“$\mathbb{P}^1$-GLUING” FOR LOCAL COMPLETE INTERSECTIONS

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ABSTRACT. We prove an analogue of the Affine Horrocks' Theorem for local complete intersection ideals of height n in $R[T]$, where R is a regular domain of dimension d, which is essentially of finite type over an infinite perfect field of characteristic unequal to 2, and $2n \geq d+3$.

Dedicated to Professor S. M. Bhatwadekar on his seventieth birthday.

1. INTRODUCTION

Traditionally, many questions on the theory of algebraic vector bundles on affine varieties (or, more generally, projective modules over commutative Noetherian rings) have been motivated from topology. On the other hand, in nice situations, algebraic vector bundles are deeply connected with local complete intersection ideals of the associated coordinate ring of the variety. Therefore, results on algebraic vector bundles inspire one to ask analogous questions on these ideals. For example, the classical Quillen-Suslin Theorem [Q, Su] asserts that any algebraic vector bundle on the affine space $\mathbb{A}^n_k$ (k field) is trivial. The following conjecture of M. P. Murthy, in some sense, may be regarded as an analogue of the Quillen-Suslin Theorem. Recall that the notation $\mu(-)$ stands for the minimal number of generators.

Conjecture 1.1. [Mu] *Let k be a field and let $A = k[T_1, \cdots, T_d]$ be the polynomial ring. Let $n \in \mathbb{N}$ and $I \subset A$ be an ideal such that $ht(I) = n = \mu(I/I^2)$. Then $\mu(I) = n$.*

The conjecture is still open in general. See the appendix to this paper and [F2] for the clarification on the status of Murthy's conjecture in light of recent developments. The best known answer to this conjecture is due to Mohan Kumar [Mo 2, Theorem 5]:

Theorem 1.2. [Mo 2] *Let k be a field and let $A = k[T_1, \cdots, T_d]$. Let $I \subset A$ be an ideal such that $\mu(I/I^2) = n \geq dim(A/I) + 2$. Then $\mu(I) = n$.*

It is well-known that applying a change of variables one can assume that I contains a monic polynomial in T_d with coefficients from $R := k[T_1, \cdots, T_{d-1}]$. Mohan Kumar actually proves:

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Theorem 1.3. [Mo 2] *Let R be a commutative Noetherian ring and $I \subset R[T]$ be an ideal containing a monic polynomial. Let $\mu(I/I^2) = n \geq dim(R[T]/I) + 2$. Then there is a projective $R[T]$-module P of rank n and a surjective $R[T]$-linear map $\alpha : P \twoheadrightarrow I$.*

Since projective $k[T_1, \cdots, T_d]$-modules are free by the Quillen-Suslin Theorem, Mohan Kumar obtains (1.2) as a corollary of (1.3). Later, Mandal improves (1.3) in [Ma 1], by showing that P can be taken to be free (under the same assumptions as in (1.3)), and therefore, $\mu(I) = n$. A closer inspection of Mandal's proof shows that he essentially proves the following:

Theorem 1.4. [Ma 1] *Let R be a commutative Noetherian ring and $I \subset R[T]$ be an ideal containing a monic polynomial. Let $I = (f_1, \cdots, f_n) + I^2$, where $n \geq dim(R[T]/I) + 2$. Then there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $g_i - f_i \in I^2$. In other words, the n generators of I/I^2 can be lifted to a set of n generators of I.*

We now recall one result on algebraic vector bundles which is closely associated to the Quillen-Suslin Theorem. It is the Affine Horrocks' Theorem [Q]: *Let R be any commutative ring and $\mathcal{E}$ be a vector bundle on $\mathbb{A}^1_R$. If $\mathcal{E}$ extends to a vector bundle on $\mathbb{P}^1_R$, then $\mathcal{E}$ is extended from Spec(R).* In particular, (in algebraic terms) we have:

Theorem 1.5. [Q] *Let R be a commutative ring and P be a projective $R[T]$-module. Assume that the projective $R(T)$-module $P \otimes_{R[T]} R(T)$ is free. Then P is a free $R[T]$-module. (Here $R(T)$ is the ring obtained from $R[T]$ by inverting all the monic polynomials).*

The above theorem may be termed as a *monic inversion principle* for (freeness of) projective modules. From around the time it was proved, the monic inversion principle became a recurrent theme in various allied areas. It has been glaringly missing from the area of complete intersections. Motivated by the preceding discussion on Murthy's conjecture, we therefore pose the following natural question.

Question 1.6. *Let R be a commutative Noetherian ring of dimension $d \geq 2$. Let $I \subset R[T]$ be an ideal of height n such that $\mu(I/I^2) = n$. Let $I = (f_1, \cdots, f_n) + I^2$ be given. Assume that $IR(T) = (G_1, \cdots, G_n)$ with $G_i - f_i \in I^2R(T)$. Then, do there exist $F_1, \cdots, F_n \in I$ such that $I = (F_1, \cdots, F_n)$ with $F_i - f_i \in I^2$?*

Remark 1.7. When $d = n = 2$, the answer to the above question is negative (see [BRS 1, Example 3.15]). However, in this case it follows from [D, Section 7] that $\mu(I) = 2$. When $d = n \geq 3$, the above question has an affirmative answer if either R is local, or if R is an affine domain over an algebraically closed field of characteristic zero [D, Proposition 5.8].

We settle Question 1.6 affirmatively in the following form (Theorem 5.11 in the text).

Theorem 1.8. *Let R be a regular domain of dimension d which is essentially of finite type over an infinite perfect field k of characteristic unequal to 2. Let $I \subset R[T]$ be an ideal of height n such that $\mu(I/I^2) = n$, where $2n \geq d+3$. Let $I = (f_1, \cdots, f_n) + I^2$ be given. Assume that $IR(T) = (G_1, \cdots, G_n)$ with $G_i - f_i \in I^2R(T)$. Then, there exist $F_1, \cdots, F_n \in I$ such that $I = (F_1, \cdots, F_n)$ with $F_i - f_i \in I^2$.*

As our methods involve quadratic forms and orthogonal groups, we require $2R = R$. We now spend a few words on the line of proof. We have $I = (f_1, \cdots, f_n) + I^2$. Let $\omega_I : (R[T]/I)^n \twoheadrightarrow I/I^2$ be the corresponding surjective map. Applying Nakayama Lemma we can find $h \in I^2$ such that $h - h^2 = f_1g_1 + \cdots + f_ng_n$, for some $g_1, \cdots, g_n \in R[T]$. Now consider the following pointed set

$$Q'_{2n}(R[T]) = \{(x_1, \cdots x_n, y_1, \cdots, y_n, z) \in R[T]^{2n+1} \mid \sum_{i=1}^{n} x_iy_i + z^2 = 1\}$$

with a base point $(0, \cdots, 0, 0, \cdots, 0, 1)$. Let $O_{2n+1}(R[T])$ be the orthogonal group preserving the quadratic form $X_1Y_1 + \cdots + X_nY_n + Z^2$. Let $EO_{2n+1}(R[T])$ be the elementary subgroup of $O_{2n+1}(R[T])$. To the pair (I, ω_I) we associate the $EO_{2n+1}(R[T])$-orbit of $v = (2f_1, \cdots, 2f_n, 2g_1, \cdots, 2g_n, 1-2h)$ in the orbit space $Q'_{2n}(R[T])/EO_{2n+1}(R[T])$. This association does not depend on the choices made for h and $g_1, \cdots, g_n$ above. Further, we show that, to lift ω_I to a surjection $\varphi : R[T]^n \twoheadrightarrow I$, it is enough to prove the following "quadratic" analogue of a monic inversion principle of Ravi Rao [Ra, Theorem 1.1]. See Theorem 5.8 below.

Theorem 1.9. *Let A be a regular domain containing a field k of characteristic $\neq 2$. Let $n \geq 2$ and $v = (\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \gamma) \in Q'_{2n}(A[T])$. Suppose that there is a monic polynomial $f \in A[T]$ such that $v \equiv (0, \cdots, 0, 1)$ mod $EO_{2n+1}(A[T]_f)$. Then $v \equiv (0, \cdots, 0, 1)$ mod $EO_{2n+1}(A[T])$.*

Our methods in this article do not extend to non-smooth set-up (see Example 4.10). On the other hand, lifting the restriction $2n \geq d+3$ (which is concurrent with Mohan Kumar's range $n \geq \dim(R[T]/I) + 2$) looks to be an enormous task because then it would give a complete solution to Murthy's conjecture, which seems elusive at the moment.

2. HOMOTOPY

In this section we recall the definitions of two pointed sets and their homotopy orbits from [F1]. In this article, by *'homotopy'* we shall mean *'naive homotopy'*, as defined below.

Definition 2.1. Let F be a functor originating from the category of rings to the category of sets. For a given ring R, two elements $F(u_0), F(u_1) \in F(R)$ are said to be homotopic if there is an element $F(u(T)) \in F(R[T])$ such that $F(u(0)) = F(u_0)$ and $F(u(1)) = F(u_1)$.

Definition 2.2. Let F be a functor from the category of rings to the category of sets. Let R be a ring. Consider the equivalence relation on $F(R)$ *generated by* homotopies (the relation is easily seen to be reflexive and symmetric but is not transitive in general). The set of equivalence classes will be denoted by $\pi_0(F(R))$.

Example 2.3. Let R be a ring. Two matrices $\sigma, \tau \in GL_n(R)$ are *homotopic* if there is a matrix $\theta(T) \in GL_n(R[T])$ such that $\theta(0) = \sigma$ and $\theta(1) = \tau$. Of particular interest are the matrices in $GL_n(R)$ which are *homotopic to identity*.

Definition 2.4. Recall that $E_n(R)$ is the subgroup of $GL_n(R)$ generated by all elementary matrices $E_{ij}(\lambda_{ij})$ (whose diagonal entries are all 1, $i \neq j$, and ij-th entry is $\lambda_{ij} \in R$).

Remark 2.5. Any $\theta \in E_n(R)$ is homotopic to identity. To see this, let $\theta = \prod E_{ij}(\lambda_{ij})$. Define $\Theta(T) := \prod E_{ij}(T\lambda_{ij})$. Then, clearly $\Theta(T) \in E_n(R[T])$ and we observe that $\Theta(1) = \theta$, $\Theta(0) = I_n$.

2.1. **The pointed set $Q'_{2n}(R)$ and its homotopy orbits:** Let R be any commutative Noetherian ring where 2 is invertible. Let $n \geq 2$ and consider the set

$$Q'_{2n}(R) = \{(x_1, \cdots x_n, y_1, \cdots, y_n, z) \in R^{2n+1} \mid \sum_{i=1}^{n} x_i y_i + z^2 = 1\}$$

with a base point $(0, \cdots, 0, 0, \cdots, 0, 1)$. Let $O_{2n+1}(R)$ be the group of orthogonal matrices preserving the quadratic form $\sum_{i=1}^{n} X_i Y_i + Z^2$. Then there is a natural action of $O_{2n+1}(R)$ and its subgroup $SO_{2n+1}(R)$ on the set $Q'_{2n}(R)$. Let $EO_{2n+1}(R)$ be the elementary subgroup of $SO_{2n+1}(R)$. As $n \geq 2$, the subgroup $EO_{2n+1}(R)$ is normal in $SO_{2n+1}(R)$ (see [VP, Lemma 4]). Indeed, the group $EO_{2n+1}(R)$ also acts on the set $Q'_{2n}(R)$.

Theorem 2.6. *Let R be a regular ring containing a field k with Char$(k) \neq 2$. Then, for any $n \geq 2$ there is a bijection*

$$\pi_0(Q'_{2n}(R)) \xrightarrow{\sim} Q'_{2n}(R)/EO_{2n+1}(R).$$

Proof. As R is regular and contains a field, by [MaMi, Theorem 4.2], for any $f(T) \in Q'_{2n}(R[T])$, there exists $\tau(T) \in O_{2n+1}(R[T])$ such that $\tau(0) = I_{2n+1}$ and $f(T)\tau(T) = f(0)$. Applying [St, Theorem 1.3] we further note that $\tau(T) \in EO_{2n+1}(R[T])$. □

The following corollary is now obvious.

Corollary 2.7. *Let R be a regular ring containing a field k with Char$(k) \neq 2$. Then, for $n \geq 2$, the relation induced by homotopy on $Q'_{2n}(R)$ is an equivalence relation.*

2.2. **The pointed set $Q_{2n}(R)$ and its homotopy orbits:** Let R be any commutative Noetherian ring where 2 is invertible. Let $n \geq 2$ and consider the set

$$Q_{2n}(R) = \{(x_1, \cdots x_n, y_1, \cdots, y_n, z) \in R^{2n+1} \mid \sum_{i=1}^{n} x_i y_i = z - z^2\}$$

with a base point $(0, \cdots, 0, 0, \cdots, 0, 0)$. It is proved in [F1] that there is a bijection $\beta_n : Q_{2n}(R) \to Q'_{2n}(R)$ and its inverse $\alpha_n : Q'_{2n}(R) \to Q_{2n}(R)$ given by

- $\beta_n(x_1, \cdots x_n, y_1, \cdots, y_n, z) = (2x_1, \cdots 2x_n, 2y_1, \cdots, 2y_n, 1 - 2z)$
- $\alpha_n(x_1, \cdots x_n, y_1, \cdots, y_n, z) = \frac{1}{2}(x_1, \cdots x_n, y_1, \cdots, y_n, 1 - z)$

Note that both α_n and β_n preserve the base points of the respective sets. They induce bijections between the sets $\pi_0(Q_{2n}(R))$ and $\pi_0(Q'_{2n}(R))$ (will use the same notations). By transporting the action of $EO_{2n+1}(R)$ on $Q'_{2n}(R)$ through the bijections given above one sees that $EO_{2n+1}(R)$ also acts on $Q_{2n}(R)$ in the following way:

$$Mv := \alpha_n((\beta_n(v))M),$$

for $v \in Q_{2n}(R)$ and $M \in EO_{2n+1}(R)$. Further, note that the bijections α_n, β_n induce bijections between the sets $\pi_0(Q_{2n}(R))$ and $\pi_0(Q'_{2n}(R))$. Combining these with Theorem 2.6, one obtains the following result.

Theorem 2.8. *Let R be a regular ring containing a field k with Char$(k) \neq 2$. Then, for any $n \geq 2$ there is a bijection*

$$\pi_0(Q_{2n}(R)) \xrightarrow{\sim} Q_{2n}(R)/EO_{2n+1}(R).$$

We shall require the following corollary later.

Corollary 2.9. *Let R be a regular ring containing a field k with Char$(k) \neq 2$. Then, for $n \geq 2$, the relation induced by homotopy on $Q_{2n}(R)$ is an equivalence relation.*

3. The Euler class groups and homotopy

Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. In this section we recollect definitions of the Euler class groups from [BRS 1]. However, our emphasis will also be on the definition of the Euler class group given by M. V. Nori in terms of homotopy (as appeared in [BRS 1]). In [DK], the first named author and Manoj Keshari investigated in detail the relation between these two equivalent definitions and their consequences. We reproduce some of those results at one place for the convenience of the reader.

Let R be as above. Let B be the set of pairs (m, ω_m) where m is a maximal ideal of R and $\omega_m : (R/m)^d \twoheadrightarrow m/m^2$. Let G be the free abelian group generated by B. Let

$J = m_1 \cap \cdots \cap m_r$, where m_i are maximal ideals of R. Any $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ induces surjections $\omega_i : (R/m_i)^d \twoheadrightarrow m_i/m_i^2$ for each i. We associate $(J, \omega_J) := \sum_1^r (m_i, \omega_i) \in G$. Now,

Definition 3.1. (Nori) Let S be the set of elements $(I(1), \omega(1)) - (I(0), \omega(0))$ of G where (i) $I \subset R[T]$ is a local complete intersection ideal of height d; (ii) Both $I(0)$ and $I(1)$ are reduced ideals of height d; (iii) $\omega(0)$ and $\omega(1)$ are induced by $\omega : (R[T]/I)^d \twoheadrightarrow I/I^2$. Let H be the subgroup generated by S. The d-th Euler class group $E^d(R)$ is defined as $E^d(R) := G/H$.

Definition 3.2. (Bhatwadekar-Sridharan) Let S_1 be the set of elements (J, ω_J) of G for which ω_J has a lift to a surjection $\theta : R^d \twoheadrightarrow J$ and H_1 be the subgroup of G generated by S_1 . The Euler class group $E^d(R)$ is defined as $E^d(R) := G/H_1$.

Remark 3.3. We shall refer to the elements of the Euler class group as *Euler cycles*.

Remark 3.4. The above definitions appear to be slightly different than the ones given in [BRS 1]. However, note that if $(J, \omega_J) \in S$ (resp. S_1) and if $\overline{\sigma} \in E_d(R/J)$, then the element $(J, \omega_J \overline{\sigma})$ is also in S (resp. S_1). For details, see [DK, Remark 5.4] and [DZ, Proposition 2.2].

Remark 3.5. Bhatwadekar-Sridharan proved (see [BRS 1, Remark 4.6]) that $H = H_1$ and therefore the above definitions of the Euler class group are equivalent.

The following theorem collects a few results at one place (see [BRS 1, 4.11], [K, 4.2], [DK, Theorem 5.13] for details).

Theorem 3.6. *Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. Let $J \subset R$ be a reduced ideal of height d and $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ be a surjection. Then, the following are equivalent:*

(1) *The image of $(J, \omega_J) = 0$ in $E^d(R)$*
(2) *ω_J can be lifted to a surjection $\theta : R^d \twoheadrightarrow J$.*
(3) *$(J, \omega_J) = (I(0), \omega(0)) - (I(1), \omega(1))$ in G where (i) $I \subset R[T]$ is a local complete intersection ideal of height n; (ii) Both $I(0)$ and $I(1)$ are reduced ideals of height n, and (iii) $\omega(0)$ and $\omega(1)$ are induced by $\omega : (R[T]/I)^n \twoheadrightarrow I/I^2$.*

A series of remarks are in order.

Remark 3.7. Let $(J, \omega_J), (J', \omega_{J'})$ be such that both J, J' are reduced ideals and they represent the same element in $E^d(R)$. In other words, $(J, \omega_J) - (J', \omega_{J'}) \in H_1$ (where H_1 is as in (3.2). It is easy to see from the proof of [K, 4.1] that $(J, \omega_J) - (J', \omega_{J'})$ actually belongs to S_1 (where S_1 is as in (3.2)). This observation will be crucially used in the proof of Proposition 4.2.

Remark 3.8. Let $J \subset R$ be an ideal of height d which is not necessarily reduced and let $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ be a surjection. Then also one can associate an element (J, ω_J) in $E^d(R)$ and prove the above theorem for (J, ω_J). See [BRS 1, Remark 4.16] for details.

Remark 3.9. All the definitions and results in this section can be easily extended to the case when R is a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k. The key result on which this entire section depends (including the equivalence of the two definitions of $E^d(R)$) is the *"homotopy theorem"* of Bhatwadekar-Sridharan [BRS 1, Theorem 3.8]. This homotopy theorem, which was proved for smooth affine domains over an infinite perfect field, has an obvious extension to regular domains essentially of finite type over such fields.

Remark 3.10. All the definitions and results in this section can also be extended to a much more relaxed range. Let R be a regular domain which is essentially of finite type over an infinite perfect field k. Let n be an integer such that $2n \geq d + 3$. Then the n-th Euler class group $E^n(R)$ has been defined in [BRS 4]. To extend the results of this section one has to use [BK, Theorem 4.13], which is a generalization of the homotopy theorem of Bhatwadekar-Sridharan mentioned above. We are not working out the details here as the process is routine.

4. A BIJECTION

Let R be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k. Recall from Section 2 that we have a set

$$Q_{2d}(R) := \{(x_1, \cdots x_d, y_1, \cdots, y_d, z) \in R^{2d+1} \mid \sum x_i y_i = z - z^2\}$$

Recall further that $\pi_0(Q_{2d}(R))$ denotes the set of equivalence classes of $Q_{2d}(R)$ with respect to the equivalence relation generated by homotopies. The homotopy orbit of $v = (x_1, \cdots x_d, y_1, \cdots, y_d, z) \in Q_{2d}(R)$ will be denoted by $[v] = [(x_1, \cdots x_d, y_1, \cdots, y_d, z)]$.

4.1. **A set-theoretic map.** We first define a set-theoretic map from the Euler class group $E^d(R)$ to $\pi_0(Q_{2d}(R))$. By [BRS 1, Remark 4.14] we know that an arbitrary element of $E^d(R)$ can be represented by a single Euler cycle (J, ω_J), where J is a reduced ideal of height d. Now $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ is given by $J = (a_1, \cdots, a_d) + J^2$, for some $a_1, \cdots, a_d \in J$. Applying the Nakayama Lemma one obtains $s \in J^2$ such that $J = (a_1, \cdots, a_d, s)$ with $s - s^2 = a_1 b_1 + \cdots + a_d b_d$ for some $b_1, \cdots, b_d \in R$ (see [Mo 1] for a proof). We associate to (J, ω_J) the homotopy class $[(a_1, \cdots, a_d, b_1, \cdots, b_d, s)]$ in $\pi_0(Q_{2d}(R))$.

Remark 4.1. For an ideal I with a given set of generators of I/I^2, the idea of associating the whole data (as above) to an affine quadric, goes back to Mohan Kumar and Nori (to the best of our knowledge). We refer to [Sw, Section 17] for an excellent exposition of their innovative method. The novelty of the approach in [F1] is to associate the information upto naive homotopy, which we borrow in this article.

The following proposition has been proved in [AF, MaMi]. However, our arguments are quite different.

Proposition 4.2. *Let R be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k. The association $(J, \omega_J) \mapsto [(a_1, \cdots, a_d, b_1, \cdots, b_d, s)]$ is well defined and gives rise to a set-theoretic map $\theta_d : E^d(R) \to \pi_0(Q_{2d}(R))$. The map θ_d takes the trivial Euler cycle to the homotopy orbit of the base point $(0, \cdots, 0)$ of $Q_{2d}(R)$.*

Proof. We need to check the following:

(1) If ω_J is also given by $J = (\alpha_1, \cdots, \alpha_d) + J^2$ and if $\tau \in J^2$ is such that $\tau - \tau^2 = \alpha_1\beta_1 + \cdots + \alpha_d\beta_d$, then $[(a_1, \cdots, a_d, b_1, \cdots, b_d, s)] = [(\alpha_1, \cdots, \alpha_d, \beta_1, \cdots, \beta_d, \tau)]$ in $\pi_0(Q_{2d}(R))$.
(2) If $\overline{\sigma} \in E_d(R/J)$, then the image of $(J, \omega_J\overline{\sigma})$ in $Q_{2d}(R)$ is homotopic to the image of (J, ω_J).
(3) If (J, ω_J) is also represented by $(J', \omega_{J'})$ in $E^d(R)$, then their images are homotopic in $Q_{2d}(R)$.

Proof of (1) : This has been proved in [F1, Theorem 2.0.2].
Proof of (2) : Suppose that (J, ω_J) is given by $J = (a_1, \cdots, a_d) + J^2$, and $s \in J^2$ be such that $J = (a_1, \cdots, a_d, s)$ with $s - s^2 = a_1b_1 + \cdots + a_db_d$ for some $b_1, \cdots, b_d \in R$. Let $\sigma \in E_d(R)$ be a lift of $\overline{\sigma}$ and write $(a_1, \cdots, a_d)\sigma = (\alpha_1, \cdots, \alpha_d)$. Then $J = (\alpha_1, \cdots, \alpha_d) + J^2$ and $J = (\alpha_1, \cdots, \alpha_d, s)$. If we write $(b_1, \cdots, b_d)(\sigma^{-1})^t = (\beta_1, \cdots, \beta_d)$ (here t stands for transpose), then it is easy to see that $s(1 - s) = \alpha_1\beta_1 + \cdots + \alpha_d\beta_d$. Now, note that

$$\lambda = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & (\sigma^{-1})^t & 0 \\ 0 & 0 & 1 \end{pmatrix} \in E_{2d+1}(R)$$

and $(a_1, \cdots, a_d, b_1, \cdots, b_d, s)\lambda = (\alpha_1, \cdots, \alpha_d, \beta_1, \cdots, \beta_d, s)$. Since elementary matrices are homotopic to identity (see Remark 2.5), we are done in this case.

Proof of (3) : Note that by Remark 3.7, $(J, \omega_J) - (J', \omega_{J'}) \in S_1$, where S_1 is as in Definition 3.2. Therefore, there exists $(K, \omega_K) \in S_1$ such that $(J, \omega_J) - (J', \omega_{J'}) = (K, \omega_K)$ in G. This implies that $(J, \omega_J) = (J', \omega_{J'}) + (K, \omega_K)$ in G and therefore, $J = J' \cap K$. Since J is reduced, we have $J' + K = R$. Assume that ω_J is induced by $J = (a_1, \cdots, a_d) + J^2$.

From the definition of the Euler class group above it is clear that $\omega_{J'}$ is induced by $J' = (a_1, \cdots, a_d) + J'^2$ and ω_K is induced by $K = (a_1, \cdots, a_d) + K^2$.

Now $(K, \omega_K) \in S_1$ and therefore, it is a complete intersection ideal and ω_K has a surjective lift $\alpha : R^d \twoheadrightarrow K$. This means that $K = (c_1, \cdots, c_d)$ such that $c_i - a_i \in K^2$. Because of (2) above, we are now free to apply elementary transformations. Applying elementary transformations on $(c_1, \cdots, c_d)$, if necessary, we may assume by [RS, Lemma 3] that $\text{ht}(c_1, \cdots, c_{d-1}) = d - 1$ and $J' + (c_1, \cdots, c_{d-1}) = R$ (Note that if we apply $\sigma \in E_d(R)$ on $(c_1, \cdots, c_d)$, we have to apply σ on $(a_1, \cdots, a_d)$ as well to retain the relations and equations). Consider the ideal $I' = (c_1, \cdots, c_{d-1}, (1 - c_d)T + c_d)$ in $R[T]$. Write $I = I' \cap J'[T]$. Using the Chinese Remainder Theorem we can then find $f_1, \cdots, f_d \in I$ such that:

(a) $I = (f_1, \cdots, f_d) + I^2$.
(b) $f_i = c_i \bmod I'^2$ for $i = 1, \cdots, d - 1$ and $f_d = (1 - c_d)T + c_d \bmod I'^2$.
(c) $f_i = a_i \bmod J'[T]^2$ for $i = 1, \cdots, d$.

Let $\omega : (R[T]/I)^d \twoheadrightarrow I/I^2$ be the surjection corresponding to $f_1, \cdots, f_d$. We then have, $I(0) = J$, $I(1) = J'$. From (b) we get $f_i(0) = c_i \bmod K^2$. Since we already have $c_i = a_i \bmod K^2$, it follows that $f_i(0) = a_i \bmod K^2$. On the other hand, from (c) we get $f_i(0) = a_i \bmod J'^2$. Combining, we have $f_i(0) = a_i \bmod J^2$. Also, from (c), we obtain $f_i(1) = a_i \bmod J'^2$.

The upshot of the above paragraph is that ω_J is induced by $J = (f_1(0), \cdots, f_d(0)) + J^2$, whereas ω'_J is induced by $J' = (f_1(1), \cdots, f_d(1)) + J'^2$.

As $I = (f_1, \cdots, f_d) + I^2$, we can find $h \in I^2$ and $g_1, \cdots, g_d \in R[T]$ such that $I = (f_1, \cdots, f_d, h)$ with $h - h^2 = f_1 g_1 + \cdots + f_d g_d$. Then $(f_1, \cdots, f_d, g_1, \cdots, g_d, h) \in Q_{2d}(R[T])$. It follows from the last paragraph and (1) that the images of $J = (a_1, \cdots, a_d) + J^2$ and $J = (f_1(0), \cdots, f_d(0)) + J^2$ are same in $\pi_0(Q_{2d}(R))$. Similarly, the images of $J' = (a_1, \cdots, a_d) + J'^2$ and $J' = (f_1(1), \cdots, f_d(1)) + J'^2$ are same in $\pi_0(Q_{2d}(R))$. Now, to complete the proof, it is enough to show that the images of $J = (f_1(0), \cdots, f_d(0)) + J^2$ and $J' = (f_1(1), \cdots, f_d(1)) + J'^2$ are homotopic. Note that $h(0) \in I(0)^2 = J^2$ with $h(0) - h(0)^2 = f_1(0)g_1(0) + \cdots + f_d(0)g_d(0)$. Similarly, we have $h(1) \in I(1)^2 = J'^2$ with $h(1) - h(1)^2 = f_1(1)g_1(1) + \cdots + f_d(1)g_d(1)$. Therefore it is easy to see that $(f_1, \cdots, f_d, g_1, \cdots, g_d, h) \in Q_{2d}(R[T])$ is the required homotopy. This concludes the proof. □

4.2. **The map θ_d is a bijection.** We now proceed to prove that the set-theoretic map $\theta_d : E^d(R) \to \pi_0(Q_{2d}(R))$ is a bijection. In order to do that, we need some preparatory lemmas. We prove a form of the so called "moving lemma" below (Lemma 4.5). This has been proved in [AF]. However, we reprove it here for the sake of completeness. Further, instead of using the *generalized dimension function* and the concept of *basic* *elements*, we use the *prime avoidance lemma*. Therefore the arguments here are probably more basic and easier to follow.

The following lemma is an easy application of the prime avoidance lemma (for a proof, see [IR, Lemma 7.1.4]).

Lemma 4.3. *Let A be a commutative Noetherian ring and $(a_1,\cdots,a_n,a)\in A^{n+1}$. Then there exist $\mu_1,\cdots,\mu_n\in A$ such that ht$(I_a)\geq n$, where $I=(a_1+a\mu_1,\cdots,a_n+a\mu_n)$. In other words, if $\mathfrak{p}\in$ Spec(A) such that $I\subset\mathfrak{p}$ and $a\notin\mathfrak{p}$, then ht$(\mathfrak{p})\geq n$.*

Remark 4.4. If A is a geometrically reduced affine algebra over an infinite field then Swan's version of Bertini theorem, as given in [BRS 2, Theorem 2.11], implies that $\mu_1,\cdots,\mu_n$ can be so chosen that the ideal $I=(a_1+a\mu_1,\cdots,a_n+a\mu_n)$ has the additional property that $(A/I)_a$ is a geometrically reduced ring.

Lemma 4.5. **(Moving Lemma)** *Let A be a commutative Noetherian ring. Let $(a,b,s)=(a_1,\cdots,a_n,b_1,\cdots,b_n,s)\in Q_{2n}(A)$. Then there exists $\mu=(\mu_1,\cdots,\mu_n)\in A^n$ such that*

(1) *The row $(a',b',s')=(a_1+\mu_1(1-s)^2,\cdots,a_n+\mu_n(1-s)^2,b_1+(1-\mu b^t),\cdots,b_n+(1-\mu b^t),s+\mu b^t(1-s))\in Q_{2n}(A)$,*
(2) *$[(a,b,s)]=[(a',b',s')]$ in $\pi_0(Q_{2n}(A))$ and*
(3) *ht$(K)\geq n$, where $K=(a_1+\mu_1(1-s)^2,\cdots,a_n+\mu_n(1-s)^2,s+\mu b^t(1-s))$.*

Proof. We consider the row $(a_1,\cdots,a_n,(1-s)^2)\in A^{n+1}$. By Lemma 4.3 there exist $\mu_1,\cdots,\mu_n\in A$ such that $\text{ht}(I_{(1-s)^2})\geq n$, where $I=(a_1+\mu_1(1-s)^2,\cdots,a_n+\mu_n(1-s)^2)$. In other words, if $\mathfrak{p}\in\text{Spec}(A)$ such that $I\subset\mathfrak{p}$ and $(1-s)\notin\mathfrak{p}$, then $\text{ht}(\mathfrak{p})\geq n$.

Set $A=a+T(1-s)^2\mu\in A[T]^n$, then an easy computation yields that

$$Ab^t(1-T\mu b^t)=(1-s)(1-T\mu b^t)-(1-s)^2(1-T\mu b^t)^2.$$

Setting $B=(1-T\mu b^t)b$, it is easy to check that $(A,B,(1-s)(1-T\mu b^t))\in Q_{2n}(A[T])$. Then it follows that $(A,B,1-(1-s)(1-T\mu b^t))=(A,B,s+T\mu b^t(1-s))\in Q_{2n}(A[T])$. Thus (1) and (2) are proved.

Now we have the following relations among the ideals:

$$\begin{aligned} I &= \big(a_1+\mu_1(1-s)^2,\cdots,a_n+\mu_n(1-s)^2\big)\\ &= \big(a+\mu(1-s)^2,(1-s)(1-\mu b^t)\big)\cap\big(a+\mu(1-s)^2,s+\mu b^t(1-s)\big)\\ &= \big(a+\mu(1-s)^2,(1-s)(1-\mu b^t)\big)\cap K\end{aligned}$$

Let $\mathfrak{p}\in\text{Spec}(A)$ such that $K\subset\mathfrak{p}$. As $s+\mu b^t(1-s)\in K\subset\mathfrak{p}$, it follows that $(1-s)(1-\mu b^t)\notin\mathfrak{p}$ and therefore, $1-s\notin\mathfrak{p}$. Note that $I\subset K\subset\mathfrak{p}$. Therefore, by the first paragraph, $\text{ht}(\mathfrak{p})\geq n$. This proves (3). $\square$

Remark 4.6. If A is a geometrically reduced affine algebra over an infinite perfect field then using Swan's Bertini theorem (see Remark 4.4 above), one can choose K to have the additional property that either $K = A$ or K is a reduced ideal.

We are now ready to prove the main theorem of this section. This theorem has also been proved independently in [AF, MaMi]. However, the proof given in [AF] is incomplete, as they are using [F1, Theorem 3.2.7] which is not valid (see the appendix to this paper). We must remark here that in [AF], the authors prove some remarkable results and settled a couple of open problems. If their other arguments are correct, then our proof validates their results. We have not verified the proof of [MaMi] in detail but one can easily see that our approach and arguments are entirely different. Further, our proof is much more straightforward.

Theorem 4.7. *Let R be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k. The set-theoretic map $\theta_d : E^d(R) \to \pi_0(Q_{2d}(R))$ is a bijection.*

Proof. Let $v = (a_1, \cdots, a_d, b_1, \cdots, b_d, s) \in Q_{2d}(R)$. Then the ideal $I(v) = (a_1, \cdots, a_d, s)$ of R need not be of height d. However, we may apply Lemma 4.5 to obtain $v' = (a'_1, \cdots, a'_d, b'_1, \cdots, b'_d, s')$ in the same homotopy class of v such that the ideal $J = (a'_1, \cdots, a'_d, s')$ has height $\geq d$ and is reduced. Then $J = (a'_1, \cdots, a'_d) + J^2$. If $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ is the corresponding map, then it follows that the image of (J, ω_J) under θ_d is $[v'] = [v]$ in $\pi_0(Q_{2d}(R))$. This shows that θ_d is surjective.

Rest of the proof is devoted to proving that θ_d is injective. Let (J, ω_J) and $(J', \omega_{J'})$ be elements of $E^d(R)$ be such that $\theta_d((J, \omega_J)) = \theta_d((J', \omega_{J'}))$. Let ω_J be given by $J = (a_1, \cdots, a_d) + J^2$. As $\text{ht}(J) = d$, applying Lemma 4.3 if necessary, we may assume that $\text{ht}(a_1, \cdots, a_d) = d$. Now there exists $s \in J^2$ such that $J = (a_1, \cdots, a_d, s)$ with $s - s^2 = a_1b_1 + \cdots + a_db_d$ for some $b_1, \cdots, b_d \in R$. Similarly, $\omega_{J'}$ is given by $J' = (a'_1, \cdots, a'_d) + J'^2$ with $\text{ht}(a'_1, \cdots, a'_d) = d$. There exists $s' \in J'^2$ be such that $J' = (a'_1, \cdots, a'_d, s')$ with $s' - s'^2 = a'_1b'_1 + \cdots + a'_db'_d$ for some $b'_1, \cdots, b'_d \in R$.

We now assume that

$$\theta((J, \omega_J)) = [(a_1, \cdots, a_d, b_1, \cdots, b_d, s)] = [(a'_1, \cdots, a'_d, b'_1, \cdots, b'_d, s')] = \theta((J', \omega_{J'}))$$

in $\pi_0(Q_{2d}(R))$. Applying Corollary 2.9 we have $V = (f_1, \cdots, f_d, g_1, \cdots, g_d, h) \in Q_{2d}(R[T])$ such that $V(0) = (a_1, \cdots, a_d, b_1, \cdots, b_d, s)$ and $V(1) = (a'_1, \cdots, a'_d, b'_1, \cdots, b'_d, s')$. If we consider the ideal $I = (f_1, \cdots, f_d, h)$ of $R[T]$ then we have $I = (f_1, \cdots, f_d) + I^2$. Let $\omega_I : (R[T]/I)^d \twoheadrightarrow I/I^2$ denote the corresponding surjection. However, the height of I need not be d, although both $I(0)$ $(= J)$ and $I(1)$ $(= J')$ have height d.

As both $\text{ht}((a_1, \cdots, a_d) = d = \text{ht}(a'_1, \cdots, a'_d)$, it follows that

$$\text{ht}(f_1, \cdots, f_d, T(T-1)) = d + 1 \qquad (*)$$

Consider $(f_1, \cdots, f_d, (T^2-T)h^2) \in R[T]^{d+1}$. By Lemma 4.3, there exist $\mu_1, \cdots, \mu_d \in R[T]$ such that $\text{ht}((F_1, \cdots, F_d)_{h^2(T^2-T)}) \geq d$, where $F_i = f_i + \mu_i h^2(T^2-T)$, for $i = 1, \cdots, d$. Note that we have $I = (F_1, \cdots, F_d) + (h)$, and $(h) \subset I^2$. Applying [BRS 3, 2.11], there exists $k \in (h)$ such that $I = (F_1, \cdots, F_d, k)$ where $k - k^2 \in (F_1, \cdots, F_d)$. We now take $K = (F_1, \cdots, F_d, 1-k)$ and write $\omega_K : (R[T]/K)^d \twoheadrightarrow K/K^2$ for the corresponding surjection. We record that $I \cap K = (F_1, \cdots, F_d)$ in $R[T]$.

Let $P \in \text{Spec}(R[T])$ be such that $K \subseteq P$. Then, as $k \in (h)$ and $1-k \in K$, we see that $h \notin P$. If $T^2 - T \notin P$, then $\text{ht}(P) \geq d$. If $T^2 - T \in P$, then by $(*)$ above, $\text{ht}(P) \geq d+1$. In any case, $\text{ht}(K) \geq d$. Note that

$$K(0) \cap I(0) = K(0) \cap J = (F_1(0), \cdots, F_d(0)) = (a_1, \cdots, a_d),$$

$$K(1) \cap I(1) = K(1) \cap J' = (F_1(1), \cdots, F_d(1)) = (a'_1, \cdots, a'_d).$$

As the height of each of the ideals involved here is d, we have

$$(J, \omega_J) + (K(0), \omega_{K(0)}) = 0 = (J', \omega_{J'}) + (K(1), \omega_{K(1)}) \text{ in } E^d(R),$$

where $\omega_{K(0)}$ is induced by $a_1, \cdots, a_d$, and $\omega_{K(1)}$ is induced by $a'_1, \cdots, a'_d$.

Therefore, $(J, \omega_J) - (J', \omega_{J'}) = (K(1), \omega_{K(1)}) - (K(0), \omega_{K(0)}) \in H$ (where H is as in (3.1)) and consequently, $(J, \omega_J) = (J', \omega_{J'})$ in $E^d(R)$. This completes the proof. □

Remark 4.8. Let R be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k. Let n be an integer such that $2n \geq d+3$. Then also we have a bijection $\theta_n : E^n(R) \to \pi_0(Q_{2n}(R))$. The proof is similar to the one as above with suitable modifications.

Remark 4.9. Let R be a commutative Noetherian ring of dimension d. Let n be an integer such that $2n \geq d+3$. The n-th Euler class group $E^n(R)$ has been defined in this case as well (see [BRS 3] for $n = d \geq 2$, and [BRS 4] for $3 \leq d \leq 2n-3$). Further, following the same set of arguments as above, one can define a well-defined set-theoretic map $\theta : E^d(R) \longrightarrow \pi_0(Q_{2d}(R))$. This map is surjective but is not injective in general. In fact, even if R is an affine algebra over an algebraically closed field, θ may not be injective. The issue here is smoothness. We elaborate with an example.

The following example of Bhatwadekar, based on an example constructed by Bhatwadekar, Mohan Kumar and Srinivas [BRS 1, Example 6.4], appeared in [DK]. We recall it verbatim.

Example 4.10. (Bhatwadekar) Let
$$B = \frac{\mathbb{C}[X, Y, Z, W]}{(X^5 + Y^5 + Z^5 + W^5)},$$
as in [BRS 1, Example 6.4]. Then B is a graded normal affine domain over $\mathbb{C}$ of dimension 3, having an isolated singularity at the origin. Let $F(B)$ be the subgroup of $\widetilde{K}_0(B)$ generated by all elements of the type $[P] - [P^*]$, where P is a finitely generated projective B-module. As B is graded, $\mathrm{Pic}(B) = 0$. Therefore, by [BRS 1, 6.1] $F(B) = F^3K_0(B)$. Since $\mathrm{Proj}(B)$ is a smooth surface of degree 5 in $\mathbb{P}^3$, it follows from a result of Srinivas that $F(B) = F^3K_0(B) \neq 0$. Therefore, there exists a projective B-module P of rank 3 with trivial determinant such that $[P] - [P^*]$ is a nonzero element of $F(B)$. This implies that P does not have a unimodular element. We now consider the ring homomorphism $f : B \to B[T]$ given by $f(x) = xT, f(y) = yT, f(z) = zT, f(w) = wT$. We regard $B[T]$ as a B-module through this map and $Q = P \otimes_B B[T]$. Then it is easy to see that Q/TQ is free and $Q/(T-1)Q = P$. Therefore, Q is a projective $B[T]$-module which is not extended from B. Now consider a surjection $\alpha : Q \twoheadrightarrow I$ where $I \subset B[T]$ is an ideal of height 3. Fix an isomorphism $\chi : B[T] \simeq \wedge^3(Q)$. Note that Q/IQ is a free $B[T]/I$-module. We choose an isomorphism $\sigma : (B[T]/I)^3 \simeq Q/IQ$ such that $\wedge^3\sigma = \chi \otimes B[T]/I$. Composing σ and $\alpha \otimes B[T]/I$, we obtain a surjection $\omega : (B[T]/I)^3 \twoheadrightarrow I/I^2$. It is now easy to see that $(I(0), \omega(0)) = 0$ in $E^3(B)$ (as Q/TQ is free), whereas $(I(1), \omega(1)) = e(Q/(T-1)Q, \chi(1)) = e(P, \chi(1))$ in $E^3(B)$ cannot be trivial (as P does not have a unimodular element). $\square$

Remark 4.11. The above example shows that there are distinct elements in $E^3(B)$ whose images in $\pi_0(Q_6(B))$ are the same.

Remark 4.12. Let R be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k of characteristic $\neq 2$. Recall from Section 2 that we have a bijection $\beta_d : \pi_0(Q_{2d}(R)) \xrightarrow{\sim} \pi_0(Q'_{2d}(R))$, and therefore we immediately obtain $\beta_d\theta_d : E^d(R) \xrightarrow{\sim} \pi_0(Q'_{2d}(R))$. On the other hand, an element of $Q'_{2d}(R)$ obviously projects to a unimodular row of length $d+1$, and eventually we obtain a set-theoretic map $\delta_R : E^d(R) \twoheadrightarrow Um_{d+1}(R)/E_{d+1}(R)$. Now let $X = \mathrm{Spec}(R)$ be a smooth affine variety of dimension d over $\mathbb{R}$, whose real points $X(\mathbb{R})$ constitute an orientable manifold. In the sequel [DTZ] we prove that δ_R is a morphism of groups. Further, we carry out complete computations for the groups $Um_{d+1}(R)/E_{d+1}(R)$, $Um_{d+1}(R)/SL_{d+1}(R)$ and the Mennicke symbol $MS_{d+1}(R)$ for such varieties.

5. The monic inversion principle

In this final section we address the following question raised in the introduction. Recall that $R(T)$ is the ring obtained from $R[T]$ by inverting all the monic polynomials.

Question 5.1. *Let R be a commutative Noetherian ring of dimension $d \geq 2$. Let $I \subset R[T]$ be an ideal of height n such that $\mu(I/I^2) = n$. Let $I = (f_1, \cdots, f_n) + I^2$ be given. Assume that $IR(T) = (G_1, \cdots, G_n)$ with $G_i - f_i \in I^2R(T)$. Then, do there exist $F_1, \cdots, F_n \in I$ such that $I = (F_1, \cdots, F_n)$ with $F_i - f_i \in I^2$?*

Remark 5.2. We first remark on the case $d = n = 2$. In this case, the answer to the above question is negative (see [BRS 1, Example 3.15]). However, in this case it follows from [D, Section 7] that there exist $F_1, F_2 \in I$ and $\alpha \in SL_2(R[T]/I)$ such that $I = (F_1, F_2)$ with $(\overline{f_1}, \overline{f_2})\alpha = (\overline{F_1}, \overline{F_2})$, where 'bar' denotes images in I/I^2.

In view of the above remark, from now on we assume that $3 \leq n \leq d \leq 2n - 3$ and write this simply as $2n \geq d + 3$. To recast the above question in the language of the Euler class groups let us recall that if R is a regular domain (containing a field) of dimension $d \geq 3$, the n-th Euler class group $E^n(R[T])$ of $R[T]$ has been defined in [DRS] (We remind the reader that this definition is not a trivial extension of the definition of the Euler class group $E^n(R)$ as available in [BRS 4]). Let $I \subset R[T]$ be an ideal of height n such that $\mu(I/I^2) = n$. Let $\omega_I : (R[T]/I)^n \twoheadrightarrow I/I^2$ be a surjection. One can associate an element (I, ω_I) in $E^n(R[T])$. The following is the relevant result for us.

Theorem 5.3. [DRS, Theorem 3.1] *Let R be a regular domain of dimension d containing a field and n be an integer such that $2n \geq d + 3$. Let $I \subset R[T]$ be an ideal of height n such that $\mu(I/I^2) = n$. Let $\omega_I : (R[T]/I)^n \twoheadrightarrow I/I^2$ be a surjection. Then $(I, \omega_I) = 0$ in $E^n(R[T])$ if and only if there is a surjection $\beta : R[T]^n \twoheadrightarrow I$ such that β lifts ω_I.*

We now consider $R(T)$. As the extension $R[T] \longrightarrow R(T)$ is flat, there is a canonical group homomorphism $\Gamma : E^n(R[T]) \longrightarrow E^n(R(T))$. The following question is a reformulation of the question above.

Question 5.4. Is the canonical map $\Gamma : E^n(R[T]) \longrightarrow E^n(R(T))$ injective?

Here we settle the question in the affirmative when R is a regular domain of dimension $d \geq 3$ which is essentially of finite type over a field k of characteristic $\neq 2$. We first prove an easy proposition. For the definitions of $Q'_{2n}(-)$ and $\pi_0(Q'_{2n}(-))$, see Section 2.

Proposition 5.5. *Let A be any commutative Noetherian ring and $n \geq 2$. Then there is a bijection $\mu : \pi_0(Q'_{2n}(A)) \simeq \pi_0(Q'_{2n}(A[T]))$.*

Proof. For $v \in Q'_{2n}(A)$ we define $\mu([v]) = [v]$, sending the orbit $[v]$ to $[v] \in \pi_0(Q'_{2n}(A[T]))$. To see that μ is well defined, let $w \in Q'_{2n}(A)$ be homotopic to v. Then there is $F(X) \in Q'_{2n}(A[X])$ such that $F(0) = v$ and $F(1) = w$. Treat $F(X)$ as an element of $Q'_{2n}(A[T, X])$ and observe that $[v] = [w]$ in $\pi_0(Q'_{2n}(A[T]))$.

We now assume that $v, w \in Q'_{2n}(A)$ be such that $[v] = [w]$ in $\pi_0(Q'_{2n}(A[T]))$. Then there is $F(T,X) \in Q'_{2n}(A[T,X])$ such that $F(T,0) = v$ and $F(T,1) = w$. Then we also have $F(0,0) = v$ and $F(0,1) = w$. Clearly, $G(X) = F(0,X) \in Q'_{2n}(A[X])$ and is the desired homotopy. This shows that μ is injective.

Let $V(T) \in Q'_{2n}(A[T])$. We show that $V(T)$ is homotopic to $V(0) \in Q'_{2n}(A)$. It is easy to see that $V(TX) \in Q'_{2n}(A[T,X])$. The substitution $X = 0$ gives $V(0)$, whereas the substitution $X = 1$ gives $V(T)$. Therefore, μ is surjective and the proof is complete. $\square$

Let R be a regular domain of dimension $d \geq 3$ which is essentially of finite type over an infinite perfect field k. Let n be an integer such that $2n \geq d+3$. Recall that in Section 4 we established a bijection $\theta_n : E^n(R) \xrightarrow{\sim} \pi_0(Q_{2n}(R))$. If we assume that $\mathrm{Char}(k) \neq 2$, then we also have a bijection $\beta_n : \pi_0(Q_{2n}(R)) \xrightarrow{\sim} \pi_0(Q'_{2n}(R))$. Therefore, $\beta_n\theta_n : E^n(R) \xrightarrow{\sim} \pi_0(Q'_{2n}(R))$. From now on we assume that our rings contain $\frac{1}{2}$.

Proposition 5.6. *Let R be a regular domain of dimension $d \geq 3$ which is essentially of finite type over an infinite perfect field k of characteristic $\neq 2$. Let n be an integer such that $2n \geq d+3$. Then we have a bijection $E^n(R[T]) \simeq \pi_0(Q'_{2n}(R[T]))$.*

Proof. It has been proved in [DRS, Theorem 3.8] that under the assumption on the ring, the group $E^n(R)$ is canonically isomorphic to $E^n(R[T])$. We combine this with $\beta_n\theta_n : E^n(R) \xrightarrow{\sim} \pi_0(Q'_{2n}(R))$ and use the above proposition. $\square$

We now have the following commutative diagram:

$$\begin{array}{ccccc}
E^n(R[T]) & \xrightarrow{\sim} & \pi_0(Q'_{2n}(R[T])) & \xrightarrow{\sim} & \dfrac{Q'_{2n}(R[T])}{EO_{2n+1}(R[T])} \\
\downarrow & & \downarrow & & \downarrow \\
E^n(R(T)) & \xrightarrow{\sim} & \pi_0(Q'_{2n}(R(T))) & \xrightarrow{\sim} & \dfrac{Q'_{2n}(R(T))}{EO_{2n+1}(R(T))}
\end{array}$$

We can thus transfer the question on the Euler class group to a question on the map $Q'_{2n}(R[T])/EO_{2n+1}(R[T]) \to Q'_{2n}(R(T))/EO_{2n+1}(R(T))$. But note that the horizontal bijections are only set-theoretic maps. However, as we will see below, for us it will be enough to prove the following statement: For $v \in Q'_{2n}(R[T])$,

$$v \overset{EO_{2n+1}(R(T))}{\sim} (0, \cdots, 0, 1) \Rightarrow v \overset{EO_{2n+1}(R[T])}{\sim} (0, \cdots, 0, 1)$$

We proceed to prove the above assertion in a more general set-up. We will need the following factorization lemma for the elementary orthogonal group.

Lemma 5.7. [St, Lemma 2.3] *Let A be a commutative Noetherian ring with $2R = R$. Let $n \geq 2$ and let $f, g \in A$ be such that $fA + gA = A$. If $\sigma \in EO_{2n+1}(A_{fg})$, then there exist $\alpha \in EO_{2n+1}(A_f)$ and $\beta \in EO_{2n+1}(A_g)$ such that $\sigma = \alpha_g\, \beta_f$.*

We now prove the following monic inversion principle which is an analogue of [Ra, Theorem 1.1]. We follow the same line of arguments as in [Ra]. It will be interesting if one can remove the regularity assumption from the version we prove.

Theorem 5.8. *Let A be a regular domain containing a field k of characteristic $\neq 2$. Let $n \geq 2$ and $v = (f_1, \cdots, f_n, g_1, \cdots, g_n, h) \in Q'_{2n}(A[T])$. Suppose that there is a monic polynomial $f \in A[T]$ such that $v \equiv (0, \cdots, 0, 1)$ mod $EO_{2n+1}(A[T]_f)$. Then $v \equiv (0, \cdots, 0, 1)$ mod $EO_{2n+1}(A[T])$.*

Proof. Under the assumptions of the theorem, by [MaMi, Theorem 4.2], v is extended from A. More precisely, from [MaMi] we have

$$v \overset{O_{2n+1}(A[T],T)}{\sim} v(0),$$

and therefore, by [St, Theorem 1.3] it follows that $v \overset{EO_{2n+1}(A[T])}{\sim} v(0)$. As $v(0)$ and $v(1)$ are homotopic, applying Theorem 2.6 we have

$$v(0) \overset{EO_{2n+1}(A)}{\sim} v(1), \text{ implying that } v \overset{EO_{2n+1}(A[T]}{\sim} v(1) \qquad (*)$$

We first treat the special case when $f = T$. In this case, $v_T \sim (0, \cdots, 0, 1)$ implies that $v \sim v(1)$ and $v(1) \sim (0, \cdots, 0, 1)$, and therefore, we are done:

$$v \overset{EO_{2n+1}(A[T])}{\sim} (0, \cdots, 0, 1).$$

We now show that one can reduce to the special case above. Assume that f is an arbitrary monic polynomial. By $(*)$ we now have $v = v(1)$ and that $v(1)\sigma = (0, \cdots, 0, 1)$ for some $\sigma \in EO_{2n+1}(A[T]_f)$.

Let $f^* = T^{-\deg f}.f \in A[T^{-1}]$. Then $f^*(T^{-1} = 0) = 1$ and note that in $A[T^{-1}]$, we have $f^*A[T^{-1}] + T^{-1}A[T^{-1}] = A[T^{-1}]$. Further, $A[T^{-1}, T]_{f^*} = A[T, T^{-1}]_f$.

Applying Lemma 5.7 for the comaximal elements f^* and T^{-1} in $A[T^{-1}]$, we have $\sigma_T = (\sigma_1)_{T^{-1}}(\sigma_2)_{f^*}$, where $\sigma_1 \in EO_{2n+1}(A[T^{-1}]_{f^*})$ and $\sigma_2 \in EO_{2n+1}(A[T^{-1}, T])$. Now we treat $v(1)$ as an element in $Q'_{2n}(A[T^{-1}]_{f^*})$. Observe that the vectors $v(1)\sigma_1$ and $(0, \cdots, 0, 1)\sigma_2^{-1}$ coincide over $A[T^{-1}, T]_{f^*}$. We can patch them to obtain $w \in Q'_{2n}(A[T^{-1}])$. Note that $w_{T^{-1}} = (0, \cdots, 0, 1)\sigma_2^{-1}$ in $A[T, T^{-1}]$. In other words,

$$w_{T^{-1}} \overset{EO_{2n+1}(A[T,T^{-1}])}{\sim} (0, \cdots, 0, 1)$$

Now apply the special case treated above to conclude that $w \overset{EO_{2n+1}(A[T^{-1}])}{\sim} (0, \cdots, 0, 1)$. For this final moment, let us write $Y = T^{-1}$. As $f^*(Y = 0) = 1$ and as $w_{f^*} = v(1)\sigma_1$, specializing at $Y = 0$ we obtain

$$w(0) = v(1)\sigma_1(0) \overset{EO_{2n+1}(A)}{\sim} (0, \cdots, 0, 1),$$

showing that $v = v(1) \overset{EO_{2n+1}(A)}{\sim} (0, \cdots, 0, 1)$. □

Corollary 5.9. *Let A be a regular domain containing a field k of characteristic $\neq 2$. Let $v \in Q'_{2n}(A[T])$ be such that v is homotopic to $(0, \cdots, 0, 1)$ in $Q'_{2n}(A(T))$. Then v is homotopic to $(0, \cdots, 0, 1)$ in $Q'_{2n}(A[T])$.*

Proof. From Theorem 2.6 we can conclude that $v \overset{EO_{2n+1}(A(T))}{\sim} (0, \cdots, 0, 1)$. We can then find a single monic polynomial $f \in A[T]$ such that

$$v \overset{EO_{2n+1}(A[T]_f)}{\sim} (0, \cdots, 0, 1)$$

Applying the above theorem and Theorem 2.6 again we are done. □

We are now ready to prove that the monic inversion principle holds for the Euler class groups.

Theorem 5.10. *Let R be a regular domain of dimension $d \geq 3$ which is essentially of finite type over an infinite perfect field k of characteristic $\neq 2$. Let n be an integer such that $2n \geq d + 3$. Then the natural map $E^n(R[T]) \longrightarrow E^n(R(T))$ is injective.*

Proof. The canonical map $E^n(R[T]) \longrightarrow E^n(R(T))$ is a group homomorphism. Therefore, it is enough to show that if $(J, \omega_J) \in E^n(R[T])$ is such that its image $(JR(T), \omega_J \otimes R(T))$ is trivial in $E^n(R(T))$, then $(J, \omega_J) = 0$ in $E^n(R[T])$. Note that the horizontal bijections in the following commutative diagram preserve base points.

$$\begin{array}{ccc} E^n(R[T]) & \xrightarrow{\sim} & \pi_0(Q'_{2n}(R[T])) \\ \downarrow & & \downarrow \\ E^n(R(T)) & \xrightarrow{\sim} & \pi_0(Q'_{2n}(R(T))) \end{array}$$

Therefore we are done by Corollary 5.9. □

The assertion of Theorem 5.10, translated back to simple ideal-theoretic terms, gives us the main theorem:

Theorem 5.11. *Let R be a regular domain of dimension d which is essentially of finite type over an infinite perfect field k of characteristic $\neq 2$. Let n be a positive integer such that $2n \geq d+3$. Let $I \subset R[T]$ be an ideal of height n such that*

$$I = (f_1, \cdots, f_n) + I^2.$$

Assume that $IR(T) = (G_1, \cdots, G_n)$ with $G_i - f_i \in I^2R(T)$ for $i = 1, \cdots, n$. Then, there exist $g_1, \cdots, g_n \in I$ such that

$$I = (g_1, \cdots, g_n), \text{ and } g_i - f_i \in I^2 \text{ for } i = 1, \cdots, n$$

6. Appendix: On Murthy's conjecture (by Mrinal Kanti Das)

Evidently, this paper is intimately related to the complete intersection conjecture of Murthy which was mentioned at the beginning of this paper. The purpose of this appendix is to clarify its present status in light of recent developments. The reader may be aware that a solution of a stronger form of Murthy's conjecture appeared in [F1, Theorem 3.2.9]. However, in October, 2016, the author of this appendix came up with the following series of arguments and communicated them privately to the author of [F1].

Proposition 6.1. *Let R be a commutative Noetherian ring and $n \geq 2$. There is a natural bijection between the sets $\pi_0(Q_{2n}(R))$ and $\pi_0(Q_{2n}(R[T]))$.*

Proof. The proof is exactly the same as that of Proposition 5.5. □

To derive corollaries, we first prove an easy lemma.

Lemma 6.2. *Let A be a Noetherian ring and $I \subset A$ be an ideal such that $I = J + K$, where J, K are ideals and $K \subset I^2$. Then, there exists $s \in K$ such that $I = (s, J)$ and $s - s^2 \in J$.*

Proof. We have $I = J + I^2$. The ideal I/J is idempotent in A/J and therefore by Nakayama lemma, there exists $e \in I$ such that I/J is generated by $e + J$, with $e - e^2 \in J$. The canonical map $K \longrightarrow I/J$ is surjective. Choose a preimage s of $e + J$ in K. Then, I/J is generated by $s + J$, implying that $I = (s, J)$. Further, $s - s^2 \in J$ as $s + J = e + J$. □

Now let R be a Noetherian ring and $n \geq 2$. Let $I \subset R[T]$ be an ideal such that $I = (f_1(T), \cdots, f_n(T)) + (I^2T)$. By the above lemma, there exist $s(T) \in (I^2T)$ and $g_1(T), \cdots, g_n(T) \in R[T]$ such that $I = (s(T), f_1(T), \cdots, f_n(T))$ and $s(T)(1 - s(T)) = f_1(T)g_1(T) + \cdots + f_n(T)g_n(T)$. With this data, we have the following:

Corollary 6.3. *The homotopy orbit of $(f_1(T), \cdots, f_n(T), g_1(T), \cdots, g_n(T), s(T))$ is trivial in $\pi_0(Q_{2n}(R[T]))$.*

Proof. By the proposition, the vector $(f_1(T), \cdots, f_n(T), g_1(T), \cdots, g_n(T), s(T))$ is homotopic to $(f_1(0), \cdots, f_n(0), g_1(0), \cdots, g_n(0), s(0))$. The latter is homotopically trivial because $s(0) = 0$. □

Corollary 6.4. *Let R be a regular ring containing an infinite field k (Char$(k) \neq 2$). Let $n \geq 2$ and $I \subset R[T]$ be an ideal such that*

$$I = (f_1(T), \cdots, f_n(T)) + (I^2T).$$

Then there exist $F_1(T), \cdots, F_n(T) \in I$ such that $I = (F_1(T), \cdots, F_n(T))$ with $F_i(T) - f_i(T) \in (I^2T)$.

Proof. The above corollary and [F1, Theorem 3.2.7]. □

But this final corollary contradicts the following example of Bhatwadekar-Sridharan. We give only an outline of the example.

Example 6.5. [BRS 1, 3.15] Let $R = \mathbb{C}[X, Y]$ and $F = X^3 + Y^3 - 1$. Consider the ideal $I = (F, T-1) \subset R[T]$. Let $\omega_I : (R[T])/I)^2 \twoheadrightarrow I/I^2$ be the surjection induced by $(F, T-1)$. Observe that $R[T]/I = R/(F)$. One can choose a suitable matrix $\overline{\sigma} \in SL_2(R/(F))$ such that its image in $SK_1(R/(F))$ is non-trivial (this part is crucial). Consider the composite $\omega_I \overline{\sigma} : (R[T])/I)^2 \twoheadrightarrow I/I^2$. Let it be given by $I = (G, H) + I^2$. As $I(0) = R$, using [BRS 1, 3.9], G, H can be lifted to obtain $I = (G', H') + (I^2 T)$. Rest of the example is devoted to showing that if G', H' can be lifted to a set of two generators of I, then the choice of $\overline{\sigma}$ is violated.

This shows that [F1, Theorem 3.2.7] is not correct in its present form. Soon thereafter, the author of this appendix could spot the gap in the arguments leading to this theorem in [F1]. It is in Lemma 3.2.3, *op. cit.*. Using notations and terms therein, we observe: *it is not clear why vM has the strong lifting property.* Indeed $I(v) = I(vM)$ but the generators have to be lifted modulo $(s - \lambda a_i)^2$. This was also privately communicated to the author of [F1], which has been kindly acknowledged in [F2].

One might hope that perhaps this gap can be fixed. However, S. Mandal, in collaboration with none other than M. P. Murthy, came up with two examples in [Ma 2], showing that assertions in [F1, Theorems 3.2.7, 3.2.8] are not valid. The strong form of Murthy's conjecture in [F1] is derived from these results. As a consequence, the conjecture is in general open and the best known solution is the one provided by Mohan Kumar forty years ago.

Remark 6.6. In the context of the present article, the above clarification is necessary. A diligent reader would notice that if the arguments in [F1] were correct, then the arguments in this article can also be extended to answer Question 1.6 in the introduction, (without any restriction on the coheight of the ideal) for the type of rings considered here.

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