PROJECTIVE GENERATION OF CURVES (III)

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ABSTRACT. Let A be an affine domain of dimension n over a field of characteristic zero. Let $I \subset A[T]$ be a local complete intersection ideal of height n such that $\mu(I/I^2) = n$. This paper examines under what condition I is surjective image of a projective $A[T]$-module of rank n. More specifically, one is interested in knowing when is an element (I, ω_I) of the Euler class group $E(A[T])$ obtained as the Euler class of a projective $A[T]$-module. It is proved that such a phenomenon occurs if and only if the naturally induced element $(I(0), \omega_{I(0)})$ of the Euler class group $E(A)$ is obtained as the Euler class of a projective A-module.

1. INTRODUCTION

Let R be a Noetherian ring and I be an ideal of height n. It is well known (by dimension theory) that $n \leq \mu(I)$, where for a finitely generated R-module M, $\mu(M)$ denotes the minimal number of generators of M. In view of this, it is of some interest to know when $n = \mathrm{ht}(I) = \mu(I)$. Obvious condition is that $\mu(I/I^2) = n$. Now suppose that $\mu(I/I^2) = n$. Even then I need not be generated by n elements, as can be easily seen by taking a non-principal ideal in a Dedekind domain (which is not PID). Such examples lead us to consider a modified question, namely, if $\mu(I/I^2) = n$, is I at least surjective image of a projective R-module P of rank n? If it is so and P has trivial determinant (i.e., $\wedge^n(P) \simeq R$), then we say that I is projectively generated. It is easy to see that if $n = 2$ then I is projectively generated. Therefore we always assume that $n \geq 3$. It can be shown that even if R is a ring of dimension $n \geq 3$ and $\mathfrak{m}$ is a maximal ideal of R such that $R_{\mathfrak{m}}$ is regular (hence $\mu(\mathfrak{m}/\mathfrak{m}^2) = n$), $\mathfrak{m}$ need not be projectively generated. For example, let R be the coordinate ring of the real three sphere and $\mathfrak{m}$ a real maximal ideal of R. It is well known that every projective R-module of rank three is free and $\mathfrak{m}$ is not generated by 3 elements.

On the other hand, if R is a polynomial algebra of a Noetherian ring, there are some interesting affirmative results. Let $R = A[T]$ where A is a Noetherian ring and let I be an ideal of R of height n such that (1) $\mu(I/I^2) = n$, (2) $n \geq \dim(R/I) + 2$, (3) I contains a monic polynomial. Then, by a result of Mohan Kumar [Mo], I is projectively

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generated. A subsequent (remarkable) theorem of Mandal [Ma] says that, under the above assumptions, in fact I is efficiently generated (i.e. $\mu(I) = n$). In particular, if $n \geq \dim(R) - 1 \geq 3$, then I is efficiently generated. Before proceeding further we would like to point out that the above results of Mohan Kumar and Mandal are not true without condition (3). To see this, take A to be the coordinate ring of the real three sphere, $\mathfrak{m}$ to be a real maximal ideal, $R = A[T]$ and $I = \mathfrak{m}R$.

Keeping this background in mind, one needs to address the following question: *Let A be an affine domain of dimension $n \geq 3$ over a field k. Let $I \subset A[T]$ be a local complete intersection ideal of height n such that $\mu(I/I^2) = n$ which does not contain any monic polynomial. When is I projectively generated?*

In this paper we give a (partial) solution to the above question. Before stating our result we need some discussion. For $\lambda \in k$, let $I(\lambda) := \{f(\lambda) | f(T) \in I\}$. Note that $I(\lambda)$ is an ideal of A and $I(\lambda) = I + (T - \lambda)/(T - \lambda) \simeq I/I \cap (T - \lambda)$. Now assume that k is a field of characteristic zero. It is easy to see that for almost all λ, $I(\lambda)$ is an ideal of A of height n such that $I(\lambda)/I(\lambda)^2$ is generated by n elements. It is obvious that if I is projectively generated then so also is $I(\lambda)$. Now suppose, for some $\lambda \in k$, say $\lambda = 0$, that there exists a projective A-module Q of rank n with trivial determinant and a surjection $\phi : Q \to I(0)$. Then, since $I(0) \simeq I/(I \cap (T))$, ϕ can be lifted to a map θ from $Q[T]$ to $I/(I^2 \cap (T))$ (here $Q[T]$ is $Q \otimes_A A[T]$). The core idea of our main result (Theorem 3.5 below) essentially says that, if for some suitable ϕ a *surjective* lift θ exists then I is projectively generated. More precisely, the following is implicit in the proof of Theorem 3.5.

Theorem 1.1. *Let A be an affine domain of dimension $n \geq 3$ over a field k of characteristic zero. Let $I \subset A[T]$ be a local complete intersection ideal of height n such that $\mu(I/I^2) = n$. Assume that $I(0)$ is of height n. If there exists a projective A-module Q of rank n with trivial determinant and a surjection from $Q[T]$ to $I/(I^2 \cap (T))$, then I is projectively generated.*

Remark 1.2. Note that if $I(0) = A$ then in the above theorem we can take Q to be free and then by [BRS1, 3.9] there is a surjection $Q[T] \twoheadrightarrow I/(I^2T)$, proving that I is projectively generated. We do not mention this case in the results below.

Applying our main theorem we prove the following results (see 3.7, 3.11, 3.10).

Corollary 1.3. *Let A be an affine domain of dimension $n \geq 3$ over a field k of characteristic zero. Let $I \subset A[T]$ be a local complete intersection ideal of height n such that I/I^2 is generated by n elements. Then I is projectively generated in the following cases :*

(1) *n is even and $I(0)$ is a projectively generated ideal of height n;*
(2) *k is a C_1 field and $I(0)$ is a projectively generated ideal of height n;*
(3) *k is algebraically closed.*

Remark 1.4. The third part of the above corollary has been already proved in [BRS2, Theorem 2.7]. In fact this paper may be considered as a sequel to [BRS1, BRS2] and the method of proof adopted here are refinements of techniques from [BRS2].

A diligent reader who is also familiar with the theory of the Euler class groups, upon seeing the above discussion, will immediately realize that we have proved the following result : *Let* (I,ω_I) *be an element of the Euler class group* $E(A[T])$*, where* ω_I *is a local orientation of* I*. Assume that for the induced local orientation* $\omega_{I(0)}$ *of* $I(0)$*, the element* $(I(0),\omega_{I(0)})$ *of the Euler class group* $E(A)$ *is obtained as the Euler class of a projective* A*-module. Then, there exist a projective* $A[T]$*-module whose Euler class is* (I,ω_I)*.*

For the complete statement, see (Theorem 3.5). In the proof of this theorem we have used some basic techniques from the Euler class theory. We shall urge the general reader to look up Section 2 for the definitions and only a handful of relevant results for references.

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2. Preliminaries

We begin this section with a collection of useful lemmas. The following lemma is a consequence of a theorem of Eisenbud-Evans [E-E] and is proved in [BRS4].

Lemma 2.1. [BRS4, 2.13] *Let* A *be a Noetherian ring and* P *be a projective* A*-module of rank* n*. Let* $(\alpha,a)\in(P^*\oplus A)$*. Then there exists an element* $\beta\in P^*$ *such that* $ht(I_a)\geq n$*, where* $I=(\alpha+a\beta)(P)$*. In particular, if the ideal* $(\alpha(P),a)$ *has height* $\geq n$*, then* $ht(I)\geq n$*. Further, if* $(\alpha(P),a)$ *is an ideal of height* $\geq n$ *and* I *is a proper ideal of* A*, then* $ht(I)=n$*.*

The following couple of lemmas are easy. However, we prove them for the sake of completeness.

Lemma 2.2. *Let* A *be a Noetherian ring,* $I\subset A$ *an ideal and* P *be a projective* A*-module. Assume that there exists* $\alpha\in Hom_A(P,A)$ *such that* $\alpha(P)\subset I$*. Then,* $\alpha:P\longrightarrow I$ *is a surjection if and only if* $I=\alpha(P)+I^2$ *and* $V(\alpha(P))=V(I)$ *in* $Spec(A)$*.*

Proof. We write $J=\alpha(P)$. Let $\overline{I}=I/J$, where bar denotes reduction modulo J. Then $\overline{I}=\overline{I}^2$ and therefore $\overline{I}$ is generated by an idempotent $\overline{e}\in\overline{I}$. If $e\in I$ is a preimage of $\overline{e}$, then $I=(J,e)$ and $e(1-e)\in J$.

To prove that $J=I$, it is enough to check that $I_{\mathfrak{p}}=J_{\mathfrak{p}}$, for all $\mathfrak{p}\in\text{Spec}(R)$. If $\mathfrak{p}\not\supset J$, then $\mathfrak{p}\not\supset I$, and therefore, $I_{\mathfrak{p}}=A_{\mathfrak{p}}=J_{\mathfrak{p}}$. Now let $\mathfrak{p}\supset J$. As $V(I)=V(J)$, we have $\mathfrak{p}\supset I$, and since $I=(J,e)$ with $e(1-e)\in J$ and $A_{\mathfrak{p}}$ is a local ring, we have $I_{\mathfrak{p}}=J_{\mathfrak{p}}$. $\square$

Lemma 2.3. *Let B be a Noetherian ring with dim$(B) = d$ and P be a projective B-module of rank $n \geq d+1$. Let I, L be ideals of B such that $L \subseteq I^2$. Assume that there is a surjection $\phi : P \twoheadrightarrow I/L$. Then ϕ can be lifted to a surjection $\Psi : P \twoheadrightarrow I$.*

Proof. Let $\psi : P \longrightarrow I$ be a lift of ϕ. Then we have $\psi(P) + L = I$. As $L \subseteq I^2$, by [BRS1, Lemma 3.2] there exists $e \in L$ such that $\psi(P) + (e) = I$.

We now apply Lemma 2.1 to the element (ψ, e) of $P^* \oplus B$ to obtain $\theta \in P^*$ such that if $K = (\psi + e\theta)(P)$, then ht$(K_e) \geq n$. As dim$(B) = d \leq n-1$, it follows that $K_e = B_e$ and consequently, $e^l \in K$ for some positive integer l.

We have $K + (e) = I$ and $e \in L \subseteq I^2$. Applying Lemma 2.2 it is easy to see that $K = I$. We rename $\psi + e\theta$ as Ψ. Then $\Psi : P \twoheadrightarrow I$ is a surjection. Since $e \in L$, clearly Ψ lifts ϕ. $\square$

Lemma 2.4. *Let A be a Noetherian ring with dim$(A) = d$. Let $I \subset A[T]$ be an ideal with ht $(I) \geq 2$. Assume that there is a projective A-module Q of rank $n \geq d+1$ such that there is a surjection $\phi : Q[T] \twoheadrightarrow I/(I^2T)$. Then ϕ can be lifted to a surjection $\Phi : Q[T] \twoheadrightarrow I$.*

Proof. We can choose $s \in I^2 \cap A$ such that ht $(s) = 1$. Let bar denote reduction modulo (s). Applying Lemma 2.3 it is easy to see that $\overline{\phi}$ has a lift to a surjection $\overline{\theta} : \overline{Q}[T] \twoheadrightarrow \overline{I}$. Therefore, if θ is a lift of $\overline{\theta}$ to Hom$(Q[T], I)$ and ϕ_1 is a lift of ϕ to Hom$(Q[T], I)$, then we have $(\theta - \phi_1)(Q[T]) \subset (I^2T) + (s)$. If we write $\eta = \theta(0) - \phi_1(0)$, then we observe that $\eta(Q) \subset (s)$. Let $\theta - \eta(T)$ be denoted by θ', where $\eta(T)$ stands for $\eta \otimes A[T]$. From the above relations we have: (1) $I = (\theta'(Q[T]), s)$; (2) $\theta'(0) = \phi_1(0) = \phi(0)$, and hence $\theta'(0)(Q) = I(0)$. A local-global argument, similar to the one given in the proof of Lemma 2.2 above, implies that $I = (\theta'(Q[T]), sT)$. Applying Lemma 2.1 we can find $\tau \in$ Hom$(Q[T], A[T])$ such that the ideal $I'' = (\theta' + sT\tau)(Q[T])$ satisfies the following properties:

(1) $I'' = I \cap I'$ where I' is an ideal of height $\geq d+1$;
(2) $I' + (sT) = A[T]$.

Clearly I' contains a monic polynomial and therefore, by [L, Lemma 1.1, p. 79], we have $I' \cap A + (s) = A$, implying that I' contains an element of the form $1 + sa$ where $a \in A$.

Let us write $\alpha = \theta' + sT\tau$. We have the surjection $\alpha_{1+sA} : Q[T]_{1+sA} \twoheadrightarrow I_{1+sA}$. On the other hand we have a surjection $\phi(0)_s(T) : Q_s[T] \twoheadrightarrow I(0)_s[T] = A_s[T] = I_s$. It is easy to check that the kernels of the induced surjections $\alpha_{s(1+sA)}$ and $\phi(0)_{s(1+sA)}(T)$ are both extended. Since these two surjections are the same modulo T, by [BRS4, Lemma 2.9] there is an automorphism $\sigma(T)$ of $Q_{s(1+sA)}[T]$ such that $\sigma(0) =$ id and $\alpha_{s(1+sA)}\sigma(T) = \phi(0)_{s(1+sA)}(T)$. As $\sigma(0) =$ id, one can now apply [BRS4, Lemma 2.10] and patch the surjections α_{1+sA} and $\phi(0)_s(T)$ to obtain a surjective map $\Phi : Q[T] \twoheadrightarrow I$. Using [BRS4, Lemma 2.10] it can be easily checked that Φ lifts ϕ. $\square$

Lemma 2.5. *Let R be a Noetherian ring of dimension $n \geq 3$ and $I \subset R[T]$ be an ideal of height n. Let Q be a projective R-module of rank n such that there is a surjection $\alpha : Q[T] \twoheadrightarrow I/(I^2T)$. Let $b \in I^2 \cap R$ be any non-zerodivisor. Then we can find a lift $\beta \in Hom(Q[T], I)$ of α with the following properties :*

(1) $I'' + (bT) = I$, *where* $I'' = \beta(Q[T])$*;*
(2) $I'' = I \cap I'$, *where ht* $(I') \geq n$*;*
(3) $I' + (bT) = R[T]$.

Proof. The proof is very similar to the first part of the proof of the above lemma. Write $A = R/(b)$ and let "bar" denote reduction modulo (b). Then $\dim(A) \leq n-1$. Applying the above lemma we obtain a surjection $\Phi : \overline{Q}[T] \twoheadrightarrow \overline{I}$ which lifts $\overline{\alpha}$. If ϕ is a lift of Φ to $\text{Hom}(Q[T], I)$ and α_1 is a lift of α to $\text{Hom}(Q[T], I)$, then we have $(\phi - \alpha_1)(Q[T]) \subset (I^2T) + (b)$. If we write $\eta = \phi(0) - \alpha_1(0)$, then we observe that $\eta(Q) \subset (b)$. Let $\phi - \eta(T)$ be denoted by α'. From the above relations we have: (1) $I = (\alpha'(Q[T]), b)$; (2) $\alpha'(0) = \alpha_1(0) = \alpha(0)$ and hence $\alpha'(0)(Q) = I(0)$. A simple local-global argument implies that $I = (\alpha'(Q[T]), bT)$.

We can now apply Lemma 2.1 to conclude the rest. $\square$

Remark 2.6. In the above lemma, if R is a geometrically reduced affine algebra over an infinite field then applying Swan's Bertini Theorem [BRS3, 2.11] in place of Eisenbud-Evans theorem, we may further obtain that I' is reduced.

We now recall the definitions of the Euler class groups and collect the results relevant to this paper. For a Noetherian ring A of dimension n containing $\mathbb{Q}$, the n^{th} Euler class group $E^n(A)$ has been defined in [BRS4]. This definition has been extended to the definition of the n^{th} Euler class group $E^n(A[T])$ in [D1]. We refer to these two papers for further details and unexplained terms related to the Euler class theory. Also, since in this paper we are only concerned about the n^{th} Euler class groups, we shall simply write them as $E(A)$ and $E(A[T])$.

Definition 2.7. (**The Euler class group** $E(A)$) Let A be a commutative Noetherian ring (containing $\mathbb{Q}$) of dimension $n \geq 2$. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Two surjections α, β from $(A/J)^n$ to J/J^2 are said to be related if there exists $\sigma \in SL_n(A/J)$ such that $\alpha\sigma = \beta$. Clearly this is an equivalence relation on the set of surjections from $(A/J)^n$ to J/J^2. Let $[\alpha]$ denote the equivalence class of α. Such an equivalence class $[\alpha]$ is called a *local orientation* of J. By abuse of notation, we shall identify an equivalence class $[\alpha]$ with α. A local orientation α is called a *global orientation* if $\alpha : (A/J)^n \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : A^n \twoheadrightarrow J$. Let G be the free abelian group on the set of pairs $(\mathcal{N}, \omega_{\mathcal{N}})$ where $\mathcal{N}$ is an $\mathcal{M}$-primary ideal for some maximal ideal $\mathcal{M}$ of height n such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements and $\omega_{\mathcal{N}}$ is a local orientation of $\mathcal{N}$. Now let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements and ω_J be a local orientation of J. Let $J = \cap_i \mathcal{N}_i$ be the (irredundant) primary decomposition of J. We associate to the pair (J, ω_J), the element $\sum_i (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ of G where $\omega_{\mathcal{N}_i}$ is the local orientation of $\mathcal{N}_i$ induced by ω_J. By abuse of notation, we denote $\sum_i (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ by (J, ω_J). Let H be the subgroup of G generated by the set of pairs (J, ω_J), where J is an ideal of height n and ω_J is a global orientation of J. The Euler class group of A is $E(A) \stackrel{\text{def}}{=} G/H$.

Definition 2.8. (**The Euler class of a projective** A**-module**) Let P be a projective A-module of rank n such that $A \simeq \wedge^n(P)$ and let $\chi : A \stackrel{\sim}{\to} \wedge^n P$ be an isomorphism. Let $\varphi : P \twoheadrightarrow J$ be a surjection where J is an ideal of height n. Therefore we obtain an induced surjection $\overline{\varphi} : P/JP \twoheadrightarrow J/J^2$. Let $\overline{\gamma} : (A/J)^n \stackrel{\sim}{\to} P/JP$ be an isomorphism such that $\wedge^n(\overline{\gamma}) = \overline{\chi}$. Let ω_J be the local orientation of J given by $\overline{\varphi}\,\overline{\gamma} : (A/J)^n \twoheadrightarrow J/J^2$. Let $e(P, \chi)$ be the image in $E(A)$ of the element (J, ω_J) of G. The assignment sending the pair (P, χ) to the element $e(P, \chi)$ of $E(A)$ is well defined (see [BRS4]). The *Euler class* of (P, χ) is defined to be $e(P, \chi)$.

Theorem 2.9. [BRS4] *Let A be a Noetherian ring (containing $\mathbb{Q}$) of dimension $n \geq 3$. Let $I \subset A$ be an ideal of A of height n such that I/I^2 is generated by n elements and ω_I be a local orientation of I. Let P be a rank n projective A-module with trivial determinant with a trivialization $\chi : A \simeq \wedge^n(P)$.*

(a) Suppose that the image of (I, ω_I) is zero in $E(A)$. Then ω_I is a global orientation of I.
(b) Suppose that $e(P, \chi) = (I, \omega_I)$ in $E(A)$. Then there exists a surjection $\alpha : P \twoheadrightarrow I$ such that ω_I is induced by α and χ (as described above).
(c) P has a unimodular element if and only if $e(P, \chi) = 0$ in $E(A)$.

Definition 2.10. (**The Euler class group** $E(A[T])$) Let A be a Noetherian ring of dimension $n \geq 3$ containing $\mathbb{Q}$. Let $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Two surjections α and β from $(A[T]/I)^n \twoheadrightarrow I/I^2$ are said to be *related* if there exists $\sigma \in SL_n(A[T]/I)$ such that $\alpha\sigma = \beta$. This is an equivalence relation on the set of surjections from $(A[T]/I)^n$ to I/I^2. Let $[\alpha]$ denote the equivalence class of α. We call such an equivalence class $[\alpha]$ a *local orientation of* I. It was shown in [D1, Proposition 4.4], that if $\alpha : (A[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$ then so can any β equivalent to α. We call a local orientation $[\alpha]$ of I a *global orientation of* I if the surjection $\alpha : (A[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection

$\theta : A[T]^n \twoheadrightarrow I$. Let G be the free abelian group on the set of pairs (I, ω_I) where $I \subset A[T]$ is an ideal of height n such that $\text{Spec}(A[T]/I)$ is connected, I/I^2 is generated by n elements and $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ is a local orientation of I. Let $I \subset A[T]$ be an ideal of height n and ω_I a local orientation of I. Now I can be decomposed uniquely as $I = I_1 \cap \cdots \cap I_r$, where the I_k's are ideals of $A[T]$ of height n, pairwise comaximal, such that $\text{Spec}(A[T]/I_k)$ is connected for each k. Clearly ω_I induces local orientations ω_{I_k} of I_k for $1 \leq k \leq r$. By (I, ω_I) we mean the element $\Sigma(I_k, \omega_{I_k})$ of G. Let H be the subgroup of G generated by set of pairs (I, ω_I), where I is an ideal of $A[T]$ of height n generated by n elements and ω_I is a global orientation of I given by the set of generators of I. We define the Euler class group of $A[T]$, denoted by $E(A[T])$, to be G/H.

Remark 2.11. For details on this remark the reader is referred to [D1, D2]. There is a canonical injective group homomorphism $\Phi : E(A) \longrightarrow E(A[T])$. It has been proved in [D2, Theorem 3.3] that there is a surjective group homomorphism $\Psi : E(A[T]) \twoheadrightarrow E(A)$ such that the composite map $\Psi\Phi$ is identity on $E(A)$. Further, the map Ψ has the property that if $(I, \omega_I) \in E(A[T])$ is such that $\text{ht}(I(0)) = n$, then $\Psi((I, \omega_I)) = (I(0), \omega_{I(0)})$, where $I(0) = \{f(0) \mid f \in I\}$ and $\omega_{I(0)}$ is the local orientation of $I(0)$, naturally induced by ω_I.

Definition 2.12. (**The Euler class of a projective $A[T]$-module**) Let P be a projective $A[T]$-module of rank n with trivial determinant. Fix a trivialization $\chi : A[T] \overset{\sim}{\to} \wedge^n(P)$. Let $\alpha : P \twoheadrightarrow I$ be a surjection such that I is an ideal of height n. Note that, since P has trivial determinant and $\dim(A[T]/I) \leq 1$, P/IP is a free $A[T]/I$-module. Composing $\alpha \otimes A[T]/I$ with an isomorphism $\gamma : (A[T]/I)^n \overset{\sim}{\to} P/IP$ with the property $\wedge^n(\gamma) = \chi \otimes A[T]/I$, we get a local orientation, say ω_I, of I. Let $e(P, \chi)$ be the image in $E(A[T])$ of the element (I, ω_I) of G. (We say that (I, ω_I) is obtained from the pair (α, χ)). It can be proved that the assignment sending the pair (P, χ) to $e(P, \chi)$ is well defined (see [D1, Lemma 4.6]). We define the *Euler class* of P to be $e(P, \chi)$.

Theorem 2.13. [D1] *Let A be a Noetherian ring (containing $\mathbb{Q}$) of dimension $n \geq 3$. Let $I \subset A[T]$ be an ideal of $A[T]$ of height n such that I/I^2 is generated by n elements and ω_I be a local orientation of I. Let P be a rank n projective $A[T]$-module with trivial determinant with a trivialization $\chi : A[T] \simeq \wedge^n(P)$.*

(a) Suppose that the image of (I, ω_I) is zero in $E(A[T])$. Then ω_I is a global orientation of I.
(b) Suppose that $e(P, \chi) = (I, \omega_I)$ in $E(A[T])$. Then there exists a surjection $\alpha : P \twoheadrightarrow I$ such that ω_I is induced by α and χ (as described above).
(c) P has a unimodular element if and only if $e(P, \chi) = 0$ in $E(A[T])$.

We need the following remark for subsequent discussions. Let us first explain some of the notations used in this remark and the next section.

Let R be a Noetherian ring of dimension n, containing $\mathbb{Q}$, and let $(J, \omega) \in E(R)$. Let u be a unit in R/J. Let $\tau \in GL_n(R/J)$ be a matrix with $\det(\tau) = u$. By the notation $u\omega$ we mean the local orientation of J given by the composite: $(R/J)^n \underset{\tau}{\overset{\sim}{\to}} (R/J)^n \underset{\omega}{\twoheadrightarrow} J/J^2$. Note that $(J, u\omega) \in E(R)$ does not depend on the choice of τ.

Remark 2.14. Let R be a Noetherian ring of dimension n, containing $\mathbb{Q}$, and let $(J, \omega) \in E(R)$. Assume that there is a projective R-module P of rank n and an isomorphism $\chi : R \overset{\sim}{\to} \wedge^n(P)$ such that $e(P, \chi) = (J, u\omega)$ in $E(R)$, where u is a unit in R. Then if we take χ_1 to be the composite $R \underset{u^{-1}}{\overset{\sim}{\to}} R \underset{\chi}{\overset{\sim}{\to}} \wedge^n(P)$, it is easy to see that $e(P, \chi_1) = (J, \omega)$.

The following proposition and the subsequent remark are crucial to the next section where we prove the main theorem. These results assert that it is enough to prove the main theorem for reduced rings. As in the main theorem we are considering affine algebras over a field of characteristic zero, it further reduces the problem to geometrically reduced rings. The main advantage of this reduction is that we can then apply Swan's Bertini theorem (see remark 2.6) and move to the residual ideal which is reduced. The whole proof of the main theorem hinges on this special type of residual ideal.

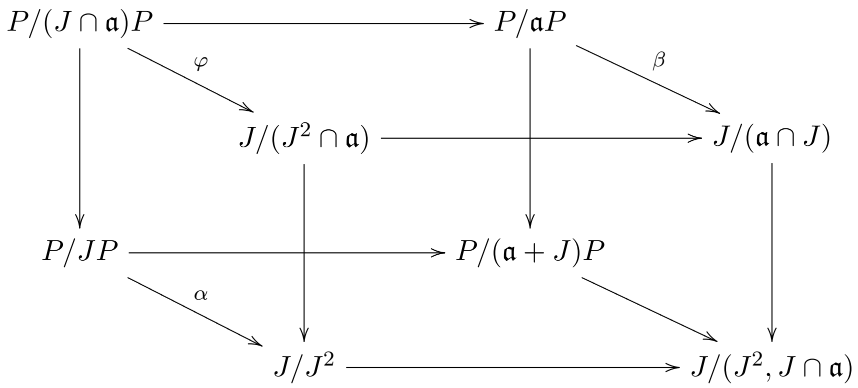
Proposition 2.15. *Let R be a Noetherian ring of dimension n, containing $\mathbb{Q}$ and $\Phi : E(R) \simeq E(R_{red})$ be the canonical isomorphism as illustrated in* [BRS4, 4.6]. *Let $(J, \omega) \in E(R)$ be such that $\Phi((J, \omega)) = e(P', \chi')$, where P' is a projective R_{red}-module of rank n and $\chi' : R_{red} \simeq \wedge^n P'$ an isomorphism. Then there exists a projective R-module P of rank n and an isomorphism $\chi : R \simeq \wedge^n P$ such that $e(P, \chi) = (J, \omega)$ in $E(R)$.*

Proof. Let $\mathfrak{a}$ be the nilradical of R. So $R_{\text{red}} = R/\mathfrak{a}$. Since there is a bijection between the isomorphism classes of finitely generated projective R-modules and the isomorphism classes of finitely generated projective $R/\mathfrak{a}$-modules, there exists a projective R-module P of rank n such that $P \otimes R/\mathfrak{a} = P/\mathfrak{a}P \simeq P'$ (see [W, Lemma 2.2, page 70]). Since the canonical map from R^* to $(R/\mathfrak{a})^*$ is surjective, in view of the above remark we may assume that $P/\mathfrak{a}P = P'$ and χ' is induced by some isomorphism $\chi : R \simeq \wedge^n(P)$. Under the map Φ, the element (J, ω) goes to $(\overline{J}, \overline{\omega})$, where $\overline{J} = (J + \mathfrak{a})/\mathfrak{a} = J/(J \cap \mathfrak{a})$ and $\overline{\omega}$ is naturally induced by ω (for details see [BRS4, 4.6]). Denoting $R/\mathfrak{a}$ by $\overline{R}$, $P/\mathfrak{a}P$ by $\overline{P}$ and χ' by $\overline{\chi}$, we rewrite the hypothesis of the proposition as : $e(\overline{P}, \overline{\chi}) = (\overline{J}, \overline{\omega})$ in $E(\overline{R})$.

We choose an isomorphism $\sigma : (P/JP) \overset{\sim}{\to} (R/J)^n$ such that $\wedge^n\sigma = \chi \otimes R/J$ and obtain the surjection $\alpha : P/JP \twoheadrightarrow J/J^2$ which is the composite

$$P/JP \overset{\sigma}{\underset{\sim}{\to}} (R/J)^n \overset{\omega}{\twoheadrightarrow} J/J^2.$$

Now σ will induce $\overline{\sigma} : \overline{P}/\overline{JP} \overset{\sim}{\to} (\overline{R}/\overline{J})^n$ and similarly we will get $\overline{\alpha}$. The hypothesis $e(\overline{P}, \overline{\chi}) = (\overline{J}, \overline{\omega})$ implies that there is a surjection $\beta : \overline{P} \twoheadrightarrow \overline{J}$ such that it lifts $\overline{\alpha}$. We now consider the following patching diagram, patch α and β to obtain a surjection $\varphi : P/(J \cap \mathfrak{a})P \twoheadrightarrow J/(J^2 \cap \mathfrak{a})$.



As P is projective, we can find $\theta : P \to J$ which is a lift of φ. As φ lifts α and β, so does θ. Therefore, we have : (1) $\theta(P) + J^2 = J$; and (2) $\theta(P) + (\mathfrak{a} \cap J) = J$. As the ideal $\mathfrak{a}$ is nilpotent, it follows from (2) that $V(J) = V(\theta(P))$ in Spec(R). Now the proposition can be easily deduced from Lemma 2.2. □

Remark 2.16. It has been proved in [D3, Proposition 2.15] that $E(R[T]) \simeq E(R_{\text{red}}[T])$. Exactly the same proof as above will show that Proposition 2.15 is true for $E(R[T])$ as well.

3. THE MAIN THEOREM

In this section, by a ring we shall mean a commutative Noetherian ring. Further, we make the blanket assumption that the rings considered here contain $\mathbb{Q}$, although some of the results may be true without this hypothesis.

We need some preparatory results to prove the main theorem.

Lemma 3.1. *Let R be a ring of dimension $n \geq 2$, $J \subset R$ be an ideal of height $\geq n-1$ and Q be a projective R-module with trivial determinant. Then there exists $b \in J^2$ such that $ht(b) = 1$ and Q_{1+b} is a free R_{1+b}-module.*

Proof. As the determinant of Q is trivial and $\dim(R/J^2) \leq 1$, it follows that Q/J^2Q is a free R/J^2-module. Consequently, Q_{1+J^2} is a free R_{1+J^2}-module. Therefore, there exists $b \in J^2$ such that Q_{1+b} is free. If $\operatorname{ht}(b) = 0$ then we can find $c \in J^2$ such that $\operatorname{ht}(b + cb + c) = 1$. Now since $(1 + b)(1 + c) = 1 + b + bc + c$, we can assume, without loss of generality, that $\operatorname{ht}(b) = 1$ and Q_{1+b} is free. $\square$

Before stating our next lemma, we recall from Remark 2.11 that there is a surjective group homomorphism $\Psi : E(A[T]) \twoheadrightarrow E(A)$ with the property that if $(I, \omega_I) \in E(A[T])$ is such that $\operatorname{ht}(I(0)) = n$, then $\Psi((I, \omega_I)) = (I(0), \omega_{I(0)})$, where $I(0) = \{f(0) \mid f \in I\}$ and $\omega_{I(0)}$ is the local orientation of $I(0)$, naturally induced by ω_I.

Lemma 3.2. *Let R be a ring of dimension $n \geq 3$ and Q be a projective R-module of rank n with trivial determinant. Fix $\chi : R \xrightarrow{\sim} \wedge^n(Q)$. Let $(I, \omega_I) \in E(R[T])$ be such that ht $I(0) = n$ and $(I(0), \omega_{I(0)}) \in E(R)$, where $\omega_{I(0)}$ is the local orientation of $I(0)$ induced by ω_I. Assume that there is a surjection $\alpha : Q \twoheadrightarrow I(0)$ such that (α, χ) induces $e(Q, \chi) = (I(0), \omega_{I(0)})$. Then there is a surjective map $\theta : Q[T] \twoheadrightarrow I/(I^2T)$ such that $\theta(0) = \alpha$.*

Proof. As Q has trivial determinant, we have $Q[T]/IQ[T]$ free over $R[T]/I$. We choose an isomorphism $\overline{\sigma} : Q[T]/IQ[T] \xrightarrow{\sim} (R[T]/I)^n$ such that $\wedge^n(\overline{\sigma}) = (\chi \otimes R[T]/I)^{-1}$. Therefore, the composite surjection

$$\theta' : Q[T] \twoheadrightarrow Q[T]/IQ[T] \overset{\overline{\sigma}}{\xrightarrow{\sim}} (R[T]/I)^n \overset{\omega_I}{\twoheadrightarrow} I/I^2$$

is such that $\theta'(0) \otimes R/I(0) = \alpha \otimes R/I(0)$. Applying [BRS1, 3.9] we can lift θ' to a surjection $\theta : Q[T] \twoheadrightarrow I/(I^2T)$. $\square$

Proposition 3.3. *Let $R \hookrightarrow B$ be a flat extension of rings such that dim(R) = dim$(B) = n \geq 3$. Let Q be a projective R-module of rank n with trivial determinant such that $Q \otimes_R B$ is a free B-module and P' be a projective $B[T]$-module of rank n with trivial determinant such that P'/TP' is a free B-module. Let $\chi : R \xrightarrow{\sim} \wedge^n(Q)$, $\chi' : B[T] \xrightarrow{\sim} \wedge^n(P')$ be isomorphisms. Let $I \subset R[T]$ be an ideal of height n such that ht$(I(0)) = n$ and both $IB[T]$ and $I(0)B$ are proper ideals. Let $\omega : R[T]^n \twoheadrightarrow I/I^2$ be a local orientation of I. Assume that there are surjections $\alpha : Q \twoheadrightarrow I(0)$ and $\beta : P' \twoheadrightarrow IB[T]$ such that (β, χ') induces $e(P', \chi') = (IB[T], \omega \otimes B) \in E(B[T])$, whereas (α, χ) induces $e(Q, \chi) = (I(0), \omega(0)) \in E(R)$, where $\omega(0)$ is the local orientation of $I(0)$ induced by ω. Then, there is an isomorphism $\psi : P'/TP' \xrightarrow{\sim} Q \otimes B$ and a surjection $\eta : P' \twoheadrightarrow IB[T]$ such that $\eta(0) = (\alpha \otimes B)\psi$ (where $\alpha \otimes B : Q \otimes B \twoheadrightarrow I(0)B$ is induced by α).*

Proof. Let us denote P'/TP' by P'_0. Let ω be given by : $I = (f_1, \cdots, f_n) + I^2$.

First we need to briefly explain what it means when we say, “the pair (β, χ') induces $e(P', \chi') = (IB[T], \omega \otimes B)$ in $E(B[T])$”. In what follows ‘tilde’ will denote reduction modulo $IB[T]$ and 'bar' will denote reduction modulo $I(0)B$. Note that $P'/IB[T]P'$ is a free $B[T]/IB[T]$-module of rank n. We now choose a basis $\widetilde{p}_1, \cdots, \widetilde{p}_n$ of $P'/IB[T]P'$ such that $\widetilde{p}_1 \wedge \cdots \wedge \widetilde{p}_n = \chi' \otimes B[T]/IB[T]$ and we have

$$\widetilde{\beta} : P'/IB[T]P' \twoheadrightarrow IB[T]/I^2B[T]$$

where $\widetilde{p}_i$ goes to $\widetilde{f}_i$. Reducing modulo T, we have a basis $\overline{p}_1(0), \cdots, \overline{p}_n(0)$ of the free $B/I(0)B$-module $P'_0/I(0)BP'_0$. This basis of $P'_0/I(0)BP'_0$ can be lifted to a basis $p^0_1, \cdots, p^0_n$ of the free B-module P'_0. Note that if we reduce $\widetilde{\beta}$ modulo T, the induced map takes $\overline{p}_i(0)$ to $\overline{f}_i(0)$ for $i = 1, \cdots, n$.

On the other hand we have that $e(Q, \chi) = (I(0), \omega_{I(0)})$ in $E(R)$ and there is a surjection $\alpha : Q \twoheadrightarrow I(0)$ such that (α, χ_0) induces the Euler class $(I(0), \omega_{I(0)})$. We choose a basis $q_1, \cdots, q_n$ of the free B-module $Q \otimes B$ such that $q_1 \wedge \cdots \wedge q_n = \chi \otimes B$ and we have

$$\overline{\alpha \otimes B} : \frac{(Q \otimes B)}{I(0)B(Q \otimes B)} \twoheadrightarrow \frac{I(0)B}{I(0)^2B}$$

where $\overline{q_i}$ goes to $\overline{f_i(0)}$, $i = 1, \cdots, n$.

Let us take the two free B-modules $Q \otimes B$ and P'_0, with bases $q_1, \cdots, q_n$ and $p^0_1, \cdots, p^0_n$, respectively. Let $\psi : P'_0 \xrightarrow{\sim} Q \otimes B$ be the isomorphism which sends p^0_i to q_i for $i = 1, \cdots, n$.

Consider the composite map $\varphi = (\alpha \otimes B)\psi : P'_0 \twoheadrightarrow I(0)B$. By the way this map is constructed, it follows that $\beta(0) \otimes B/I(0)B = \varphi \otimes B/I(0)B$. Therefore, we have a surjection $\theta : P' \twoheadrightarrow IB[T]/(I^2T)B[T]$ such that $\beta \otimes B[T]/IB[T] = \theta \otimes B[T]/IB[T]$ (i.e. β lifts θ modulo $I^2B[T]$).

Now we move to the ring $B(T)$, which is obtained from $B[T]$ by inverting all the monic polynomials. Here we have the induced map

$$\theta \otimes B(T) : P' \otimes B(T) \twoheadrightarrow IB(T)/I^2B(T),$$

of which $\beta \otimes B(T)$ is clearly a lift onto $IB(T)$. Now applying [D1, 4.8] we see that there exists a surjection $\eta : P' \twoheadrightarrow IB[T]$ such that η lifts θ. We further note that the map $\eta(0) : P'_0 \twoheadrightarrow I(0)B$ is precisely φ. □

We shall also need the following proposition to prove the main theorem. The proof of this proposition can be easily derived following the proof of [B1, Proposition 3.2]; it is essentially contained there. One crucial ingredient for the proof is [B1, Proposition 3.1], which asserts that if R is a semilocal ring of dimension one containing $\mathbb{Q}$, then Pic$(R[T])$ is a divisible group. This result is proved using the Milnor conductor diagram for Pic, details on which can be found in [B1].

Lemma 3.4. *Let $\widetilde{B}$ be a semilocal ring of dimension one containing $\mathbb{Q}$. Then, Pic$(\widetilde{B}[T])$ is a divisible group. Let M be an invertible ideal of $\widetilde{B}[T]$ such that dim$(\widetilde{B}[T]/M) = 0$. Let*

$\mathfrak{b} = M \cap \widetilde{B}$ and let $(0) = \mathfrak{b} \cap \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of $\widetilde{B}$ such that $M + \mathfrak{a}[T] = \widetilde{B}[T]$. Then, given any positive integer d, there exists an invertible ideal N of $\widetilde{B}[T]$ such that

(1) $N + M\mathfrak{a}[T] = \widetilde{B}[T]$;
(2) $N^d \cap M = (\widetilde{f})$, *for some non-zerodivisor* $\widetilde{f} \in \widetilde{B}[T]$;
(3) *dim*$(\widetilde{B}[T]/N) = 0$.

We now prove the main theorem.

Theorem 3.5. *Let R be an affine algebra of dimension $n \geq 2$ over a field k of characteristic zero. Let $(I, \omega_I) \in E(R[T])$. Let $\lambda \in k$ be such that ht $(I(\lambda)) \geq n$ and there exists a projective R-module Q of rank n and an isomorphism $\chi : R \overset{\sim}{\to} \wedge^n(Q)$ such that $e(Q, \chi) = (I(\lambda), \omega_{I(\lambda)})$ in $E(R)$. Then there is a projective $R[T]$-module P of rank n and an isomorphism $\chi_1 : R[T] \overset{\sim}{\to} \wedge^n(P)$ such that $e(P, \chi_1) = (I, \omega_I)$ in $E(R[T])$. Moreover, $P/TP \simeq Q$.*

Proof. If $n = 2$, then any $(I, \omega_I) \in E(R[T])$ is the Euler class of a projective module, without any hypothesis on $I(\lambda)$ (see[D1]). Therefore, in what follows, we assume $n \geq 3$. We can make the transformation $T \mapsto T - \lambda$ and replace $I(\lambda)$ by $I(0)$. By Remark 2.16 we may further assume that R is reduced. Since $\mathbb{Q} \subset R$, it follows that R is geometrically reduced (we shall need this in applying Swan's Bertini Theorem in Step1).

Step 1. We have $e(Q, \chi) = (I(0), \omega_{I(0)})$ in $E(R)$. Therefore, there is a surjection $\alpha : Q \twoheadrightarrow I(0)$ such that (α, χ) induces the Euler class $(I(0), \omega_{I(0)})$. Applying (3.2) we obtain a surjection $\theta : Q[T] \twoheadrightarrow I/(I^2T)$ such that $\theta(0) = \alpha$.

Let ω_I be given by: $I = (f_1, \cdots, f_n) + I^2$. Note that if $I(0) = R$, then we can take Q to be free and a simple argument shows that there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n) + (I^2T)$ where $g_i - f_i \in I^2$. The argument given in (3.2) is a generalization of this phenomenon. Keeping this remark in mind, in what follows we assume that ht $(I(0)) = n$ and work with the set up as obtained in the last paragraph.

Let $J = I \cap R$. By (3.1) there exists a non-zerodivisor $b \in J^2$ such that Q_{1+b} is a free R_{1+b}-module of rank n.

Recall from above that we have a surjection $\theta : Q[T] \twoheadrightarrow I/(I^2T)$. As R is geometrically reduced, applying (2.5, 2.6), it is easy to see that there is a lift $\gamma \in \text{Hom}(Q[T], R[T])$ of θ such that

(1) $I = I'' + (bT)$, where $\gamma(Q[T]) = I''$;
(2) $I'' = I \cap I_1$;
(3) $I_1 + (bT) = R[T]$;
(4) ht $(I_1) = n$ and $R[T]/I_1$ is reduced.

From (1) and (2) we further have,

(5) $e(Q[T], \chi \otimes R[T]) = (I, \omega_I) + (I_1, \omega_{I_1})$ in $E(R[T])$, where ω_{I_1} is induced by (γ, χ).

Step 2. Let $B = R_{1+bR}$. We first note that if $I_1B[T] = B[T]$, then we have $\gamma \otimes B[T] : Q \otimes B[T] \twoheadrightarrow IB[T]$ which lifts $\theta \otimes B[T]$ and we are done by [D1, 3.8]. Therefore we assume that $I_1B[T]$ is a proper ideal. As $I_1B[T] + bB[T] = B[T]$ and bB is contained in the Jacobson radical of B, we conclude that $I_1B[T]$ is intersection of finitely many maximal ideals of height n in $B[T]$. Hence if $K = B \cap I_1B[T]$ then K is a reduced ideal of height $n-1$ and $K + bB$ is an ideal of height n. Moreover if $\mathfrak{q}$ is a minimal prime ideal of K then $B_\mathfrak{q}$ is a regular local ring. Now the image of b belongs to the Jacobson radical of the one-dimensional reduced ring B/K and hence $(B/K)_b$ is a product of fields. Therefore, we can find $a_1, \cdots, a_{n-1} \in K$ such that $\operatorname{ht}(a_1, \cdots, a_{n-1}) = n-1$ and $\operatorname{ht}(a_1, \cdots, a_{n-1}, b) = n$. Further, note that $K_b = (a_1, \cdots, a_{n-1})_b + K_b^2$ and therefore, $K_\mathfrak{q} = (a_1, \cdots, a_{n-1})_\mathfrak{q}$ for all minimal prime ideals $\mathfrak{q}$ of K. Let $(a_1, \cdots, a_{n-1}) = K \cap K_1$ be a reduced primary decomposition .

Step 3. Let $\widetilde{B} = B/(a_1, \cdots, a_{n-1})$. Then $\widetilde{B}$ is semilocal of dimension one and in $\widetilde{B}$, we have $(0) = \widetilde{K} \cap \widetilde{b}\widetilde{K}_1$. Moreover, $\widetilde{I}_1$ is an invertible ideal such that $\widetilde{I}_1 + \widetilde{b}\,\widetilde{K}_1[T] = \widetilde{B}[T]$. Now note that $\widetilde{B}$ is a subring of $\widetilde{B}/\widetilde{K} \oplus \widetilde{B}/\widetilde{b}\,\widetilde{K}_1$ with the conductor ideal $\widetilde{K} + \widetilde{b}\,\widetilde{K}_1$.

Applying Lemma 3.4 we can find an invertible ideal N of $\widetilde{B}[T]$ such that

(a) $N + \widetilde{I}_1\,\widetilde{b}\,\widetilde{K}_1[T] = \widetilde{B}[T]$;
(b) $N^d \cap \widetilde{I}_1\widetilde{B}[T] = (\widetilde{f})$ for some non-zerodivisor $\widetilde{f} \in \widetilde{B}[T]$;
(c) $\dim(\widetilde{B}[T]/N) = 0$.

Since in $\widetilde{B}$ we have $\widetilde{K}.\widetilde{K}_1 = \widetilde{(0)}$ and $N + \widetilde{b}\,\widetilde{K}_1[T] = \widetilde{B}[T]$, it is easy to check that any maximal ideal of $\widetilde{B}[T]$ containing N must contain $\widetilde{K}[T]$.

Let I_2 be the inverse image of N in $B[T]$. Let M be a maximal ideal of $B[T]$ containing I_2. Then $K \subset M \cap B$. Now $M \cap B = \mathfrak{q}$ is a prime ideal of B of height $n-1$ containing K and hence it is minimal over K. Therefore $B_\mathfrak{q}$ is a regular local ring and consequently $B[T]_M$ is also regular. This shows that the ideal I_2 has finite projective dimension and it is locally generated by a regular sequence of length n. Moreover $I_1 \cap I_2^{(d)}$ is a complete intersection of height n. As $I_1 \cap I_2^{(d)} = (a_1, \cdots, a_{n-1}, f)$, it follows that both I_1 and $I_2^{(d)}$ are independent of the local orientations (this is implicit in the proof of Proposition 3.12 of [D3]). Therefore, in particular, if we denote the local orientation of $I_2^{(d)}$ induced by $(a_1, \cdots, a_{n-1}, f)$ as ω, we have

(6) $(I_1, \omega_{I_1}) + (I_2^{(d)}, \omega) = 0$ in $E(B[T])$.

By a result of Murthy [Mu, 2.2], there exists a projective $B[T]$-module P' of rank n with trivial determinant such that the following hold:

(7) $[P'] - [B[T]^n] = -[B[T]/I_2]$ in $K_0(B[T])$, and
(8) There is a surjection $\delta : P' \twoheadrightarrow I_2^{(d)}$.

Choose any isomorphism $\chi' : B[T] \simeq \wedge^n(P')$ and observe that $e(P', \chi') = (I_2^{(d)}, \omega)$. Since $QB[T]$ is free, from (5) we have $(I, \omega_I) + (I_1, \omega_{I_1}) = 0$ in $E(B[T])$. Therefore, $e(P', \chi') = (I, \omega_I)$ in $E(B[T])$. By [D1, 4.10], there exists a surjection $\beta : P' \twoheadrightarrow IB[T]$ such that (β, χ') induces $e(P', \chi') = (I, \omega_I)$ in $E(B[T])$.

Since I_2 is comaximal with $bB[T]$ and b belongs to the Jacobson radical of B, we have $I_2 + (T) = B[T]$. In other words, $I_2(0) = B$. Therefore, specializing equation (7) above at $T = 0$ we obtain $[P'/TP'] - [B^n] = 0$ in $K_0(B)$, implying that P'/TP' is a stably free B-module. As the Jacobson radical of B has height at least one, we further conclude that P'/TP' is free.

As $\dim(B_b) = \dim R_{b(1+bR)} \leq n - 1$, the ideal I_{2b} is a complete intersection. So we have, $[P'_b] - [B_b[T]^n] = -[B_b[T]/I_{2b}] = 0$ in $K_0(B_b[T])$. Therefore P'_b is stably free $B_b[T]$-module of rank n and as $\dim(B_b) \leq n - 1$, it is actually free.

Step 4. Applying (3.3) we obtain an isomorphism $\psi : P'/TP' \xrightarrow{\sim} Q \otimes B$ and a surjection $\eta : P' \twoheadrightarrow IB[T]$ such that $\eta(0) = \alpha\psi$.

Now consider the two surjections : (1) $\alpha_b(T) : Q_b[T] \twoheadrightarrow I_b(= I(0)_b[T])$, induced by α and (2) $\eta : P' \twoheadrightarrow IR_{1+bR}[T]$. Over the ring $R_{b(1+bR)}[T]$ we have

$$\alpha(T)_{b(1+bR)} : Q_{b(1+bR)}[T] \twoheadrightarrow I_{b(1+bR)}(= R_{b(1+bR)}[T]),$$

$$\eta_b : P'_b \twoheadrightarrow I_{b(1+bR)}(= R_{b(1+bR)}[T])$$

Now recall that both $Q_{b(1+bR)}[T], P'_b$ are free $R_{b(1+bR)}[T]$-modules. Let us write $K_1 = \ker(\alpha(T)_{b(1+bR)})$ and $K_2 = \ker(\eta_b)$. Since $\dim R_{b(1+bR)} \leq n - 1$ and $\mathbb{Q} \subset R$, by [R, 2.5] K_1, K_2 are both locally free and therefore, by Quillen's local-global principle [Q], they are extended from $R_{b(1+bR)}$. Further, reducing modulo T we observe that $\alpha_{b(1+bR)}\, \psi_b = \eta_b(0)$. This implies that K_1 and K_2 are isomorphic and there is an isomorphism $\Psi : P'_b \simeq Q_{b(1+bR)}[T]$ such that $\Psi(0) = \psi_b$. By a standard patching argument the result follows. $\square$

As mentioned in the introduction, we have essentially proved the following result, which is apparent from the proof of Theorem 3.5 above. The reader may note that at the beginning of Step 1 in the proof of (3.5), the hypothesis of the following theorem is achieved.

Theorem 3.6. *Let A be an affine domain of dimension $n \geq 3$ over a field k of characteristic zero. Let $I \subset A[T]$ be a local complete intersection ideal of height n such that $\mu(I/I^2) = n$. Assume that $I(0)$ is of height n. If there exists a projective A-module Q of rank n with trivial determinant and a surjection from $Q[T]$ to $I/I^2 \cap (T)$, then I is projectively generated.*

When the dimension of the ring is even, we have the following stronger conclusion. We also give an example (3.9) to show that a similar result as below does not hold when the dimension is odd.

Corollary 3.7. *Let R be an affine algebra of dimension n over a field k of characteristic zero, where n is even. Let $I \subset R[T]$ be an ideal such that I/I^2 is generated by n elements. Let $\lambda \in k$ be such that ht $(I(\lambda)) \geq n$ and there exists a projective R-module Q of rank n (with trivial determinant) such that Q maps onto $I(\lambda)$. Then there exists a projective $R[T]$-module P of rank n (with trivial determinant) such that P maps onto I.*

Proof. Without loss of generality, as in the theorem, we may assume that $\lambda = 0$.

Let us choose a set of generators of I/I^2, say, $I = (f_1, \cdots, f_n) + I^2$ and denote the corresponding local orientation by ω. Therefore (I, ω) is an element of the Euler class group $E(R[T])$. If $I(0) = R$, by [BRS1, 3.9], ω can be lifted to a surjection $R[T]^n \twoheadrightarrow I/(I^2T)$. The result then follows from the above theorem by taking Q to be free.

Now assume that $I(0)$ is proper. Then $(I(0), \omega(0)) \in E(R)$, where $\omega(0)$ is induced by ω. It is given that there is a surjection $\alpha : Q \twoheadrightarrow I(0)$ where Q is a projective R-module of rank n with trivial determinant. Fix $\chi : R \simeq \wedge^n(Q)$. Then (α, χ) will induce a local orientation, say, σ of $I(0)$ and we have $e(Q, \chi) = (I(0), \sigma)$ in $E(R)$. By [B2, 2.2] there exists a unit $a \in (R/I(0))^*$ such that $a\sigma = \omega(0)$ (by this expression we mean that $\omega(0)$ is the composition of σ with a matrix in $GL_n(R/I(0))$ whose determinant is a). Now, by [BRS4, 5.1], there exists a projective R-module Q' where $Q \oplus R \xrightarrow{\sim} Q' \oplus R$ and an isomorphism $\chi' : R \simeq \wedge^n(Q')$ such that $e(Q', \chi') = (I(0), a^{n-1}\sigma)$ in $E(R)$. As n is even, we have $(I(0), a^{n-1}\sigma) = (I(0), a\sigma)$ in $E(R)$ by [BRS4, Lemma 5.4]. Therefore, $e(Q', \chi') = (I(0), \omega(0))$ in $E(R)$. Now we can apply the above theorem. $\square$

Remark 3.8. In the above corollary P/TP is stably isomorphic to Q.

The following example is based on [BRS1, 5.2], [BRS3, 5.9] and we request the reader to look at these papers for the details. The ideal J_1 below is the same as the one considered in [G, p. 39].

Example 3.9. Let $R = \frac{\mathbb{R}[X_0, X_1, X_2, X_3]}{(X_0^2 + X_1^2 + X_2^2 + X_3^2 - 1)} = \mathbb{R}[x_0, x_1, x_2, x_3]$, the coordinate ring of the real 3-sphere. Consider the following ideals in R

$$J_0 = (x_1, x_2, x_3), \; J_1 = (x_0x_1, x_2(1 - x_2) - x_0^2, x_1(1 - x_2), x_3).$$

Then J_0 is a reduced ideal of height 3 in R and J_1 is an $\mathfrak{m}$-primary ideal, where $\mathfrak{m} = (x_0, x_1, x_2 - 1, x_3)$. Moreover, J_1 is a local complete intersection which is not generated by 3 elements.

Now let $\widetilde{R} = \frac{\mathbb{R}[X_0,X_1,X_2,X_3,X_4]}{(X_0^2+X_1^2+X_2^2+X_3^2+X_4-1)} = \mathbb{R}[x_0,x_1,x_2,x_3,x_4]$. Consider

$$\widetilde{J}_0 = (x_1,x_2,x_3,x_4),\ \widetilde{J}_1 = (x_0x_1, x_2(1-x_2)-x_0^2, x_1(1-x_2), x_3, x_4).$$

Then $\widetilde{R}/(x_4) = R$, $\widetilde{J}_0/(x_4) = J_0$ and $\widetilde{J}_1/(x_4) = J_1$. It has been proved in [BRS1, 5.2] that $\widetilde{J}_1$ is not generated by 4 elements.

Let $S = \frac{\mathbb{R}[X_0,X_1,X_2]}{(X_0^2+X_1^2+X_2^2-1)} = \mathbb{R}[x_0,x_1,x_2]$. Then we have $S = R/(x_3) = \widetilde{R}/(x_3,x_4)$. Let $K_0 = (x_1,x_2) = \widetilde{J}_0/(x_3,x_4)$ and $K_1 = \widetilde{J}_1/(x_3,x_4)$. Then both K_0, K_1 are surjective images of the projective S-module $Q = S^3/(x_0e_0+x_1e_1+x_2e_2)$, where e_0,e_1,e_2 are standard basis elements of the free module S^3 (see [BRS1, 5.2]). Further, as proved in [BRS1, 5.2], there exists a prime ideal N of $S[T]$ of height two such that: (1) N is a surjective image of $Q[T]$ and (2) $N(0) = K_0$, $N(1) = K_1$. Note that $\mu(N/N^2) = 2$.

Let $\widetilde{I}$ be the inverse image of N under the surjection from $\widetilde{R}[T]$ to $S[T]$. Similarly, let I be the inverse image of N under the surjection from $R[T]$ to $S[T]$. Then $\widetilde{I}$ is a prime ideal of $\widetilde{R}[T]$ with $\text{ht}(\widetilde{I}) = 4$ such that $\widetilde{I}(0) = \widetilde{J}_0$ and $\widetilde{I}(1) = \widetilde{J}_1$. Now, from [BRS1, 5.2] we know that $\widetilde{J}_1$ is not generated by four elements. Since $\widetilde{I}(1) = \widetilde{J}_1$, it follows that $\widetilde{I}$ is not generated by four elements. It is easy to see that $I = \widetilde{I}/(x_4)$ and $I/(x_3) = N$. Therefore $I \subset R[T]$ is a prime ideal with $\text{ht}(I) = 3$ and $\mu(I/I^2) = 3$. But I is not generated by three elements. As projective $R[T]$-modules of rank 3 are free, it is equivalent to saying that I is not projectively generated. Note that $I(0) = J_0 = (x_1,x_2,x_3)$. $\square$

For general n we have the following corollaries.

Corollary 3.10. *Let R be an affine algebra of dimension n over a C_1 field k of characteristic zero. Let $I \subset R[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Let $\lambda \in k$ be such that ht $(I(\lambda)) \geq n$ and there exists a projective R-module Q of rank n (with trivial determinant) such that Q maps onto $I(\lambda)$. Then there exists a projective $R[T]$-module P of rank n with trivial determinant and a surjection $\phi : P \twoheadrightarrow I$.*

Proof. There exists $\lambda \in k$ such that ht $(I(\lambda)) \geq n$. We can make the transformation $T \mapsto T-\lambda$ and assume that ht $(I(0)) \geq n$.

Let us choose a set of generators of I/I^2, say, $I = (f_1,\cdots,f_n) + I^2$ and denote the corresponding local orientation by ω. Therefore (I,ω) is an element of the Euler class group $E(R[T])$. If $I(0) = R$, by [BRS1, 3.9], ω can be lifted to a surjection $R[T]^n \twoheadrightarrow I/(I^2T)$. The result then follows from the above theorem by taking Q to be free.

Now assume that $I(0)$ is proper. Then $(I(0),\omega(0)) \in E(R)$, where $\omega(0)$ is induced by ω. By assumption, there exists a projective R-module Q of rank n with trivial determinant such that there is a surjection $\alpha : Q \twoheadrightarrow I(0)$. Let $\chi : R \xrightarrow{\sim} \wedge^n(Q)$ be an isomorphism. Now the weak Euler class $e(Q)$ is $(I(0))$ in $E_0(R)$. As $E(R) \simeq E_0(R)$

(see [D2, 5.2] for a proof), it follows that the Euler class of Q induced by (α, χ) is $e(Q, \chi) = (I(0), \omega(0))$ in $E(R)$. The result now follows from the above theorem. $\square$

The following result was proved in [BRS2, 2.7].

Corollary 3.11. *Let R be an affine algebra of dimension n over an algebraically closed field k of characteristic zero. Let $I \subset R[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Then there exists a projective $R[T]$-module P of rank n with trivial determinant and a surjection $\phi : P \twoheadrightarrow I$.*

Proof. Without loss of generality we take $\lambda = 0$ and we further assume that $I(0)$ is an ideal of height n (the case $I(0) = R$ being clear).

By [D1, Proof of 5.8], there exists a projective R-module Q of rank n such that the determinant of Q is trivial and there is a surjection $\alpha : Q \twoheadrightarrow I(0)$. The result now follows from Corollary 3.10. $\square$

Corollary 3.12. *Let k be a field of characteristic zero and R be a k-algebra which is essentially of finite type over k and is semilocal of dimension n. Let $I \subset R[T]$ be any ideal of height n such that I/I^2 is generated by n elements. Then there is a projective $R[T]$-module P of rank n with trivial determinant and a surjection $\phi : P \twoheadrightarrow I$.*

Proof. The proof is similar to the above corollaries and therefore we just give a sketch. If $I(0) = R$, then we are done. In the case when $I(0)$ is proper, note that as R is semilocal, any set of generators of $I(0)/I(0)^2$ can be lifted to a set of generators of $I(0)$. This means that any $\omega : R[T]^n \twoheadrightarrow I/I^2$ can be lifted to $\theta : R[T]^n \twoheadrightarrow I/(I^2T)$. Therefore, applying the main theorem, the result follows. $\square$

Let R be a commutative Noetherian ring of dimension $n \geq 2$, containing $\mathbb{Q}$. Let us now mention one of the most intriguing open problems in the Euler class theory. Let $\mathcal{P}roj_n(R)$ denote the set of projective R-modules of rank n. Consider the following subset of $E(R)$

$$H = \{(J, \omega_J) \in E(R) |\, \exists P \in \mathcal{P}roj_n(R), \chi : R \xrightarrow{\sim} \wedge^n P \text{ with } e(P, \chi) = (J, \omega_J)\},$$

i.e., H is the set of all those elements of $E(R)$ which occur as the Euler class of some projective module. The following question is then natural.

Question 3.13. Is H a subgroup of $E(R)$?

If $n = 2$, then it can be deduced from [BRS1, 2.14] that $H = E(R)$. If R is an affine algebra over an algebraically closed field then also $H = E(R)$, showing that the answer is in the affirmative (see [D1, Proof of 5.8]). On the other hand, if R is the coordinate ring of the 3-dimensional real sphere, then H is the trivial subgroup. The question is open in general. In the special case when R is the coordinate ring of the n-dimensional real sphere, this question has been studied in [Ma-S].

Similarly we can consider an analogous subset of $E(R[T])$, which we call as K, and ask the same question. Obviously $H \subset K$ and there is an example [BRS1, 6.4] from where one can see that H may not be equal to K. Here we prove,

Proposition 3.14. *Let R be an affine algebra of dimension n over a field k of characteristic zero. Then, H is a subgroup of $E(R)$ if and only if K is a subgroup of $E(R[T])$.*

Proof. Let H be a subgroup of $E(R)$. Let $(I_1, \omega_1), (I_2, \omega_2) \in K$ be two arbitrary elements. To prove that K is a subgroup of $E(R[T])$, we have to show that $(I_1, \omega_1) - (I_2, \omega_2) \in K$. By [D1, 6.2], there exists $I_3 \subset R[T]$ of height n and a local orientation ω_3 such that $I_3 + I_1 \cap I_2 = R[T]$ and $(I_2, \omega_2) + (I_3, \omega_3) = 0$. Let $I_4 = I_1 \cap I_3$ and ω_4 be the local orientation induced by ω_1 and ω_3. We then have

$$(I_4, \omega_4) = (I_1, \omega_1) + (I_3, \omega_3) = (I_1, \omega_1) - (I_2, \omega_2)$$

in $E(R[T])$. Since k is infinite, we can choose $\lambda \in k$ such that $\operatorname{ht} I_i(\lambda) \geq n$, for $i = 1, \cdots, 4$. Using the transform $T \mapsto T - \lambda$ we may assume that $\lambda = 0$.

There is a group homomorphism $\Psi : E(R[T]) \twoheadrightarrow E(R)$ which roughly works like specializing at $T = 0$ (see [D2] for details). It is easy to check that the elements $(I_1(0), \omega_1(0)), (I_2(0), \omega_2(0)) \in H$. Applying Ψ to the above equations we have

$$(I_4(0), \omega_4(0)) = (I_1(0), \omega_1(0)) + (I_3(0), \omega_3(0)) = (I_1(0), \omega_1(0)) - (I_2(0), \omega_2(0))$$

in $E(R)$. By hypothesis, H is a subgroup of $E(R)$ and hence,

$$(I_4(0), \omega_4(0)) = (I_1(0), \omega_1(0)) - (I_2(0), \omega_2(0)) \in H.$$

Therefore, there exists a projective R-module Q of rank n together with $\chi : R \simeq \wedge^n(Q)$ such that $e(Q, \chi) = (I_4(0), \omega_4(0))$. By the main theorem, there exists a projective $R[T]$-module P of rank n and an isomorphism $\chi_1 : R[T] \simeq \wedge^n(P)$ such that $e(P, \chi_1) = (I_4, \omega_4)$ in $E(R[T])$. Therefore, $(I_1, \omega_1) - (I_2, \omega_2) = (I_4, \omega_4) \in K$.

It is easy to see that the converse is obvious and is true even when R is a Noetherian ring containing $\mathbb{Q}$. $\square$

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