) = (I, \omega'_I) + (I, -\omega'_I)$ in $E(A, L)$. Thus the proof is complete. ∎

Proposition 3.7 *There is a group homomorphism* $E_0(A, L) \to E(A, L)$ *which sends* (I) *to* $(I, \omega_I) + (I, -\omega_I)$. *As a consequence, if* $(I) = 0$ *in* $E_0(A, L)$ *then the element* $(I, \omega_I) + (I, -\omega_I) = 0$ *in* $E(A, L)$.

Proof Recall that $E_0(A, L) = F/K$, where F is the free abelian group on the set S of ideals $\mathcal{N} \subset A$ such that $\mathcal{N} \subset A$ is an $\mathcal{M}$-primary ideal for some maximal ideal $\mathcal{M}$ of height n and $\mathcal{N}/\mathcal{N}^2$ is generated by n elements.

Let $\mathcal{N} \in S$ be arbitrary. Take any local L-orientation $\omega_{\mathcal{N}}$ of $\mathcal{N}$. Then, by (3.6), the association $(\mathcal{N}) \mapsto (\mathcal{N}, \omega_{\mathcal{N}}) + (\mathcal{N}, -\omega_{\mathcal{N}})$ does not depend on the choice of $\omega_{\mathcal{N}}$ and it is easy to see that it extends to a group homomorphism $\psi' : F \longrightarrow E(A, L)$.

Now K is the subgroup of F generated by elements (I) where I is an ideal of height n such that there exists a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow I$. Then α induces a global L-orientation, say, ω'_I of I. Therefore, $(I, \omega'_I) = 0 = (I, -\omega'_I)$ in $E(A, L)$ and clearly $\psi'((I)) = 0$. Hence ψ' induces a group homomorphism $\psi : E_0(A, L) \longrightarrow E(A, L)$.

Last statement of the proposition is now obvious. ∎

Remark 3.8 The last conclusion of the above proposition was earlier proved in [B-RS 2, Corollary 7.9] in the case $L = A$.

Let A, I be as in (3.3). Further assume that I is a local complete intersection ideal (of height n). In this situation it is easy to see that for any choice of $I^{(r)}$, the ideal $I^{(r)}$ is also a local complete intersection ideal. Moreover, for a suitable choice of $I^{(r)}$ we have $[A/I^{(r)}] = r[A/I]$ in $K_0(A)$. This can be proved as follows. Since I/I^2 is generated by n elements, using standard arguments about avoidance of prime ideals, one can show that there exists an A-regular sequence $\{a_1, \cdots, a_{n-1}\}$ such that if $J = (a_1, \cdots, a_{n-1})$ then $J \subset I$, I/J is a local complete intersection ideal of A/J of height 1 and $I/(J + I^2)$ is a free A/I-module of rank 1. Now $I^{(r)} = J + I^r$ will do the job. Moreover, in this case, if $r = (n-1)!$ then, by [Mu, Theorem 2.2], there exists a projective A-module P of rank n such that $I^{(r)}$ is a surjective image of P and $[P] - [A^r] = -[A/I]$ in $K_0(A)$. In what follows we tacitly assume that for a local complete intersection ideal I (of height n) our choice of $I^{(r)}$ is such that $[A/I^{(r)}] = r[A/I]$ in $K_0(A)$.

Theorem 3.9 *Let A be a ring of dimension $n \geq 2$ containing the field $\mathbb{Q}$ of rational numbers and let J be a local complete intersection ideal of height n. If the element $[A/J] \in K_0(A)$ is torsion then J is set theoretically generated by n elements.*

Proof Let t be a positive integer such that $t[A/J] = 0$ and let $J^{(t)}$ be the ideal as constructed above. Hence $[A/J^{(t)}] = t[A/J]$ in $K_0(A)$. Since the ideals $J^{(t)}, J$ have the same radical, replacing J by $J^{(t)}$ we assume that $[A/J] = 0$ in $K_0(A)$. Let $I = J^{((n-1)!)}$. Then, by [Mu, Theorem 2.2], there exists a projective A-module P of rank n such that I is a surjective image of P and $[P] - [A^n] = -[A/J]$ in $K_0(A)$. Since $[A/J] = 0$, P is stably free and hence, as I being a surjective image of P, $(I) = 0$ in $E_0(A)$. Therefore, by (3.7), $(I, \omega_I) + (I, -\omega_I) = 0$ in $E(A)$ for a local orientation ω_I of I. But then, by (3.6), $(I^{(2)}, *) = 0$ in $E(A)$. Hence, by (2.5), $I^{(2)}$ is a complete intersection ideal. Since $I^{(2)}, I, J$ have the same radical, J is a set theoretic complete intersection ideal. ∎

Remark 3.10 In the case J is not a local complete intersection ideal, J will be still set theoretically generated by n elements if we assume that (J) is a torsion element of $E_0(A)$. The following example shows that even if A is a smooth affine domain over an algebraically closed field, an ideal J

can be set theoretically complete intersection without $[A/J]$ being torsion in $K_0(A)$.

Example 3.11 Let $A = \mathbb{C}[X_1, X_2, X_3]/({X_1}^4 + {X_2}^4 + {X_3}^4 - 1)$ and let $F^2K_0(A)$ denote the subgroup of $K_0(A)$ generated by elements of the type $[A/M]$ where M is a maximal ideal of A.

Claim: $F^2K_0(A)$ is a nonzero torsion-free group.

Proof. Since $\dim(A) = 2$, $F^2K_0(A)$ is precisely the kernel $SK_0(A)$ of the group homomorphism $\widetilde{K_0}(A) \overset{\det}{\twoheadrightarrow} \mathrm{Pic}(A)$. Moreover the canonical map $CH_0(\mathrm{Spec}(A)) \to F^2K_0(A)$ sending $[M]$ to $[A/M]$ is an isomorphism.

Let Y be the smooth hypersurface in $\mathbf{P^3}(\mathbb{C})$ defined by the equation $U^4 + V^4 + W^4 - T^4 = 0$. Then $\mathrm{Spec}(A)$ is an affine open subset of Y. Since $p_g(Y) \neq 0$, by a result of Mumford [M], the group $A_0(Y)$ of zero cycles of Y of degree zero modulo rational equivalence is not finite dimensional. Hence, by [Mu-S, Theorem 2], $F^2K_0(A) \neq 0$. Since $F^2K_0(A)$ is isomorphic to $CH_0(\mathrm{Spec}(A))$, by [Bl-M-S, Proposition 2.1], it is torsion-free. Thus the claim is proved.

Since $F^2K_0(A) \neq 0$ there exists a maximal ideal M of A such that $[A/M] \neq 0$ in $K_0(A)$. By (2.2), there exists a prime element f in A such that $A/(f)$ is a smooth affine domain of dimension 1 over $\mathbb{C}$. Therefore $\mathrm{Pic}(A/(f))$ is a divisible group. Since $[A/M] \neq 0$, $m = M/(f)$ can not be a principal ideal. Putting these facts together we see that there exists an invertible ideal K of $A/(f)$ comaximal with m such that $m \cap K$ is not principal but $m \cap K^2$ is a principal ideal. Let I be the inverse image of K in A and let $I^{(2)} = (f) + I^2$. Then it is easy to see that $M \cap I^{(2)}$ is generated by 2 elements. This shows that the ideal $M \cap I$ is set theoretically generated by 2 elements. Since $F^2K_0(A)$ is torsion-free, if $[A/(M \cap I)]$ is a torsion element of $K_0(A)$ then $[A/(M \cap I)] = 0$.

Claim: $[A/(M \cap I)] \neq 0$.

Proof. Since M and I are comaximal, $[A/(M \cap I)] = [A/M] + [A/I]$. By similar argument, we see that $[A/(M \cap I^{(2)})] = [A/M] + [A/I^{(2)}] = [A/M] + 2[A/I]$. Since $M \cap I^{(2)}$ is generated by 2 elements we see that $[A/M] + [A/I^{(2)}] = 0$ in $K_0(A)$. But $[A/M] \neq 0$. Putting these facts together we see that $[A/(M \cap I)] \neq 0$. ∎

In the rest of this section we discuss some interesting applications of the results proved above.

Let A be a ring of dimension n and let P be a projective A-module of rank n with $\wedge^n(P) \simeq A$. It is known that if $P \simeq A \oplus Q$ for some projective A-module Q and if an ideal J of height n is a surjective image of P, then J is generated by n elements (see [MK]). In this context, the following converse question is natural.

Question. *Let A be a ring of dimension n and let P be a projective A-module of rank n with $\wedge^n(P) \simeq A$. Let $J \subset A$ be an ideal of height n such that J is generated by n elements. Assume that there is a surjective map $\alpha : P \twoheadrightarrow J$. Then, is $P \simeq Q \oplus A$?*

The question has an affirmative answer when A is an affine algebra over an algebraically closed field (see [MK]). However, it does not have an affirmative answer in general as can be seen by the following example.

Example 3.12 Let $A = \mathbb{R}[X_0, \cdots, X_n]/(\sum_{i=0}^n {X_i}^2 - 1)$ be the coordinate ring of the real n-sphere S^n. Assume that n is an even integer. Let $n = 2d$. Let v be the unimodular row $[x_0, \cdots, x_n]$ of A^{n+1} and let P be the projective A-module A^{n+1}/Av. Let J denote the ideal $(x_1, x_2, \cdots, x_n)$ of A. It is easy to see that $\text{ht}(J) = n$. The surjection $A^{n+1} \longrightarrow\!\!\!\!\rightarrow J$ defined by

$$e_0 \mapsto 0,\ e_{2i-1} \mapsto -x_{2i},\ e_{2i} \mapsto x_{2i-1},\ 1 \le i \le d$$

induces a surjection $\alpha : P \longrightarrow\!\!\!\!\rightarrow J$. Since $A^{n+1} \simeq P \oplus A$ we have $\wedge^n(P) \simeq A$. However, since P corresponds to the tangent bundle of S^n and n $(= 2d)$ is even, it follows that $P \not\simeq Q \oplus A$.

The following theorem shows that the above question has an affirmative answer if one of the generators of J is a square. Interestingly, in this case we do not need to assume that $\wedge^n(P) \simeq A$.

Theorem 3.13 *Let A be a ring containing $\mathbb{Q}$ and let $J \subset A$ be an ideal of height n such that $J = (f_1, \cdots, f_{n-1}, f_n^2)$. Let P be a projective A-module of rank n. Assume that there is a surjective map $\phi : P \twoheadrightarrow J$. Then $P \simeq Q \oplus A$ for some projective A-module Q.*

Proof Let $\wedge^n(P) \simeq L$ and let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism. The pair (ϕ, χ) induces a local L-orientation, say ω_J of J and therefore in terms of Euler class we have $e(P, \chi) = (J, \omega_J)$ in $E(A, L)$. By (2.5), it is enough to prove that $(J, \omega_J) = 0$ in $E(A, L)$.

Let $I = (f_1, \cdots, f_{n-1}, f_n)$. We first show that with respect to L, J can be realised as $I^{(2)}$. Let $K = (f_1, f_2, \cdots, f_{n-1})$. We can clearly assume that K has height $n-1$ and K/IK is generated by $n-1$ elements. Therefore, there exists a surjection $\beta : L \oplus A^{n-2} \twoheadrightarrow K/IK$ (note that $\dim A/I = 0$ and hence L/IL is free). Let $\alpha : L \oplus A^{n-2} \to K$ be a lift of β and let $K_1 = \alpha(L \oplus A^{n-2})$. We can clearly assume that height of K_1 is $n-1$. Now $K_1 + I^2 = K_1 + IK + I^2 = K + I^2 = (f_1, f_2, \cdots, f_{n-1}, f_n^2) = J$ and $I = K_1 + (f_n) + I^2$. This shows that $J = I^{(2)}$ with respect to L.

Since I is generated by n elements, $(I) = 0$ in $E_0(A)$. Since, by (2.9), $E_0(A) \xrightarrow{\sim} E_0(A, L)$ it follows that $(I)_L = 0$ in $E_0(A, L)$. Consequently, by (3.6) and (3.7) it follows that

$$(J, \omega_J) = (I^{(2)}, *) = (I, \omega_I) + (I, -\omega_I) = 0$$

in $E(A, L)$. This is what we wanted to prove. ∎

Theorem 3.14 *Let A be a ring of dimension $n \geq 2$ containing $\mathbb{Q}$ and I be an ideal of A of height n such that $I = (f_1, \cdots, f_{n-1}, f_n) + I^2$. Let $J = (f_1, \cdots, f_{n-1}) + I^2$. Let P be a projective A-module of rank n. Assume that there is a surjective map $\phi : P \twoheadrightarrow J$. Then any surjection $\Psi : P \twoheadrightarrow J/J^2$ can be lifted to a surjection $\psi : P \twoheadrightarrow J$.*

Proof Let $K = (f_1, \cdots, f_{n-1})$. Then $J = I^{(2)}$ with respect to K. Let $L = \wedge^n(P)$. Following the same arguments as in (3.13), it follows that there exists an ideal $K_1 \subset J$ of height $n-1$ such that K_1 is a surjective image of $L \oplus A^{n-2}$ and $J = K_1 + I^2$. This means $J = I^{(2)}$ with respect to K_1. Therefore, by Lemma 3.4, $(J, \omega) = (J, \omega')$ in $E(A, L)$ for any two local L-orientations of J.

Let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism. The pair (χ, ϕ) induces a local L-orientation, say ω_J of J, and hence $e(P, \chi) = (J, \omega_J)$ in $E(A, L)$.

Now consider the surjective map $\Psi : P \twoheadrightarrow J/J^2$. It is easy to see that there exists a surjection $\theta : P \twoheadrightarrow J \cap J_1$, where : (i) $J_1 \subset A$ is an ideal of height at least n; (ii) $J+J_1 = A$; (iii) $\theta \otimes A/J = \Psi$. Computing the Euler class of P using (χ, θ) we will have $e(P, \chi) = (J, \omega'_J) + (J_1, \omega_{J_1})$ in $E(A, L)$, where ω'_J, ω_{J_1} are local L-orientations of J, J_1 respectively, induced by (χ, θ). To show that Ψ lifts to a surjection $\psi : P \twoheadrightarrow J$, by [B-RS 2, Corollary 4.3], it is enough to show that $e(P, \chi) = (J, \omega'_J)$ in $E(A, L)$. But as mentioned above, by (3.4) it follows that $e(P, \chi) = (J, \omega_J) = (J, \omega'_J)$ in $E(A, L)$. This proves the theorem. ∎

4 The Main Theorem

In this section we prove our main theorem (4.31).

We first set up some notations which will be used throughout this section unless otherwise specified.

Notations. In what follows, varieties are assumed to be irreducible.
$X = \mathrm{Spec}\,(\mathrm{A})$ will always denote a smooth affine variety over the field $\mathbb{R}$ of real numbers, $X(\mathbb{R})$ will denote the set of real points of X and $\mathbb{R}(X)$ will denote the localization A_S of A with respect to the multiplicatively closed subset S of A consisting of all elements which do not belong to any real maximal ideal. The group of zero cycles of X modulo rational equivalence will be denoted by $CH_0(X)$.

Throughout this section we assume that A has at least one real maximal ideal and $\dim(A) = n \geq 2$.

Let L be a projective A-module of rank one. We denote, by abuse of notation, the Euler class group of $\mathbb{R}(X)$ with respect to $L \otimes_A \mathbb{R}(X)$ by $E(\mathbb{R}(X), L)$.

Remark 4.2 $X(\mathbb{R})$ is a real smooth manifold (C^∞-manifold) of dimension n with finitely many connected components. Since $X(\mathbb{R}) \neq \emptyset$, $\mathbb{R}(X)$ has dimension n. It is easy to deduce from well known topological results that the group $\mathrm{Pic}(\mathbb{R}(\mathrm{X}))$ of rank 1 projective $\mathbb{R}(X)$-modules is a 2-torsion group.

To begin with we derive a structure theorem for $E(\mathbb{R}(X), L)$ which plays a major role in our proof of the main theorem. The following lemma is crucial for this structure theorem.

Lemma 4.3 *Let L_1 be a projective $\mathbb{R}(X)$-module of rank 1. Let m be a maximal ideal of $\mathbb{R}(X)$ and ω_m be a local L_1-orientation of m. Then $(m, \omega_m) + (m, -\omega_m) = 0$ in $E(\mathbb{R}(X), L_1)$. Moreover, if $\widetilde{\omega}_m$ is another local L_1-orientation of m then either $(m, \widetilde{\omega}_m) = (m, \omega_m)$ or $(m, \widetilde{\omega}_m) = (m, -\omega_m)$ in $E(\mathbb{R}(X), L_1)$.*

Proof For simplicity of notation we write $B = \mathbb{R}(X)$. Since B is a localization of the regular ring A, L_1 is a projective B-module of rank 1, and $B/m = \mathbb{R}$, there exists a projective A-module L of rank 1 and a real maximal ideal M of A such that $L_1 = L \otimes_A B$ and $MB = m$.

Let $F = L \oplus A^{n-2}$. Then, by Swan's Bertini theorem (2.2), it is easy to see that there exists an A-linear map $\phi : F \to A$ such that if $J = \phi(F)$ then $J \subset M$ and A/J is a smooth affine domain of dimension 1 over $\mathbb{R}$. Let $K = JB$. Then, $\overline{m} = m/K$ is an invertible ideal of the Dedekind domain B/K.

Let $\overline{X} = \mathrm{Spec}(A/J)$. Then it is easy to see that $\mathbb{R}(\overline{X}) = B/K$ and hence every rank one projective B/K-module is 2-torsion. This implies that the invertible ideal $\overline{m}^2$ is a principal ideal. Therefore $m^{(2)} = K + m^2$ is a surjective image of $F \otimes_A B \oplus B = L_1 \oplus B^{n-1}$. Now the result follows from (3.6).

Let $\widetilde{\omega}_m$ be another local orientation of m. Then, since $\mathbb{R}(X)/m = \mathbb{R}$, it follows (from definition of orientation) that there exists $\lambda \in \mathbb{R}^*$ such that $\widetilde{\omega}_m = \lambda\omega_m$. If $\lambda > 0$ then there exists $\delta \in \mathbb{R}$ such that $\lambda = \delta^2$ and hence, by (2.6), $(m, \widetilde{\omega}_m) = (m, \omega_m)$. If $\lambda < 0$, by similar argument we see that $(m, \widetilde{\omega}_m) = (m, -\omega_m)$. ∎

To deduce a structure theorem for $E(\mathbb{R}(X), L)$ we need to set up some machinery.

Let $\mathcal{E}$ be a rank 1 projective A-module. Let $D = \bigoplus_{-\infty < i < \infty} \mathcal{E}^i$ and let $Z = \mathrm{Spec}(D)$. It is easy to see that Z is a smooth affine variety of dimension $n+1$ over $\mathbb{R}$ and hence the set $Z(\mathbb{R})$ of all real points of Z is a real manifold of dimension $n+1$. Moreover the canonical map $Z \to X$ induces the map $\Pi : Z(\mathbb{R}) \to X(\mathbb{R})$ which is surjective, open and continuous in the Euclidean topology. Note that the projective module $\mathcal{E}$ gives rise to a topological line bundle over the manifold $X(\mathbb{R})$ (which, by abuse of notation, we still denote by $\mathcal{E}$) and $Z(\mathbb{R})$ is the complement of the zero section of $\mathcal{E}$. Let C be a connected component of $X(\mathbb{R})$ and let $\mathcal{E}_C$ denote the restriction of $\mathcal{E}$ to C. Then it is easy to see that $\Pi^{-1}(C)$ is nothing but the complement of the zero section of $\mathcal{E}_C$. Hence, if $\mathcal{E}_C \simeq C \times \mathbb{R}$, then $\Pi^{-1}(C)$ is a disjoint union $(C \times \mathbb{R}^+) \bigcup (C \times \mathbb{R}^-)$.

It is a well known topological fact that if C is compact and $\mathcal{E}_C \not\simeq C \times \mathbb{R}$, then the complement $\Pi^{-1}(C)$ is connected.

Let N be a real maximal ideal of D and let $M = N \cap A$. Then the canonical map $\mathcal{E}/M\mathcal{E} \to D/N (= \mathbb{R})$ is an isomorphism of vector spaces. Hence there exists a unique non-zero element e of $\mathcal{E}/M\mathcal{E}$ which goes to 1 under this isomorphism. Let us consider the following set :

$$T' = \{(M, e) | M \in X(\mathbb{R}), e \in \mathcal{E}/M\mathcal{E} - (0)\}.$$

It is easy to see that the correspondence $N \mapsto (M, e)$ gives a bijective map from $Z(\mathbb{R})$ to T'.

Hence onward we work in the following set up.

Let L be a projective A-module of rank 1 and let $\mathcal{E} = L \otimes_A K_A$ where K_A denotes the canonical module $\wedge^n(\Omega_{A/\mathbb{R}})$. Let D, Z, T' be as defined above.

Now consider the set T of pairs (M, ω_M) where M is a real maximal ideal of A and ω_M is a local L-orientation of M. The next lemma shows that there exists a bijection between T and $Z(\mathbb{R})$. Before proceeding to prove this lemma, we make the following remark.

Remark 4.4 Since maximal ideals of $\mathbb{R}(X)$ are in one to one correspondence with real maximal ideals of A, we identify T with the set of pairs (m, ω_m) where m is a maximal ideal of $\mathbb{R}(X)$ and ω_m is a local $L \otimes_A \mathbb{R}(X)$-orientation of m. This identification remains intact even if we replace A by A_g for $g \in A$ which does not belong to any real maximal ideal. Hence we can (and will) replace A by A_g whenever necessary.

Since $\mathrm{Pic}(\mathbb{R}(\mathrm{X}))$ is a 2-torsion group, there exists $g \in A$ such that g does not belong to any real maximal ideal of A and $L^2 \otimes_A A_g \simeq A_g$ where $L^2 = L \otimes_A L$. Therefore, replacing A by A_g if necessary, we always assume that $L^2 \simeq A$. In what follows, we fix a generator κ of L^2.

Lemma 4.5 *There exists a bijection (of sets) $\Theta : Z(\mathbb{R}) \to T$.*

Proof We have already seen that there is a bijective map between the sets $Z(\mathbb{R})$ and T'. Therefore it is enough to show that there is a bijection from T to T'.

Let M be a real maximal ideal. Then the differential map $d : A \to \Omega_{A/\mathbb{R}}$ gives rise to an isomorphism

$$\wedge^n(\overline{d}) : \wedge^n(M/M^2) \xrightarrow{\sim} K_A/MK_A.$$

Recall that a local L-orientation ω_M on M can be thought of as an isomorphism

$$\omega_M : L/ML \xrightarrow{\sim} \wedge^n(M/M^2).$$

Composing ω_M with $\wedge^n(\overline{d})$, we get an isomorphism, say,

$$\phi_{\omega_M} : L/ML \xrightarrow{\sim} K_A/MK_A.$$

Let $\Delta_{\omega_M} : L/ML \otimes L/ML (= L^2/ML^2) \xrightarrow{\sim} L/ML \otimes K_A/MK_A (= \mathcal{E}/M\mathcal{E})$ be the isomorphism defined as:

$$\Delta_{\omega_M}(l \otimes l') = l \otimes \phi_{\omega_M}(l'); \quad l, l' \in L/ML.$$

Let $\overline{\kappa}$ denote the image of κ in L^2/ML^2 and let $\Delta_{\omega_M}(\overline{\kappa}) = e$. It is easy to see that $(M, \omega_M) \mapsto (M, e)$ defines a bijection from T to T'. ∎

Using the bijection Θ, we now define a relation $\sim$ on $Z(\mathbb{R})$ as follows:

Definition 4.6 Let $z_1, z_2 \in Z(\mathbb{R})$ and let $\Theta(z_i) = (M_i, \omega_{M_i}),\ \ i = 1, 2$. We say $z_1 \sim z_2$ if image of (M_1, ω_{M_1}) = image of (M_2, ω_{M_2}) in $E(\mathbb{R}(X), L)$. Obviously $\sim$ is an equivalence relation on $Z(\mathbb{R})$.

Our aim is to show that $\sim$ is an open relation on the manifold $Z(\mathbb{R})$. For this we need some preliminaries.

For the sake of simplicity of notation, from now on we will denote K_A by K. Suppose $f \in A$ be such that $L_f \simeq A_f \simeq K_f$. Let us fix generators τ and ρ of L_f and K_f respectively. Then $\mathcal{E}_f$ is generated by $\tau \otimes \rho$ and therefore $D_f = D \otimes_A A_f = A_f[T, T^{-1}]$; $T = (\tau \otimes \rho)$. Let $Z' = \mathrm{Spec}(D_f)$. With respect to the pair (τ, ρ), we assign to every $z \in Z'(\mathbb{R})$ an element of the group $\{1, -1\}$ (and call it the *signature* of z) as follows.

Let sgn denote the group homomorphism from $\mathbb{R}^*$ to $\{1, -1\}$ defined as $sgn(\mu) = 1$ if $\mu > 0$ and $sgn(\mu) = -1$ if $\mu < 0$.

Definition 4.7 Let (M, e) be an element of T' associated to z. Then $f \notin M$ and hence $e = \lambda\overline{\tau} \otimes \overline{\rho}$ for some $\lambda \in \mathbb{R}^*$. We define the *signature* of z to be $sgn(\lambda)$ and denote it by $sgn_{(\tau,\rho)}(z)$.

Let $\Theta(z) = (M, \omega_M)$ where M is a real maximal ideal of A such that $f \notin M$ and ω_M is a local L-orientation of M. We call $sgn_{(\tau,\rho)}(z)$ to be the signature of (M, ω_M) and denote it $sgn_{(\tau,\rho)}(M, \omega_M)$.

Remark 4.8 Let M be a real maximal ideal of A and let $\omega_M, \widetilde{\omega}_M$ be two local L-orientations of M. Then there exists $\alpha \in \mathbb{R}^*$ such that $\widetilde{\omega}_M = \alpha\omega_M$. If $f \notin M$ then it is easy to see that $sgn_{(\tau,\rho)}(M, \widetilde{\omega}_M) = sgn(\alpha)sgn_{(\tau,\rho)}(M, \omega_M)$.

Every $z \in Z'(\mathbb{R})$ corresponds to a real maximal ideal N of $A_f[T, T^{-1}]$, $T = (\tau \otimes \rho)$. It is easy to see that, if $N = (M, T - \delta)$; $\delta \in \mathbb{R}^*$ and $(M, e) \in T'$ is associated to z, then $e = \delta^{-1}\overline{\tau} \otimes \overline{\rho}$. Hence $sgn(\delta) = sgn_{(\tau,\rho)}(z)$.

Let $X' = \text{Spec}(A_f)$. Then $Z'(\mathbb{R}) = \Pi^{-1}(X'(\mathbb{R}))$ can be thought of as a disjoint union $(X'(\mathbb{R}) \times \mathbb{R}^+) \bigcup (X'(\mathbb{R}) \times \mathbb{R}^-)$ where $z \in X'(\mathbb{R}) \times \mathbb{R}^+$ if $sgn_{(\tau,\rho)}(z) = 1$ otherwise it belongs to $X'(\mathbb{R}) \times \mathbb{R}^-$.

To show that $\sim$ is an open relation on $Z(\mathbb{R})$, we need couple of lemmas. The following lemma is standard and hence we omit the proof.

Lemma 4.9 *Let A be a smooth affine domain of dimension n over $\mathbb{R}$ and let M be a real maximal ideal of A. Let L be a rank 1 projective A-module. Assume that A is a surjective image of $\mathbb{R}^{[l]}$ where $\mathbb{R}^{[l]}$ denotes a polynomial algebra in l variables. Then there exists a set of variables $\{T_1, \cdots, T_l\}$ (i.e. $\mathbb{R}^{[l]} = \mathbb{R}[T_1, \cdots, T_l]$) and $f \notin M$ such that A is a finite module over $\mathbb{R}[T_1, \cdots, T_n]$, $\Omega_{A_f/\mathbb{R}[T_1,\cdots,T_n]} = 0$ and $L_f \simeq A_f$.*

The following lemma, which plays a very important role in our proof, is essentially proved in ([B-RS 1], Lemma 4.6).

Lemma 4.10 *Let $F = (f_1(t), \cdots, f_l(t)) : (a, b) \to \mathbb{R}^l$ be a C^∞ embedding. Assume that for every $t \in (a, b)$, $({f_1}'(t), \cdots, {f_l}'(t)) \neq (0, \cdots, 0)$. Then for every $t_0 \in (a, b)$ and a Euclidean neighbourhood $W \subset \mathbb{R}^l$ of $F(t_0)$, there exists a neighbourhood (c, d) of t_0 contained in (a, b), such that :*

1. *For any point $t_2 \in (c,d)$, the distance (square) function $G : (c,d) \longrightarrow \mathbb{R}$ given by $G(t) = \| F(t) - F(t_2) \|^2$ has the property that $G'(t) > 0$ for $t > t_2$ and $G'(t) < 0$ for $t < t_2$.*

2. *As a consequence, for any two points $t_1, t_3 \in (c,d)$ with $t_1 < t_3$, there exists a sphere S^{l-1} in $\mathbb{R}^l$ contained in W having center $F(t_2)$ where $t_1 < t_2 < t_3$, which intersects $F(a,b)$ in precisely two points $F(t_1)$ and $F(t_3)$. Moreover, the intersections are transversal.*

Now we are ready to prove that the relation $\sim$ defined in (4.6) is an open equivalence relation. Before doing so we summarize the set up for the convenience of a reader.

1. $X = \mathrm{Spec}(A)$ is a smooth affine variety of dimension $n \geq 2$ over $\mathbb{R}$.

2. L is a projective A-module of rank one. We assume that $L^2 = L \otimes_A L \simeq A$. We fix a generator κ of L^2.

3. K denotes the canonical module $\wedge^n \Omega_{A/\mathbb{R}}$. Let $\mathcal{E} = L \otimes_A K$. Let $D = \bigoplus_{-\infty < i < \infty} \mathcal{E}^i$ and $Z = \mathrm{Spec}(D)$.

4. There is a bijection Θ from the set $Z(\mathbb{R})$ of real points of Z to the set of pairs (M, ω_M) where M is a real maximal ideal and ω_M is a local L-orientation of M.

5. For two elements $z, z' \in Z(\mathbb{R})$, we say $z \sim z'$ if $\Theta(z) = \Theta(z')$ in $E(\mathbb{R}(X), L)$. Clearly $\sim$ is an equivalence relation on $Z(\mathbb{R})$.

Proposition 4.11 *With set up as above, the relation $\sim$ is an open equivalence relation on $Z(\mathbb{R})$.*

Proof Let $z \in Z(\mathbb{R})$ and let $\Theta(z) = (M, \omega_M)$ where M is a real maximal ideal of A and ω_M is a local L-orientation of M. Let $x \in X(\mathbb{R})$ be the point corresponding to M.

Since X is affine, we can assume that X is a closed subvariety of the affine space $\mathbb{R}^l$. Then, by (4.9), there exists a suitable choice of a coordinate system of $\mathbb{R}^l$, such that the projection map $\pi : \mathbb{R}^l \to \mathbb{R}^n$ when restricted to $X(\mathbb{R})$ has finite fibers. Moreover, there exists $f \in A$ such that $f \notin M$, $L_f \simeq A_f$ and $\Omega_{A_f/\mathbb{R}}$ is generated by $dT_1, \cdots, dT_n$. Therefore, by the Inverse Function Theorem, there exists a Euclidean neighbourhood U of x contained in the Zariski neighbourhood $\mathrm{Spec}(A_f)$ such that the restriction of the projection map π to U is a C^∞ diffeomorphism onto an open ball $\mathcal{B}$ in $\mathbb{R}^n$ with center $\pi(x)$.

We fix a generator τ of L_f. Note that the canonical module $K_f = K \otimes_A A_f$ is generated by $\rho = dT_1 \wedge \cdots \wedge dT_n$.

Since, $f \notin M$, z corresponds to a real maximal ideal of $D_f = D \otimes_A A_f$. Since $D_f = A_f[T, T^{-1}],\ T = (\tau \otimes \rho)$ we can assign to z its signature $sgn_{(\tau,\rho)}(z)$.

Recall that if $Z' = \mathrm{Spec}(D_f)$ and $X' = \mathrm{Spec}(A_f)$ then
$Z'(\mathbb{R}) = X'(\mathbb{R}) \times \mathbb{R}^+ \bigcup X'(\mathbb{R}) \times \mathbb{R}^-$.

Without loss of generality we may assume $sgn_{(\tau,\rho)}(z)$ is positive. Then $z \in X'(\mathbb{R}) \times \mathbb{R}^+$ and hence $z \in U \times \mathbb{R}^+$. To prove the proposition it is enough to prove that, $z \sim z'$ for every $z' \in U \times \mathbb{R}^+$. In what follows, we detail the proof in steps.

Step 1. Let $\Pi(z') = x'$. Note that $x' \in U$.

If $x = x'$ then z' corresponds to a local L-orientation $\widetilde{\omega}_M$ of M. Let $\lambda \in \mathbb{R}^*$ be such that $\widetilde{\omega}_M = \lambda\omega_M$. Since $z, z' \in U \times \mathbb{R}^+$, it follows that λ is a positive real number and hence is a square. Therefore, by (2.6), $(M, \widetilde{\omega}_M) = (M, \omega_M)$ in $E(A, L)$ and hence in $E(\mathbb{R}(X), L)$. Therefore $z \sim z'$.

Step 2. So we assume that $x \neq x'$. Let W be an open subset of $\mathbb{R}^l$ such that $W \cap X(\mathbb{R}) = U$. Since $x \neq x'$, $\pi(x) \neq \pi(x')$. Without loss of generality we assume that $\pi(x) = (0, \cdots, 0)$ and $\pi(x') = (\gamma_1, \cdots, \gamma_n)$. Moreover, without loss of generality, we assume that $\gamma_n \neq 0$. The line $\mathcal{L}$ joining the two points $\pi(x) = (0, \cdots, 0)$ and $\pi(x') = (\gamma_1, \cdots, \gamma_n)$ is given by $n-1$ equations :

$$H_i : T_i - \zeta_i T_n,\ \ \zeta_i = \gamma_i/\gamma_n,\ 1 \leq i \leq n-1.$$

Let $\mathcal{L}_1$ be the segment of $\mathcal{L}$ contained in $\pi(U) = \mathcal{B}$. Then there exists an open interval (a, b) and a C^∞ function from (a, b) to $\mathcal{L}_1$ given by :

$$t \mapsto (\zeta_1 t, \zeta_2 t, \cdots, \zeta_{n-1} t, t).$$

Composing the above function with π^{-1} we obtain a C^∞ embedding $F(t) = (f_1(t), \cdots, f_n(t), \cdots, f_l(t))$ from (a,b) to $\pi^{-1}(\mathcal{L}_1) \subset U \subset \mathbb{R}^l$. It is easy to see that $f_i(t) = \zeta_i t$ for $i = 1, \cdots, n-1$, $f_n(t) = t$ and $f_j(t) = g_j(\zeta_1(t), \cdots, \zeta_{n-1}(t), t)$ for $j = n+1, \cdots, l$.

Let $t_0 \in (a,b)$, Since ${f_n}'(t) = 1$, by (4.10), there exists a neighbourhood (c,d) of t_0 contained in (a,b) such that for $c < t_1 < t_3 < d$ there exists a sphere S^{l-1} in $\mathbb{R}^l$ contained in W having center $F(t_2)$ where $t_1 < t_2 < t_3$, which intersects $F(a,b)$ in precisely two points $F(t_1)$ and $F(t_3)$. Moreover, the intersections are transversal.

Suppose that the sphere S^{l-1} centered at the point $F(t_2) = (f_1(t_2), \cdots, f_l(t_2))$ has radius $r = \parallel F(t_1) - F(t_2) \parallel = \parallel F(t_3) - F(t_2) \parallel$. Then its defining equation is $H : \sum_{i=1}^{l} (T_i - f_i(t_2))^2 - r^2$.

Step 3. Let M_1, M_3 be real maximal ideals of A corresponding to $F(t_1), F(t_3)$ respectively. In this step we show that, replacing A by A_g ($g \in A$ belongs to only complex maximal ideals) if necessary, there exists a surjection $\beta : L \oplus A^{n-1} \twoheadrightarrow M_1 \cap M_3$.

Since $S^{l-1} \subset W$, $S^{l-1} \cap X(\mathbb{R}) \subset U$. Therefore, if h denotes the image of H in A, then the ideal (f,h) of A is contained in only complex maximal ideals of A. Since we are interested in only real maximal ideals and their local L- orientations, inverting $f^2 + h^2$ we assume that $(f) + (h) = A$.

Recall that the line $\mathcal{L}$ is given by $n-1$ equations $H_i : T_i - \zeta_i T_n,\ i = 1, \cdots, n-1$. Let h_i denote image of H_i in A. Then, since S^{l-1} intersects $F(a,b)$ in precisely two points $F(t_1)$ and $F(t_3)$ and the intersections are transversal, we get that $(h_1, h_2, \cdots, h_{n-1}, h) = M_1 \cap M_3 \cap I$ where M_1, M_3 are real maximal ideals of A corresponding to $F(t_1), F(t_3) \in U$ respectively and I is an ideal of A not contained in any real maximal ideal of A. Let $b \in I$ be such that b does not belong to any real maximal ideal of A. So, inverting b if necessary, we assume that $(h_1, h_2, \cdots, h_{n-1}, h) = M_1 \cap M_3$.

Let $J = M_1 \cap M_3$. Since $(f) + (h) = A$, we have $L/hL = L_f/hL_f$. Recall that $L_f \simeq A_f$ and τ is a generator of L_f. Therefore there exists $l \in L$ such that $l \equiv \tau \mod (h)$. We write h_n for h. Now using the fact that $L/h_nL \simeq A/(h_n)$ and $J = (h_1, \cdots, h_{n-1}, h_n)$ we see that there exists a surjection $\beta : L \oplus A^{n-1} \twoheadrightarrow J$ such that $\beta(l) - h_1 \in (h_n)$ and $\beta(e_i) = h_i;\ 2 \leq i \leq n$ where $(e_2, \cdots, e_n)$ denotes a basis of A^{n-1}.

Step 4. The surjection $\beta : L \oplus A^{n-1} \twoheadrightarrow M_1 \cap M_3$ gives rise to local L-orientations ω_{M_1} and ω_{M_3} on M_1, M_3 respectively. This step is devoted to prove the following claim about the signatures of (M_1, ω_{M_1}) and (M_3, ω_{M_3}) (for definition of signature, see (4.7)).

Claim: $sgn_{(\tau,\rho)}(M_1, \omega_{M_1}) \neq sgn_{(\tau,\rho)}(M_3, \omega_{M_3})$.

Proof of the claim: Since $\Omega_{A_f/\mathbb{R}}$ is generated by $dT_1, \cdots, dT_n$, it follows that

$$dh_1 \wedge dh_2 \cdots \wedge dh_n = \upsilon \, dT_1 \wedge dT_2 \cdots \wedge dT_n = \upsilon \rho$$

for some $\upsilon \in A_f$. So υ defines a continuous function from U to $\mathbb{R}$. Let $t \in (a, b)$. Then $F(t) \in U$. It is easy to see that $\upsilon(F(t)) = G'(t)$ where $G(t) = \parallel F(t) - F(t_2) \parallel^2$. Therefore, by (4.10), $\upsilon(F(t_1)) < 0$ and $\upsilon(F(t_3)) > 0$ as $c < t_1 < t_2 < t_3 < d$.

Let "bar" denote reduction modulo M_1. Recall that κ is a generator of L^2 and τ is a generator of L_f. Therefore $\kappa = u(\tau \otimes \tau)$ for some $u \in {A_f}^*$. Hence u defines a continuous function from U to $\mathbb{R}^*$. Since U is connected, $u(U) \subset (0, \infty)$ or $u(U) \subset (-\infty, 0)$. Note that in L/M_1L, we have $\overline{l} = \overline{\tau}$.

Since $\overline{\widetilde{h_1}} \wedge \overline{h}_2 \wedge \cdots \wedge \overline{h}_n = \overline{h}_1 \wedge \overline{h}_2 \wedge \cdots \wedge \overline{h}_n$, in $\wedge^n(M_1/{M_1}^2)$ where $\widetilde{h_1} = \beta(l)$, it clearly follows that under the isomorphism $\omega_{M_1} : L/ML \overset{\sim}{\to} \wedge^n(M_1/{M_1}^2)$, we have $\overline{\tau} \mapsto \overline{h}_1 \wedge \overline{h}_2 \wedge \cdots \wedge \overline{h}_n$. Further, $\wedge^n(\overline{d})(\overline{h}_1 \wedge \overline{h}_2 \wedge \cdots \wedge \overline{h}_n) = \overline{dh_1} \wedge \overline{dh_2} \wedge \cdots \wedge \overline{dh_n} = \overline{\upsilon} \, \overline{\rho}$ and $\overline{\upsilon} = G'(t_1)$.

Now the above discussion shows that $\Delta_{\omega_{M_1}}(\overline{\kappa}) = G'(t_1)\overline{u} \, (\overline{\tau} \otimes \overline{\rho})$ where

$$\Delta_{\omega_{M_1}} \overset{\text{def}}{=} \text{id} \otimes \wedge^n(\overline{d})\omega_{M_1} : L/M_1L \otimes L/M_1L \longrightarrow L/M_1L \otimes K/M_1K.$$

Therefore $sgn_{(\tau,\rho)}(M_1, \omega_{M_1}) = sgn(G'(t_1)\overline{u})$. Note that $\overline{u} = u(F(t_1))$ By similar argument, we see that $sgn_{(\tau,\rho)}(M_3, \omega_{M_3}) = sgn(G'(t_3)u(F(t_3)))$. Since $G'(t_1) < 0 < G'(t_3)$ and $u(F(t_1))u(F(t_3)) > 0$, we are through.

Step 5. Let $M_1\mathbb{R}(X) = m_1$ and $M_3\mathbb{R}(X) = m_3$. Local L-orientations $\omega_{M_1}, \omega_{M_3}$ induce local L-orientations $\omega_{m_1}, \omega_{m_3}$ on m_1, m_3 respectively. Then we have $(m_1, \omega_{m_1}) + (m_3, \omega_{m_3}) = 0$ in $E(\mathbb{R}(X), L)$. Since, by (4.3), we have $(m_1, \omega_{m_1}) + (m_1, -\omega_{m_1}) = 0$ we see that $(m_1, -\omega_{m_1}) = (m_3, \omega_{m_3})$ in $E(\mathbb{R}(X), L)$. Thus image of $(M_1, -\omega_{M_1})$ = image of (M_3, ω_{M_3}) in $E(\mathbb{R}(X), L)$. Moreover,

$$sgn_{(\tau,\rho)}(M_1, -\omega_{M_1}) = -sgn_{(\tau,\rho)}(M_1, \omega_{M_1}) = sgn_{(\tau,\rho)}(M_3, \omega_{M_3}).$$

Since t_1, t_3 are arbitrary elements of (c, d), we get that if $z_1, z_3 \in F(c, d) \times \mathbb{R}^+$ then $z_1 \sim z_3$.

Step 6. We define a relation on (a, b) as follows : We define two points $y, w \in (a, b)$ to be *related*, if there exists an open interval I contained in (a, b) and containing the points y, w, such that for every $z_i, z_j \in F(I) \times \mathbb{R}^+$ $z_i \sim z_j$.

The above discussion shows that the above relation is reflexive. By definition this relation is symmetric. It is easy to see that this relation is also transitive. Thus (a, b) is a disjoint union of equivalence classes. Each equivalence class is open in (a, b). Since (a, b) is connected, it follows that there is only one equivalence class viz (a, b). Therefore, since $z, z' \in F(a, b) \times \mathbb{R}^+$, $z \sim z'$.

Thus we have proved that $\sim$ is an open equivalence relation on $Z(\mathbb{R})$. ■

Now we proceed to prove series of lemmas needed to obtain a structure theorem for $E(\mathbb{R}(X), L)$. We first state a result.

Theorem 4.12 *Let $\widetilde{Y}$ be a complete smooth curve over $\mathbb{R}$ and let $\widetilde{Y}(\mathbb{R})$ be the set of real points of $\widetilde{Y}$. Let C be a connected component of $\widetilde{Y}(\mathbb{R})$ in the Euclidean topology. Let f be a nonzero rational function on $\widetilde{Y}$. Then the number of points of C at which f has odd order is even.*

Remark 4.13 The referee brought to our attention that this theorem can be proved very easily using simple results from calculus (modulo the result that any real smooth compact connected manifold of dimension 1 is diffeomorphic to the circle S^1). However, as far as we know, an algebraic proof of this theorem is due to Knebusch [Kn, Theorem 3.4] and his proof works for any real closed field.

As a consequence of above theorem, we prove the following lemma which is a generalization of ([B-RS 1], Lemma 4.2).

Lemma 4.14 *Let L be a projective A-module of rank 1. Let $J \subset A$ be a reduced ideal of height n which is a surjective image of $L \oplus A^{n-1}$. Then A/J is supported on an even number of points on each compact connected component of $X(\mathbb{R})$.*

Proof Let $F = L \oplus A^{n-2}$ and let $\beta : F \oplus A \twoheadrightarrow J$ be a surjection. Let C be a compact connected component of $X(\mathbb{R})$, such that A/J is supported on at least one point of C. Since J is reduced, we may assume (by Swan's Bertini theorem (2.2)), that if $I = \beta(F)$ then I is a prime ideal and $B = A/I$ is smooth of dimension 1. Let $\beta(0,1) = a$.

Let $Y = \operatorname{Spec}(B)$ and let $C' = C \cap Y(\mathbb{R})$. Let $C_1', \cdots, C_l'$ be connected components of C'. Note that since C is compact, C_i' are compact, $1 \le i \le l$. Let $\widetilde{Y}$ denote the smooth projective completion of Y. Then it is easy to see that C_i' is also a compact connected component of $\widetilde{Y}(\mathbb{R})$, for each i. Let a' denote the image of a in B. By (4.12), the number of points of C_i' at which a' has odd order is even. Hence the lemma follows. ■

Lemma 4.15 *Assume that $X(\mathbb{R})$ is compact. Let K be the canonical module $\wedge^n(\Omega_{A/\mathbb{R}})$ and L be a projective A-module of rank one. Let $\mathcal{E} = L \otimes_A K$, $D = \bigoplus_{-\infty<i<\infty} \mathcal{E}^i$ and $Z = \operatorname{Spec}(D)$. Let $\Pi : Z(\mathbb{R}) \to X(\mathbb{R})$ be the canonical map. Then for every equivalence class U (with respect to the relation $\sim$), the projection $\Pi(U)$ is contained in some connected component of $X(\mathbb{R})$ (in the Euclidean topology).*

Proof Let $U \subset Z(\mathbb{R})$ be an equivalence class and let C be a connected component of $X(\mathbb{R})$. In order to prove the lemma it is enough to show that either $\Pi(U) \subset C$ or $\Pi(U) \cap C = \emptyset$.

If $X(\mathbb{R})$ is connected then there is nothing to prove. So assume that $X(\mathbb{R})$ is not connected. Let C be a connected component of $X(\mathbb{R})$ such that $\Pi(U) \cap C \neq \emptyset$. If $\Pi(U) \not\subset C$ then there exists another connected component C_1 of $X(\mathbb{R})$ such that $\Pi(U) \cap C_1 \neq \emptyset$. Let $z, z_1 \in U$ be such that $\Pi(z) \in C$ and $\Pi(z_1) \in C_1$. Let $\Theta(z) = (M, \omega_M)$ and $\Theta(z_1) = (M_1, \omega_{M_1})$. Let $m = M\mathbb{R}(X)$ and $m_1 = M_1\mathbb{R}(X)$. Let $\omega,\ \omega_1$ be local $L \otimes_A \mathbb{R}(X)$-orientations on m, m_1 induced by $\omega_M,\ \omega_{M_1}$ respectively. Since $z, z_1 \in U$ $z \sim z_1$ and hence $(m, \omega) = (m_1, \omega_1)$ in $E(\mathbb{R}(X), L)$. Therefore, by (4.3), $(m, \omega) + (m_1, -\omega_1) = 0$ in $E(\mathbb{R}(X), L)$. Therefore, by (2.5), there exists $g \in A$ such that g does not belong to any real maximal ideal of A and a surjection $\beta : F_g \oplus A_g \twoheadrightarrow (M \cap M_1)_g$ where $F = L \oplus A^{n-2}$. Since A_g is smooth, by (2.2), we assume that $B = A_g/J$ is a smooth affine domain of dimension 1 where J denotes the ideal $\beta(F_g)$. Let $Y = \operatorname{Spec}(B)$ and $\widetilde{Y}$ be a smooth completion of Y.

Since g does not belong to any real maximal ideal of A, it is easy to see that $Y(\mathbb{R})$ is a closed subset of $X(\mathbb{R})$ and hence compact. This implies that $Y(\mathbb{R}) = \widetilde{Y}(\mathbb{R})$.

Let $x, x_1 \in Y(\mathbb{R})$ be points corresponding to maximal ideals $M_g/J, M_{1g}/J$. Since $x \in C \cap Y(\mathbb{R})$ and $x_1 \in C_1 \cap Y(\mathbb{R})$, x, x_1 belong to different connected components of $Y(\mathbb{R})$. It is easy to see that $\overline{b} \in B$ vanishes at only two points of $Y(\mathbb{R})$ viz. x and x_1 and it has order 1 at these points, where $\overline{b}$ is the image of $\beta(0,1)$ in B. But this is a contradiction by (4.12) as $Y(\mathbb{R}) = \widetilde{Y}(\mathbb{R})$. This completes the proof. ∎

Let $X, A, L, K, \mathcal{E}, Z$ be as above. Let $z \in Z(\mathbb{R})$ and let $\Theta(z) = (M, \omega_M)$. Let $m = M\mathbb{R}(X)$ and let ω_m be the induced local $L \otimes_A \mathbb{R}(X)$- orientation of m. In what follows, depending upon context, we regard (m, ω_m) as $\Theta(z)$.

Lemma 4.16 *Let $X, A, L, K, \mathcal{E}, Z, C$ be as in above lemma. Let $x, y \in C$ and let m_x, m_y be corresponding maximal ideals of $\mathbb{R}(X)$. Let ω_x and ω_y be local L-orientations of m_x and m_y respectively. Let K_C and L_C denote restriction of (induced) line bundles on $X(\mathbb{R})$ to C. If $K_C \not\simeq L_C$ then $(m_x, \omega_x) = (m_y, \omega_y)$ in $E(\mathbb{R}(X), L)$. As a consequence, in this case, $2(m_x, \omega_x) = 0$ in $E(\mathbb{R}(X), L)$. If $K_C \simeq L_C$, then either $(m_x, \omega_x) = (m_y, \omega_y)$ or $(m_x, \omega_x) = (m_y, -\omega_y)$ in $E(\mathbb{R}(X), L)$.*

Proof Let $z, w \in Z(\mathbb{R})$ be such that $\Theta(z) = (m_x, \omega_x)$ and $\Theta(w) = (m_y, \omega_y)$ respectively. Since $\Pi(z) = x, \Pi(w) = y$, we have $z, w \in \Pi^{-1}(C)$. If $K_C \not\simeq L_C$ then $\mathcal{E}_C \not\simeq C \times \mathbb{R}$. Therefore $\Pi^{-1}(C)$ is connected and hence contained in an equivalence class (w.r.t. $\sim$). Hence $z \sim w$, implying $(m_x, \omega_x) = (m_y, \omega_y)$ in $E(\mathbb{R}(X), L)$. Now suppose $w' \in Z(\mathbb{R})$ be such that $\Theta(w') = (m_y, -\omega_y)$. Then $\Pi(w') \in C$ and hence $w' \in \Pi^{-1}(C)$. Therefore, as before we conclude that $z \sim w'$. Hence $(m_x, \omega_x) = (m_y, -\omega_y)$ in $E(\mathbb{R}(X), L)$. Since, by (4.3), $(m_y, \omega_y) + (m_y, -\omega_y) = 0$ in $E(\mathbb{R}(X), L)$, we have $2(m_x, \omega_x) = (m_y, \omega_y) + (m_y, -\omega_y) = 0$ in $E(\mathbb{R}(X), L)$.

If $K_C \simeq L_C$, then $\mathcal{E}_C \simeq C \times \mathbb{R}$. Therefore $\Pi^{-1}(C)$ can be thought of as a disjoint union $(C \times \mathbb{R}^+) \bigcup (C \times \mathbb{R}^-)$. Without loss of generality we assume that $z \in C \times \mathbb{R}^+$. Since $y \in C$, there exists $w' \in C \times \mathbb{R}^+$ such that $\Pi(w') = y$. Since $C \times \mathbb{R}^+$ is a connected component of $Z(\mathbb{R})$, as before we see that it is contained in an equivalence class. Therefore $z \sim w'$. This means that if $\Theta(w') = (m_y, \widetilde{\omega}_y)$ then $(m_x, \omega_x) = (m_y, \widetilde{\omega}_y)$ in $E(\mathbb{R}(X), L)$.

Since, by 4.3, $(m_y, \widetilde{\omega}_y) = (m_y, \omega_y)$ or $(m_y, \widetilde{\omega}_y) = (m_y, -\omega_y)$ we are through. ■

As a consequence of above lemma we deduce the following result.

Lemma 4.17 *Assume that $X(\mathbb{R})$ is not compact and let C be a noncompact connected component of $X(\mathbb{R})$. Let L be a rank 1 projective module over A. Let $x \in C$ and let m be the corresponding maximal ideal of $\mathbb{R}(X)$. Let ω_x be a local L-orientation of m. Then $(m, \omega_x) = 0$ in $E(\mathbb{R}(X), L)$.*

Proof Let $\widetilde{X}$ be the smooth projective completion of X. It is easy to see that there exists an affine open subset $X_1 = \operatorname{Spec}(A_1)$ of $\widetilde{X}$ such that $X_1(\mathbb{R}) = \widetilde{X}(\mathbb{R})$. Therefore $(X \cap X_1)(\mathbb{R}) = X(\mathbb{R})$. Since $X \cap X_1$ is an affine open subset of X_1 and $\operatorname{Pic}(\mathbb{R}(X_1))$ is a 2-torsion group, it is easy to see that $\mathbb{R}(X \cap X_1)$ is a localization of $\mathbb{R}(X_1)$. Now since $X \cap X_1$ is an open subset of X and $(X \cap X_1)(\mathbb{R}) = X(\mathbb{R})$, we have $\mathbb{R}(X \cap X_1) = \mathbb{R}(X)$. Let L_1 be a rank 1 projective over A_1 such that L_1 and L define the same projective module over $\mathbb{R}(X)$. Note that since $\widetilde{X}$ is projective, $X_1(\mathbb{R})$ is compact.

Since $X \cap X_1(\mathbb{R}) = X(\mathbb{R})$, we regard C as a connected subset of $X_1(\mathbb{R})$. Therefore there exists a connected component C_1 of $X_1(\mathbb{R})$ such that $C \subset C_1$. Since C_1 is compact but C is not compact, there exists $y \in C_1$ such that $y \notin C$. Note that $C_1 \not\subset X(\mathbb{R})$ (otherwise C_1 being connected, $C_1 \subset C$). Therefore we can assume that $y \notin X(\mathbb{R})$.

Let m_y denote the corresponding maximal ideal of $\mathbb{R}(X_1)$. Since $y \notin X(\mathbb{R})$, we have $m_y\mathbb{R}(X) = \mathbb{R}(X)$. Let ω_y be a local L_1-orientation of m_y. Since $x, y \in C_1$, by (4.16), either $(m_x, \omega_x) = (m_y, \omega_y)$ or $(m_x, \omega_x) = (m_y, -\omega_y)$ in $E(\mathbb{R}(X_1, L_1)$. Since $\mathbb{R}(X) = \mathbb{R}(X \cap X_1)$ is a localization of $\mathbb{R}(X_1)$, there exists a (surjective) group homomorphism from $E(\mathbb{R}(X_1), L_1)$ to $E(\mathbb{R}(X), L)$. Since, under this group homomorphism, $(m_y, \omega_y) \mapsto 0$ in $E(\mathbb{R}(X), L)$ we are through. ■

Lemma 4.18 *Let $\{C_1, \cdots, C_t\}$ denote the set of all compact connected components of $X(\mathbb{R})$ in the Euclidean topology. Let L be a projective A-module of rank 1. Then, $E(\mathbb{R}(X), L)$ is generated by t elements.*

Proof Let $x_i \in C_i,\ 1 \le i \le t$ and let m_i be a maximal ideal of $\mathbb{R}(X)$ corresponding to x_i. Let ω_{x_i} be a local L-orientation of m_i for $1 \le i \le t$. Let F be a free abelian group with a basis $(e_1, \cdots, e_t)$. Let $\Delta : F \to E(\mathbb{R}(X), L)$ be a group homomorphism defined by $\Delta(e_i) = (m_i, \omega_i);\ 1 \le i \le t$. We show that Δ is surjective.

Since $E(\mathbb{R}(X), L)$ is generated by elements of the type (m, ω_m) where m is a maximal ideal of $\mathbb{R}(X)$ and ω_m is a local L-orientation of m, it is enough to show that (m, ω_m) is in the image of Δ.

Let $x \in X(\mathbb{R})$ be the corresponding element. If $x \notin C_i, \forall i,\ 1 \le i \le t$ then x belongs to a noncompact component of $X(\mathbb{R})$ and hence, by (4.17), $(m, \omega_m) = 0$.

Now assume that $x \in C_i$. Let K denote the canonical module of A and let $\mathcal{E} = L \otimes_A K$. If $\mathcal{E}_{C_i} \not\simeq C_i \times \mathbb{R}$ then, by (4.16), $(m, \omega_m) = (m_i, \omega_{x_i}) = \Delta(e_i)$. If $\mathcal{E}_{C_i} \simeq C_i \times \mathbb{R}$, then, again by 4.16, $(m, \omega_m) = (m_i, \omega_{x_i}) = \Delta(e_i)$ or $(m, \omega_m) = (m_i, -\omega_{x_i}) = -(m_i, \omega_{x_i}) = -\Delta(e_i)$.

Thus, in any case, (m, ω_m) belongs to the image of Δ and hence we are through. ∎

Lemma 4.19 *Let C be a compact connected component of $X(\mathbb{R})$. Let $\mathbb{R}(C)$ be the ring obtained by inverting all elements of A which do not vanish at any point $x \in C$. Let $\mathcal{E}$ be a projective A-module of rank 1. Then $\mathcal{E} \otimes_A \mathbb{R}(C) \simeq \mathbb{R}(C)$ if and only if the restriction $\mathcal{E}|_C$ of (induced) line bundle on $X(\mathbb{R})$ to C is trivial.*

Proof Let $\mathcal{C}(C, \mathbb{R})$ denote the ring of real valued continuous functions on C. Then, since there is a ring homomorphism $\mathbb{R}(C) \to \mathcal{C}(C, \mathbb{R})$, it is easy to see that if $\mathcal{E} \otimes_A \mathbb{R}(C) \simeq \mathbb{R}(C)$, then $\mathcal{E}|_C$ is a trivial line bundle on C.

Now suppose $\mathcal{E}|_C$ is a trivial line bundle on C. To prove that $\mathcal{E} \otimes_A \mathbb{R}(C) \simeq \mathbb{R}(C)$ we can assume without loss of generality that $X(\mathbb{R})$ is compact. This can be seen as follows.

If $X(\mathbb{R})$ is not compact then (taking a smooth projective completion of X), we see as in (4.17) that there exists a smooth affine variety $X_1 = \mathrm{Spec}(A_1)$ birational to X such that $X_1(\mathbb{R})$ is compact and $X(\mathbb{R})$ is its open subset. Since C is a compact connected component of $X(\mathbb{R})$ it follows that C is a connected component of $X_1(\mathbb{R})$ also. Note also that $\mathbb{R}(X)$ is a localization of $\mathbb{R}(X_1)$ and $\mathbb{R}(C)$ is a localization of $\mathbb{R}(X)$. In fact it is easy to see that $\mathbb{R}(C)$ is obtained by inverting all elements of A_1 which do not vanish at any point of C. Moreover, there exists a rank 1 projective A_1-module $\mathcal{E}_1$ such that $\mathcal{E}_1 \otimes_{A_1} \mathbb{R}(C) = \mathcal{E} \otimes_A \mathbb{R}(C)$. Therefore, replacing X by X_1 if necessary, we can assume that $X(\mathbb{R})$ is compact.

Since C is a connected component of $X(\mathbb{R})$ and $\mathcal{E}|_C$ is trivial, there exists a continuous section $\sigma : X(\mathbb{R}) \to \mathcal{E}$ such that $\sigma(x) = 1\ \forall x \in C$ and $\sigma(y) = 0\ \forall y \notin C$. Let U be an open neighbourhood of $\sigma(X(\mathbb{R}))$ such that $U \cap (\mathcal{E}|_C) = C \times (0, \infty)$. Then, by ([Bo-C-R], Theorem 12.3.1), there exists an algebraic section $s : X(\mathbb{R}) \to \mathcal{E}$ such that $s(X(\mathbb{R})) \subset U$. This means that there exists $e \in \mathcal{E}$ such that if $x \in C$ and M_x is the corresponding real maximal ideal of A then $e \notin M_x\mathcal{E}$. Therefore $\mathcal{E} \otimes_A \mathbb{R}(C)$ is generated by $e \otimes 1$.

Thus $\mathcal{E} \otimes_A \mathbb{R}(C) \simeq \mathbb{R}(C)$. ∎

Let $X = \mathrm{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over the field $\mathbb{R}$ of real numbers. Let K be the canonical module of A. Let M be a real maximal ideal of A and let $\omega_M : K/MK \simeq \wedge^n(M/M^2)$ be a local K-orientation of M. Recall that the differential map $d : A \to \Omega_{A/\mathbb{R}}$ gives rise to an isomorphism $\wedge^n(\overline{d}) : \wedge^n(M/M^2) \xrightarrow{\sim} K/MK$ and hence composing ω_M with $\wedge^n(\overline{d})$, we get an isomorphism, say, $\phi_{\omega_M} : K/MK \xrightarrow{\sim} K/MK$. Therefore $\exists \lambda \in \mathbb{R}^*$ such that ϕ_{ω_M} is multiplication by λ. We denote by $sgn((M, \omega_M))$ the signature $sgn(\lambda)$. Recall that $sgn(\lambda)$ takes value in the multiplicative group $\{1, -1\}$. This assignment is very similar to one we have already stated in the case $K \otimes_A K \simeq A$.

Though a proof of the following lemma is implicit in [B-RS 1, Proposition 4.12] for the sake of completeness we prove it.

Lemma 4.20 *Let A, X, K be as above. Let C be a compact connected component of $X(\mathbb{R})$. Let $I = \cap M_i;\ 1 \leq i \leq l$ where for $\forall i,\ M_i$ is a real maximal ideal of A corresponding to $x_i \in C$. Let $\beta : K \oplus A^{n-1} \twoheadrightarrow I$ be a surjection and ω_i be a local K- orientation of M_i induced by β. Then $\sum_{i=1}^{l} sgn((M_i, \omega_i)) = 0$.*

Proof Let $\{e_2, e_3, \cdots, e_n\}$ be a basis of A^{n-1} and let $\beta(e_i) = a_i,\ 2 \leq i \leq n$. By (2.2), we can assume that if $J = (\beta(L), a_2, \cdots, a_{n-1})$ then $Y = \mathrm{Spec}(A/J)$ is a smooth a affine curve. Let "tilde" denote reduction modulo the ideal

J. Then we have the following exact sequence

$$0 \longrightarrow J/J^2 \overset{\widetilde{a}\mapsto \widetilde{d(a)}}{\longrightarrow} \Omega_{A/\mathbb{R}}/J(\Omega_{A/\mathbb{R}}) \longrightarrow \Omega_{\widetilde{A}/\mathbb{R}} \longrightarrow 0.$$

Let $F = K \oplus A^{n-2}$. Then, since $\beta(F) = J$ we get an isomorphism $\widetilde{\beta} : F/JF \simeq J/J^2$. Since $\widetilde{A}$ is smooth, $\Omega_{\widetilde{A}/\mathbb{R}}$ is a projective $\widetilde{A}$-module of rank one. Hence $\Omega_{A/\mathbb{R}}/J(\Omega_{A/\mathbb{R}}) \simeq F/JF \oplus \Omega_{\widetilde{A}/\mathbb{R}}$. Since $\wedge^{n-1}(F) = K = \wedge^n(\Omega_{A/\mathbb{R}})$ we get that $\Omega_{\widetilde{A}/\mathbb{R}} \simeq \widetilde{A}$.

Let Θ be a generator of $\Omega_{\widetilde{A}/\mathbb{R}}$ and let v be its preimage in $\Omega_{A/\mathbb{R}}/J(\Omega_{A/\mathbb{R}})$. Now it is easy to see that the map $K/JK \to K/JK$ given by $\widetilde{l} \mapsto \widetilde{d(\beta(l))} \wedge \widetilde{d(a_2)} \cdots \wedge \widetilde{d(a_{n-2})} \wedge v$ is an isomorphism and hence multiplication by b for some invertible element b of A/J.

Replacing v by $b^{-1}v$ and Θ by $b^{-1}\Theta$, we can assume that $b = 1$. Let $\widetilde{d} : A/J \to \Omega_{\widetilde{A}/\mathbb{R}}$ be the differential map for the smooth affine curve Y and let $\widetilde{d}(\widetilde{a_n}) = u\Theta$ where $u \in A/J$. Then, since $(\widetilde{a}_n) = \cap(M_i/J),\ 1 \leq i \leq l$ and $A/M_i = \mathbb{R}$, we see that there exists $\lambda_i \in \mathbb{R}^*$ such that λ_i is the image of u in A/M_i.

Let "bar" denote reduction modulo I. The surjection $\beta : L \oplus A^{n-1} \twoheadrightarrow I$ induces a local K-orientation $\omega_I : K/IK \simeq \wedge^n(I/I^2)$ and hence an isomorphism $K/IK \simeq K/IK$ defined as $\overline{l} \mapsto \overline{d(\beta(l))} \wedge \overline{d(a_2)} \cdots . \wedge \overline{d(a_n)}$ which is nothing but multiplication by $\overline{u}$. If ω_i is the induced local K-orientation on M_i then it is easy to see that $sgn(M_i, \omega_i) = sgn(\lambda_i)$. So we want to show that $\sum_{i=1}^{l} sgn(\lambda_i) = 0$.

Let $\widetilde{Y}$ denote the smooth projective completion of $Y = \mathrm{Spec}(A/J)$. Since C is compact, $C \cap Y(\mathbb{R})$ is also compact (though not necessarily connected). As $Y(\mathbb{R})$ is an open subset (in the Euclidean topology) of $\widetilde{Y}(\mathbb{R})$ and $C \cap Y(\mathbb{R})$ is a compact and open subset of $Y(\mathbb{R})$ (note that C is a connected component of $X(\mathbb{R})$), $C \cap Y(\mathbb{R})$ is a finite union of (compact) connected components of $\widetilde{Y}(\mathbb{R})$. Therefore, by applying ([Kn],Theorem 3.3) to the differential form $\Theta/\widetilde{a_n}$, we see that

$$\sum_{y \in C \cap Y(\mathbb{R})} res_y(\Theta/\widetilde{a_n}) = 0.$$

Let $y_1, \cdots, y_l$ be points of $C \cap Y(\mathbb{R})$ corresponding to the maximal ideals $M_1, \cdots, M_l$. Then $\widetilde{a_n}$ is a uniformising parameter precisely at these points and at any other point y of $C \cap Y(\mathbb{R})$, $\widetilde{a_n}$ is a unit. Now, since $\widetilde{d}(\widetilde{a_n}) = u\Theta$ (where $u \in A/J$), we have

$$\sum_{y \in C \cap Y(\mathbb{R})} res_y(\Theta/\widetilde{a_n}) = \sum_{i=1}^{l} (sgn({\lambda_i}^{-1})).$$

Since $sgn(\lambda) = sgn(\lambda^{-1})$, we are through. ∎

Lemma 4.21 *Let L be a projective A-module of rank 1 and let K be the canonical module of A. Let C be a compact connected component of $X(\mathbb{R})$ and let $\mathbb{R}(C)$ be the ring obtained by inverting all elements of A which do not vanish at any point $x \in C$. Suppose that $L \otimes_A \mathbb{R}(C) \simeq K \otimes_A \mathbb{R}(C)$. Then $E(\mathbb{R}(C), L) = \mathbb{Z}$.*

Proof First note that, by abuse of notation, we have written $E(\mathbb{R}(C), L)$ for $E(\mathbb{R}(C), L \otimes_A \mathbb{R}(C))$. Since $L \otimes_A \mathbb{R}(C) \simeq K \otimes_A \mathbb{R}(C)$, we have $E(\mathbb{R}(C), L) = E(\mathbb{R}(C), K)$.

Let m be a maximal ideal of $\mathbb{R}(C)$ and ω_m be a local K-orientation (more precisely local $K \otimes_A \mathbb{R}(C)$-orientation) of m. Let $M = m \cap A$ and let ω_M be a local K-orientation of M which gives rise to ω_m. We call $sgn((M, \omega_M))$ to be the signature of (m, ω_m) and denote it by $sgn((m, \omega_m))$.

Let G be a free abelian group on the set of pairs (m, ω_m), where m is a maximal ideal of $\mathbb{R}(C)$ and ω_m is a local K-orientation. Let H be the subgroup of G generated by the set of pairs (I, ω_I) where I is the intersection of finitely many maximal ideals and ω_I is a global K-orientation of I. Since $\mathbb{R}(C)$ is a localization of A (a smooth affine domain), it is easy to see that the canonical map $G/H \to E(\mathbb{R}(C), K)$ is an isomorphism.

Now consider the map $Sgn : G \twoheadrightarrow \mathbb{Z};\ (m, \omega_m) \mapsto sgn((m, \omega_m))$. Applying (4.20), we see that $Sgn(H) = 0$. Hence we get a surjection $E(\mathbb{R}(C), K) \twoheadrightarrow \mathbb{Z}$. Therefore, since $E(\mathbb{R}(C), L) = E(\mathbb{R}(C), K)$, we get a surjection $E(\mathbb{R}(C), L) \twoheadrightarrow \mathbb{Z}$.

Since $\mathbb{R}(C)$ is a localization of $\mathbb{R}(X)$ we get a surjection $E(\mathbb{R}(X), L) \twoheadrightarrow E(\mathbb{R}(C), L)$. Let $C = C_1, \cdots, C_t$ be the compact connected components of $X(\mathbb{R})$ and let $\Delta : F \to E(\mathbb{R}(X), L)$ be the surjective map defined in (4.18). Composing Δ with the surjection $E(\mathbb{R}(X), L) \twoheadrightarrow E(\mathbb{R}(C), L)$ we get a surjection $\Delta_C : F \twoheadrightarrow E(\mathbb{R}(C), L)$. Let $x \in X(\mathbb{R})$ and let m_x be the corresponding maximal ideal of $\mathbb{R}(X)$. If $x \notin C$ then $m_x \mathbb{R}(C) = \mathbb{R}(C)$. Hence

$\Delta_C(e_i) = 0$ for $2 \leq i \leq t$. This shows that $E(\mathbb{R}(C), L)$ is a cyclic group. Since $\mathbb{Z}$ is a surjective image of $E(\mathbb{R}(C), L)$ we get that $E(\mathbb{R}(C), L) = \mathbb{Z}$.

Thus the proof is complete. ■

Now we prove a structure theorem for $E(\mathbb{R}(X), L)$.

Theorem 4.22 *Let $X = \operatorname{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over the field $\mathbb{R}$ of real numbers and let $K = \wedge^n(\Omega_{A/\mathbb{R}})$ be the canonical module of A. Let L be a projective A-module of rank 1. Let $C_1, \cdots, C_r, C_{r+1}, \cdots, C_t$ be the compact connected components of $X(\mathbb{R})$ in the Euclidean topology. Let K_{C_i} and L_{C_i} denote restriction of (induced) line bundles on $X(\mathbb{R})$ to C_i. Assume that $L_{C_i} \simeq K_{C_i}$ for $1 \leq i \leq r$ and $L_{C_i} \not\simeq K_{C_i}$ for $r+1 \leq i \leq t$. Then,*

$$E(\mathbb{R}(X), L) = G_1 \oplus \cdots \oplus G_r \oplus G_{r+1} \oplus \cdots \oplus G_t,$$

where $G_i = \mathbb{Z}$ for $1 \leq i \leq r$ and $G_i = \mathbb{Z}/(2)$ for $r+1 \leq i \leq t$.

Proof Let $x_i \in C_i$, $1 \leq i \leq t$ and let m_i be a maximal ideal of $\mathbb{R}(X)$ corresponding to x_i. Let ω_i be a local L-orientation of m_i for $1 \leq i \leq t$. Let F be a free abelian group with basis $(e_1, \cdots, e_t)$. Let $\Delta : F \to E(\mathbb{R}(X), L)$ be a group homomorphism defined by $\Delta(e_i) = (m_i, \omega_i)$; $1 \leq i \leq t$. Recall that, by (4.18), Δ is surjective.

Recall that $\mathbb{R}(C_i)$ denotes the ring obtained by inverting all elements of A which do not vanish at any point $x \in C_i$. Since $\mathbb{R}(C_i)$ is a localization of $\mathbb{R}(X)$ we have a surjection $E(\mathbb{R}(X), L) \twoheadrightarrow E(\mathbb{R}(C_i), L)$. Composing this surjection with Δ we get a surjection $\Delta_{C_i} : F \twoheadrightarrow E(\mathbb{R}(C_i), L)$. Note that $\Delta_{C_i}(e_j) = 0$ if $j \neq i$. If $1 \leq i \leq r$, then, since $K_{C_i} \simeq L_{C_i}$, by (4.19) and (4.21), $E(\mathbb{R}(C_i), L) = \mathbb{Z}$. Hence $\Delta_{C_i}(e_i) = sgn(m_i, \omega_i) = \pm 1$.

Let $n_j \in \mathbb{Z}$, $1 \leq j \leq t$ be such that $\Delta(\sum_{j=1}^t n_j e_j) = 0$. Then $\Delta_{C_i}(\sum_{j=1}^t n_j e_j) = n_i \Delta_{C_i}(e_i) = 0$. Therefore, if $1 \leq i \leq r$, then $n_i = 0$.

If $r+1 \leq j \leq t$, then, since $L_{C_j} \not\simeq K_{C_j}$, by (4.16), $2(m_j, \omega_j) = 0$ and hence $2\Delta(e_j) = 0$. Therefore to complete the proof, it is enough to show that n_j is even for $\forall j,\ r+1 \leq j \leq t$.

If n_j is odd for some j, then without loss of generality we can assume that $\exists l, r+1 \leq l \leq t$ such that n_j is odd for $r+1 \leq j \leq l$ and n_j is even for $j > l$. Then, $0 = \Delta(\sum_{j=r+1}^{t} n_j e_j) = \Delta(\sum_{j=r+1}^{l} e_j)$. This implies that if $I = \cap m_j,\ r+1 \leq j \leq l$ and ω_I is the local L-orientation of I induced by $\omega_j,\ r+1 \leq j \leq l$ then $(I, \omega_I) = 0$ in $E(\mathbb{R}(X), L)$. Hence, by (2.5), I is a surjective image of $L \otimes_A \mathbb{R}(X) \oplus \mathbb{R}(X)^{n-1}$.

Let $J = I \cap A$. Then $J \subset A$ is a reduced ideal of height n. Moreover, since I is a surjective image of $L \otimes_A \mathbb{R}(X) \oplus \mathbb{R}(X)^{n-1}$, replacing A by A_g ($g \in A$ does not belong to any real maximal ideal of A) if necessary, we see that J is a surjective image of $L \oplus A^{n-1}$. But, by (4.14), this is a contradiction as A/J is supported only at points $x_j \in C_j;\ r+1 \leq j \leq l$.

Thus we have shown that n_j is even for $\forall j,\ r+1 \leq j \leq t$ and hence we are through. ■

Remark 4.23 A special case of the above theorem viz $L = K$, has been already proved by Ian Robertson (see [Ro, Theorem 12.6]).

Let $W = S^1 \times S^1 \bigcup S^1 \times S^1$ be a disjoint union where S^1 denotes (real) circle. Then the orientable manifold W has two compact connected components C_1, C_2 each isomorphic to $S^1 \times S^1$. It is easy to see that there exists a *topological* line bundle L on W such that $L|_{C_1} \simeq C_1 \times \mathbb{R}$ but $L|_{C_2} \not\simeq C_2 \times \mathbb{R}$. The following example shows that a similar situation can be realised algebraically also.

Example 4.24 Let $f(Z) = Z(Z-1)(Z-2)(Z-3) \in \mathbb{R}[Z]$ be a polynomial in one variable and let $B = \mathbb{R}[W, Z]/(W^4 + f(Z))$. It is easy to see that if $\lambda \in \mathbb{R}$ then $f(\lambda) \leq 0$ if and only if $\lambda \in [0,1] \cup [2,3]$. Therefore if $Y = \operatorname{Spec}(B)$ then $Y(\mathbb{R})$ is compact and has only two connected components C_1, C_2. In fact, if $F(Z,T) = Z(Z-T)(Z-2T)(Z-3T)$ then $W^4 + F(Z,T)$ defines a smooth curve $\widetilde{Y}$ in the projective plane $\mathbf{P}^2(\mathbb{R})$ and $Y(\mathbb{R})$ is the set of all real points of $\widetilde{Y}$.

Let $x = (0,0)$ and $y = (0,3)$ be elements of $Y(\mathbb{R})$ and let $M_x = (w, z)$, $M_y = (w, z-3)$ be corresponding real maximal ideals of B. We assume that $x \in C_1$ and hence $y \in C_2$. Note that $\operatorname{Pic}(\mathbb{R}(Y)) = \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ generated by M_x and M_y.

Let $A = B[U,V]/(U^2+V^2-1)$ and let $X = \operatorname{Spec}(A)$. Then X is a smooth affine variety of dimension 2 and the canonical module $K_A \simeq A$. Moreover the orientable manifold $X(\mathbb{R}) = C_1 \times S^1 \bigcup C_2 \times S^1$.

Let $L = M_x A$. Then L is a projective A-module of rank 1 such that the induced line bundle on $X(\mathbb{R})$ when restricted to $C_1 \times S^1$ is non-trivial but trivial when restricted to $C_2 \times S^1$.

Recall that for a smooth affine variety $X = \mathrm{Spec}(A)$ of dimension $n \geq 2$ over $\mathbb{R}$ and a rank 1 projective A-module L, we denote by $E_0(A, L)$ the weak Euler class group of A with respect to L and by $CH_0(X)$ the group of zero cycles of X modulo rational equivalence. If $L = A$, we write $E_0(A)$ for $E_0(A, A)$. Moreover, if J is an ideal of A of height n such that J/J^2 is generated by n elements, then we denote by $(J)_L$ the element in $E_0(A, L)$ associated to the ideal J and by $[J]$ the image in $CH_0(X)$ of the zero cycle associated to A/J. By (2.9) and (2.10), there exists an isomorphism $\Psi_L : E_0(A, L) \xrightarrow{\sim} CH_0(X)$ such that $\Psi_L((J)_L) = [J]$ (viz $\Psi_L = \Psi\beta_L$). Moreover, composing Ψ_L with the canonical surjection $E(A, L) \twoheadrightarrow E_0(A, L)$, we get a surjection $\Theta_L : E(A, L) \twoheadrightarrow CH_0(X)$.

Having proved a structure theorem for $E(\mathbb{R}(X), L)$, our next aim is to study the surjection Θ_L.

Since $\mathbb{R}(X)$ is a localization of A and of dimension n, it is easy to see that there exists a canonical surjective group homomorphisms $\Gamma_L : E(A, L) \twoheadrightarrow E(\mathbb{R}(X), L)$. Let $E^{\mathbb{C}}(L)$ denote the kernel of Γ_L.

Remark 4.25 Let $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ and $Y = \mathrm{Spec}(A_{\mathbb{C}})$. Then $A_{\mathbb{C}}$ is a smooth affine ring of dimension n over the field $\mathbb{C}$ of complex numbers and hence $CH_0(Y)$ is a divisible group. Let $\pi : Y \to X$ be the canonical map. Note that π is a finite morphism and it induces group homomorphisms $\pi^* : CH_0(X) \to CH_0(Y)$ and $\pi_* : CH_0(Y) \to CH_0(X)$ such that the composition $\pi_*\pi^*$ is multiplication by 2. Let $G = \pi_*(CH_0(Y))$.

Lemma 4.26 *$G = \pi_*(CH_0(Y))$ is a divisible group and if $n \geq 2$ then it is torsion-free.*

Proof Since G is a surjective image of the divisible group $CH_0(Y)$, G is divisible. Now we assume that $n \geq 2$. We want to show that G is torsion-free.

Since $n \geq 2$, by [Bl-M-S, Proposition 2.1], $CH_0(Y)$ is torsion-free. Now let $z \in CH_0(X)$ be a torsion element, say, $rz = 0$. Then, $\pi^*(rz) = r\pi^*(z) =$

0. But $\pi^*(z) \in CH_0(Y)$ and $CH_0(Y)$ is torsion-free, which implies that $\pi^*(z) = 0$. Therefore, $\pi_*\pi^*(z) = 2z = 0$. So any torsion element of $CH_0(X)$ is a 2-torsion and hence every torsion element of G is 2-torsion. But G is divisible. Putting these two facts together we see that G is torsion-free. ■

Lemma 4.27 $\Theta_L(E^{\mathbb{C}}(L)) = G$.

Proof G is generated by $\pi_*[N]$ where N is a maximal ideal of $A_{\mathbb{C}}$ and $[N]$ denotes the zero cycle in $CH_0(Y)$ associated to $A_{\mathbb{C}}/N$.

Let $M = A \cap N$. Note that if M is a complex maximal ideal of A then $[M] = \pi_*[N]$ and if M is a real maximal ideal then $2[M] = \pi_*[N]$ ($[M]$ denotes the zero cycle associated to A/M in $CH_0(X)$). This shows that if I is an ideal of A of height n such that I/I^2 is generated by n elements and I is contained in only complex maximal ideals then $[I] \in G$. Therefore, since, by (2.7), every element of $E^{\mathbb{C}}(L)$ is of the form (I, ω_I), where I is a an ideal of height n (with I/I^2 generated by n elements) contained only in complex maximal ideals and ω_I is a local L-orientation of I, we see that $\Theta_L(E^{\mathbb{C}}(L)) \subset G$.

Now to complete the proof it is enough to show that if M is real a maximal ideal of A then $2[M] \in \Theta_L(E^{\mathbb{C}}(L))$.

Let ω_M be a local L-orientation of M. Then $2[M] = \Theta_L((M, \omega_M) + (M, -\omega_M))$. By (4.3), $(M, \omega_M) + (M, -\omega_M) \in E^{\mathbb{C}}(L)$. Hence we are through.

The assertion that $2[M] \in G$ still holds even in the case dimension of A is 1 as $M^2\mathbb{R}(X)$ is a principal ideal. ■

Remark 4.28 Assume that $X(\mathbf{R})$ has precisely t compact connected components. Colliot-Thélène and Scheiderer have proved in ([CT-S], Theorem 1.3(d)) that $CH_0(X)/G$ is a vector space of dimension t over the field $\mathbb{Z}/(2)$.

Lemma 4.29 *Let B be an affine ring of dimension 1 over $\mathbb{R}$ such that B does not have any real maximal ideal. Let L be a projective B- module of rank 1 and let J be an invertible ideal of B. Then for any positive integer r there exists an invertible ideal J_1 of B such that $J + J_1 = B$ and $L \simeq J \cap {J_1}^r$.*

Proof Since B does not have any real maximal ideal, the group $\mathrm{Pic(B)}$ of rank 1 projective B-modules is divisible. This can be seen as follows:

We can assume without loss of generality that B is reduced. Let $\widetilde{B}$ be the normalization of B in the total quotient ring of B. Then $\widetilde{B}$ is a smooth affine ring of dimension 1 over $\mathbb{R}$ without any real maximal ideal and hence $\mathrm{Pic}(\widetilde{\mathrm{B}})$ is divisible. Let K be the conductor ideal of $\widetilde{B}$ over B. Then since $\widetilde{B}$ does not have any real maximal ideal it is easy to see that the group $(\widetilde{B}/K)^*$ of invertible elements of $\widetilde{B}/K$ is divisible. Now divisibility of $\mathrm{Pic(B)}$ follows from the following exact sequence:

$$(\widetilde{B}/K)^* \to \mathrm{Pic(B)} \to \mathrm{Pic}(\widetilde{\mathrm{B}}) \to 0.$$

Let $E = L \otimes_B J^{-1}$. Then, since $E \in \mathrm{Pic(B)}$ and $\mathrm{Pic(B)}$is divisible, there exists a rank 1 projective module E_1 over B such that $E \simeq {E_1}^r$. Since $\dim(B) = 1$, $E_1/JE_1 \simeq A/J$. Therefore there exists an invertible ideal J_1 such that $J + J_1 = B$ and $E_1 \simeq J_1$. Therefore $L \simeq J \otimes_B {J_1}^r = J{J_1}^r = J \cap {J_1}^r$. ■

Let A be a smooth affine domain of dimension $n \geq 2$ and L be a rank 1 projective A-module. Let I be an ideal of height n such that I/I^2 is generated by n elements. Suppose I is contained in only complex maximal ideals. Let ω_I be a local L-orientation of I and let $b \in J$ be such that b does not belong to any real maximal ideal. Then with the help of above (moving)lemma it is easy to see that there exists an ideal I_1 of height n with properties (1) $I_1 + (b) = A$, (2) $I_1/{I_1}^2$ is generated by n elements, (3) I_1 is contained in only complex maximal ideals (4) $(I, \omega_I) = (I_1, \omega_{I_1})$ in $E(A, L)$ for some local L-orientation ω_{I_1} of I_1.

Proposition 4.30 *Let $X = \mathrm{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over the field $\mathbb{R}$ of real numbers and let $K = \wedge^n(\Omega_{A/\mathbb{R}})$ be the canonical module of A. Let L be a projective A-module of rank* 1*. Let $C_1, \cdots, C_r, C_{r+1}, \cdots, C_t$ be the compact connected components of $X(\mathbb{R})$ in the Euclidean topology. Let K_{C_i} and L_{C_i} denote restriction of (induced) line bundles on $X(\mathbb{R})$ to C_i. Assume that $L_{C_i} \simeq K_{C_i}$ for $1 \leq i \leq r$ and $L_{C_i} \not\simeq K_{C_i}$ for $r+1 \leq i \leq t$. Let $\Theta_L : E(A, L) \twoheadrightarrow CH_0(X)$ be the canonical surjection. Then, $\ker(\Theta_L)$ is a free abelian group of rank r.*

Proof Recall that if $E^{\mathbb{C}}(L)$ denotes the kernel of the surjection $\Gamma_L : E(A, L) \twoheadrightarrow E(\mathbb{R}(X), L)$ then, by (4.27), $\Theta_L(E^{\mathbb{C}}(L)) = G$. Therefore we get a surjection $\Theta'_L : E(\mathbb{R}(X), L) \twoheadrightarrow CH_0(X)/G$. Since, by (4.28), $CH_0(X)/G$ is a vector space of dimension t over the field $\mathbb{Z}/(2)$ it follows from (4.22) that $\ker(\Theta'_L) = 2E(\mathbb{R}(X), L)$ which is a free abelian group of rank r.

Note that $\Gamma_L(\ker(\Theta_L)) = \ker(\Theta'_L)$. So if we show that the restriction of Γ_L to $\ker(\Theta_L)$ is injective then we are through. But this is equivalent to showing that the restriction $\Theta_L|_{E^{\mathbb{C}}(L)}$ is injective. We now proceed to prove this.

Let $x \in E^{\mathbb{C}}(L))$. Then $x = (I, \omega_I)$ where I is an ideal of A of height n contained in only complex maximal ideals. Suppose that $\Theta((I, \omega_I)) = 0$. First note that since I is contained in only complex maximal ideals, every unit of A/I is a square. Therefore, in view of (2.6), in order to prove that $(I, \omega_I) = 0$ in $E(A, L)$ it is enough to show that there is a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow I$.

Since I is contained in only complex maximal ideals there exists $b \in I$ which is contained in only complex maximal ideals. Let $\overline{A} = A/(b^2)$, and $\overline{I} = I/(b^2)$. Let $\alpha' : L/IL \oplus A/I^{n-1} \twoheadrightarrow I/I^2$ be a surjection. Then since $\dim(\overline{A}) = n - 1$, using (2.1), we see that α' can be lifted to a surjection $\overline{\alpha} : L/b^2L \oplus \overline{A}^{n-1} \twoheadrightarrow \overline{I}$. Hence if $\widetilde{\alpha} : L \oplus A^{n-1} \to I$ is a lift then $\widetilde{\alpha}(L \oplus A^{n-1}) + (b^2) = I$. Let $\widetilde{\alpha}(L) = K$ Then $I = K + (a_1, \cdots, a_{n-1}, b^2)$. By [B-RS 1, Lemma 5.1], there exists $c \in A$ such that the element $a_1 + cb^2$ does not belong to any real maximal ideal of A. So, replacing a_1 by $a_1 + cb^2$ if necessary, we can assume that a_1 does not belong to any real maximal ideal of A. Moreover, adding suitable multiples of b^2 to $a_2, \cdots, a_{n-1}$ we assume that the ideal $(a_1, \cdots a_{n-1})$ has height $n - 1$.

Let $B = A/(a_1, \cdots a_{n-1})$ and $J = I/(a_1, \cdots a_{n-1})$. Then B is an affine ring of dimension 1 over $\mathbb{R}$ which does not have any real maximal ideal and J is an invertible ideal of B. Hence, by (4.29), there exists an invertible ideal J_1 of B such that $J + J_1 = B$ and $J \cap {J_1}^2$ is a surjective image of $L \otimes_A B$. Let I_1 be an ideal of A such that $(a_1, \cdots a_{n-1}) \subset I_1$ and $I_1/(a_1, \cdots a_{n-1}) = J_1$. Let ${I_1}^{(2)} = (a_1, \cdots a_{n-1}) + {I_1}^2$. From construction it is clear that there exists a surjection $\beta : L \oplus A^{n-1} \twoheadrightarrow I \cap {I_1}^{(2)}$. So if we show that ${I_1}^{(2)}$ is a surjective image of $L \oplus A^{n-1}$, we are through.

First note that I_1 is contained in only complex maximal ideals and hence the zero cycle $[I_1] \in G$. The surjection β shows that $[I] + [{I_1}^{(2)}] = 0$ in

$CH_0(X)$. Since, by assumption, $[I] = 0$ in $CH_0(X)$, we have $2[I_1] = [{I_1}^{(2)}] = 0$. Since $[I_1] \in G$ and G is torsion-free, $[I_1] = 0$ in $CH_0(X)$. Since $\Psi_L : E_0(A, L) \to CH_0(X)$ is an isomorphism, $(I_1)_L = 0$ in $E_0(A, L)$.

Let $K = (a_1, \cdots, a_{n-1})$. Since I_1/K is an invertible ideal of $A/K = B$, there exists $a \in I_1$ such that $K + (a) + {I_1}^2 = I_1$. Since K/I_1K is free over A/I_1 of rank $n-1$, There exists a surjection $\gamma' : F/I_1F \twoheadrightarrow K/I_1K$ where $F = L \oplus A^{n-2}$. Let $\gamma : F \to K$ be a lift. Then $\gamma(F) + I_1K = K$. Since $\mathrm{ht}(K) = n-1 = \mathrm{rank}(F)$, by (2.1), there exists $c \in I_1K$ and $\delta \in \mathrm{Hom}_A(F, A)$ such that $\gamma + c\delta(F)$ is an ideal of height of $n-1$. So, replacing γ by $\gamma + c\delta$ if necessary, we assume that $K_1 = \gamma(F)$ has height $n-1$. Since $K = K_1 + I_1K \subset I_1$, we have $K_1 + {I_1}^2 = K_1 + I_1K + {I_1}^2 = K + {I_1}^2 = {I_1}^{(2)}$ and $I_1 = K_1 + (a) + {I_1}^2$

Since $(I_1) = 0$ in $E_0(A, L)$, if ω is a local L-orientation of I_1, then, by (3.6), we have $({I_1}^{(2)}, *) = (I_1, \omega) + (I_1, -\omega) = 0$ in $E(A, L)$. Hence, by (2.5), ${I_1}^{(2)}$ is a surjective image of $L \oplus A^{n-1} = F \oplus A$.

Thus the proposition is proved. ■

Now we prove our main theorem.

Theorem 4.31 *Let $X = \mathrm{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over the field $\mathbb{R}$ of real numbers. Let K denote the canonical module $\wedge^n(\Omega_{A/\mathbb{R}})$. Let P be a projective A-module of rank n and let $\wedge^n(P) = L$. Assume that $C_n(P) = 0$ in $CH_0(X)$. Then $P \simeq A \oplus Q$ in the following cases:*

1. *$X(\mathbb{R})$ has no compact connected component.*
2. *For every compact connected component C of $X(\mathbb{R})$, $L_C \not\simeq K_C$ where K_C and L_C denote restriction of (induced) line bundles on $X(\mathbb{R})$ to C.*
3. *n is odd.*

Moreover, if n is even and L is a rank 1 projective A-module such that there exists a compact connected component C of $X(\mathbb{R})$ with the property that $L_C \simeq K_C$, then there exists a projective A-module P of rank n such that $P \oplus A \simeq L \oplus A^{n-1} \oplus A$ (hence $C_n(P) = 0$) but P does not have a free summand of rank 1.

Proof Let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism and $e(P, \chi) \in E(A, L)$ be the Euler class of P. The image of $e(P, \chi)$ under the canonical surjection $\Theta_L : E(A, L) \twoheadrightarrow CH_0(X)$ is $C_n(P)$.

Proof of (1) and (2). Under the assumption of either (1) or (2), by (4.30), Θ_L is an isomorphism and hence $C_n(P) = 0$ implies that $e(P, \chi) = 0$ in $E(A, L)$. Therefore, by (2.5), $P \simeq A \oplus Q$ for some projective A-module Q of rank $n-1$. This proves (1) and (2).

Proof of (3). Since n is odd, there is an automorphism Δ of P with determinant -1. Let $\alpha : P \twoheadrightarrow I$ be a surjection where $I \subset A$ is an ideal of height n and let ω_I be the local L-orientation of of I induced by α. Then, by definition, $e(P, \chi) = (I, \omega_I)$ in $E(A, L)$. Clearly $(\alpha\Delta, \chi)$ will induce $(I, -\omega_I)$ and hence $e(P, \chi) = (I, -\omega_I)$ in $E(A, L)$. Therefore $2e(P, \chi) = (I, \omega_I) + (I, -\omega_I)$.

Now since $C_n(P) = 0$, $[I] = 0$ in $CH_0(X)$. Since $\Psi_L : E_0(A, L) \to CH_0(X)$ is an isomorphism and $\Psi_L((I)) = [I] = 0$, we get that $(I) = 0$ in $E_0(A, L)$. But then, by (3.7), $(I, \omega_I) + (I, -\omega_I) = 0$ in $E(A, L)$. Thus $2e(P, \chi) = 0$ in $E(A, L)$. Since $C_n(P) = 0$, $e(P, \chi)$ is in the kernel of the surjection $\Theta_L : E(A, L) \twoheadrightarrow CH_0(X)$ which, by (4.30), is a free abelian group. Hence $e(P, \chi) = 0$ in $E(A, L)$. Therefore as before we see that $P \simeq A \oplus Q$.

For the last part of the theorem we now suppose that n is even and there exists a compact connected component C of $X(\mathbb{R})$ with the property that $L_C \simeq K_C$.

In this case, by (4.30), $\ker(\Theta_L)$ is a free abelian group of rank ≥ 1. This shows that, since $\Psi_L : E_0(A, L) \simeq CH_0(X)$ and Θ_L factors through the canonical surjection $E(A, L) \twoheadrightarrow E_0(A, L)$, this surjection is not an isomorphism. As in [B-RS 1, Lemma 3.3] we can show that the kernel of the canonical surjection $E(A, L) \twoheadrightarrow E_0(A, L)$ is generated by all elements of the type (J, ω_J), where J is an ideal of A of height n which is a surjective image of $L \oplus A^{n-1}$ and ω_J is local L-orientation of J. Since $E(A, L)$ is not isomorphic to $E_0(A, L)$, there exists an ideal I of height n which is a surjective image of $L \oplus A^{n-1}$ and a local L-orientation ω_I such that $(I, \omega_I) \neq 0$ in $E(A, L)$. Since n is even, adapting the proof of [B-RS 1, Lemma 3.6], we see that there exists a projective A-module P of rank n which is stably isomorphic to $L \oplus A^{n-1}$ and an isomorphism $\chi : L \to \wedge^n(P)$ such that $e(P, \chi) = (I, \omega_I) \neq 0$ in $E(A, L)$.

Since $e(P, \chi) \neq 0$, P does not have a free summand of rank 1. On the other hand, since P is stably isomorphic to $L \oplus A^{n-1}$ it is clear that $C_n(P) =$

0 in $CH_0(X)$. This completes the proof.

■

Remark 4.32 Let $X = \text{Spec}(A)$ be a smooth affine surface over $\mathbb{R}$ such that $X(\mathbb{R})$ is compact and connected and the canonical module $K_{\mathbb{R}(X)}$ is not trivial. In this set up, (4.31) says that every stably free A-module of rank 2 has a free summand of rank 1 and hence all stably free A-modules are free. This yields an affirmative answer to Question 6.5 of [B-RS 1].

Remark 4.33 Let A, X, P be as in the theorem. Assume that $X(\mathbb{R})$ has at least one compact connected component, $\dim(A)$ is even and $C_n(P) = 0$. The above theorem says that, in this set up, we can not always conclude that $P \simeq A \oplus Q$. However, by [B-RS 2 Proposition 6.3], we always have $A \oplus P \simeq A \oplus A \oplus Q$. Moreover, if $X(\mathbb{R})$ is connected, then the only possible obstruction for P to split off a free summand of rank 1 is purely topological viz. the associated topological vector bundle and the manifold $X(\mathbb{R})$ have the same orientation.

Statements (1) and (3) of the theorem have been already proved in [B-RS 1] in the special case $K \otimes_A \mathbb{R}(X) \simeq \mathbb{R}(X) \simeq \wedge^n(P) \otimes_A \mathbb{R}(X)$.

We end this paper by posing the following question.

Question. *Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over a field k of characteristic 0. Let P be a projective A-module of rank n such that $C_n(P) = 0$ in $CH_0(X)$. Then, does there exist a projective A-module Q of rank $n-1$ such that $P \oplus A \simeq Q \oplus A \oplus A$?*

References

[B-RS 1] S. M. Bhatwadekar, Raja Sridharan, Zero cycles and the Euler class groups of smooth real affine varieties, *Invent. Math.* **136** (1999), 287-322.

[B-RS 2] S. M. Bhatwadekar, Raja Sridharan, The Euler class group of a Noetherian ring, *Compositio Math.* **122** (2000), 183-222.

[Bl-M-S] S. Bloch, M. P. Murthy, L. Szpiro, Zero cycles and the number of generators of an ideal, *Mém. Soc. Math. de France* **38** (1989), 51-74.

[Bo-C-R] J. Bochnak, M. Coste, M.-F. Roy, Real algebraic geometry, Ergenbisse der Mathematik, Volume 36, Springer-Verlag, 1998.

[CT-S] J.-L.Colliot-Thélène, C.Scheiderer, Zero cycles and cohomology on real algebraic varieties, *Topology* **35** (1996), 533-559.

[E-E] D. Eisenbud, E. G. Evans, Generating modules efficiently : Theorems from algebraic K-theory, *J. Algebra* **27** (1973), 278-305.

[Kn] M. Knebusch, On algebraic curves over real closed fields (I), *Math. Z.* **150** (1976), 49-70.

[M] D. Mumford, Rational equivalence of 0-cycles on surfaces, *J. Math. Kyoto Univ.* **9** (1969), 195-204.

[MK] N. Mohan Kumar, Some theorems on generation of ideals in affine algebras, *Comment. Math. Helv.* **59** (1984), 243-252.

[Mu] M. P. Murthy, Zero cycles and projective modules, *Ann. Math.* **140** (1994), 405-434.

[Mu-S] M. P. Murthy, R. G. Swan, Vector bundles over affine surfaces, *Invent. Math.* **36** (1976), 125-165.

[Pl] B. Plumstead, The conjectures of Eisenbud-Evans, *Amer. J. Math.* **105** (1983), 1417-1433.

[Ro] I. Robertson, The Euler class group of a line bundle on an affine algebraic variety over a real closed field, Ph.D. thesis, University of Chicago, 1999.

[Sw] R. G. Swan, A cancellation theorem for projective modules in the metastable range, *Invent. Math.* **27** (1974), 23-43.