1. INTRODUCTION

Let $R$ be a commutative Noetherian ring of (Krull) dimension $d \geq 2$. The group $E_{d+1}(R)$ (the subgroup of $SL_{d+1}(R)$ generated by the elementary matrices) acts on $Um_{d+1}(R)$, the set of unimodular rows of length $d + 1$ over $R$. When $d = 2$, Vaserstein [SuVa, Section 5] showed that the orbit space $Um_3(R)/E_3(R)$ carries the structure of an abelian group. Later, van der Kallen [vdK 1] extended this result to show that $Um_{d+1}(R)/E_{d+1}(R)$ has an abelian group structure for all $d \geq 2$.

The group $Um_{d+1}(R)/E_{d+1}(R)$ is intimately related to the $d^{th}$ Euler class group $E_d(R)$ studied by Bhatwadekar-Sridharan (see [BRS 3, DZ, vdK 3, vdK 4] for details on the connection between these two groups). The idea of the Euler class group was envisioned by Nori in order to detect the obstruction for a projective $R$-module of rank $d$ to split off a free summand of rank one. Although this “splitting problem” was settled by Bhatwadekar-Sridharan quite sometime back in [BRS 1, BRS 3], surprisingly, the Euler class group has not yet lost its relevance. Very recently, in [DTZ2], the current authors have succeeded in computing the structure of $Um_{d+1}(R)/E_{d+1}(R)$ for smooth affine $R$-algebras by comparing this group with the Euler class group, and appealing to the structure theorems for $E_d(R)$ available in [BRS 2] for such rings. To facilitate such a comparison, a set-theoretic map $\delta_R : E_d(R) \to Um_{d+1}(R)/E_{d+1}(R)$ was defined in [DTZ2], based on the formalism developed in [DTZ1], when $R$ is a smooth affine domain of dimension $d$ over an infinite perfect field $k$ of characteristic unequal to 2. If $k = \mathbb{R}$, it was proved in [DTZ2] that $\delta_R$ is a morphism of groups but at that time it was not clear whether $\delta_R$ is a morphism in general. In this article we prove that $\delta_R$ is indeed a morphism of groups. We believe this morphism will enable us to understand these two groups better, as it did in [DTZ2] when $k = \mathbb{R}$.

In Section 2 we recall the basics. In Section 3 we treat the special case when the group law in $Um_{d+1}(R)/E_{d+1}(R)$ is Mennicke-like (as this case is simpler and the treatment is entirely different). In Section 4 we treat the general case.

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2. Generalities: The objects and the maps

**Notation.** To avoid confusion, we shall write an ideal generated by \( f_1, \ldots, f_{d+1} \) as \( \langle f_1, \ldots, f_{d+1} \rangle \), whereas unimodular rows will be written with first brackets. Let \( v = (a_1, \ldots, a_{d+1}) \in U m_{d+1}(R) \). The orbit of \( v \) in \( U m_{d+1}(R)/E_{d+1}(R) \) will be written as \([v] = [a_1, \ldots, a_{d+1}]\).

2.1. The Euler class group. Let \( R \) be a smooth affine domain of dimension \( d \geq 2 \) over an infinite perfect field \( k \). Let \( B \) be the set of pairs \((m, \omega_m)\) where \( m \) is a maximal ideal of \( R \) and \( \omega_m : (R/m)^d \rightarrow m/m^2 \). Let \( G \) be the free abelian group generated by \( B \). Let \( J = m_1 \cap \cdots \cap m_r \), where \( m_i \) are distinct maximal ideals of \( R \). Any \( \omega_J : (R/J)^d \rightarrow J/J^2 \) induces surjections \( \omega_i : (R/m_i)^d \rightarrow m_i/m_i^2 \) for each \( i \). We associate \((J, \omega_J) := \sum_i (m_i, \omega_i) \in G \). Now, let \( S \) be the set of elements \((J, \omega_J)\) of \( G \) for which \( \omega_J \) has a lift to a surjection \( \theta : R^d \rightarrow J \) and \( H \) be the subgroup of \( G \) generated by \( S \). The Euler class group \( E^d(R) \) is defined as \( E^d(R) := G/H \).

**Remark 2.1.** The above definition appears to be slightly different from the one given in [BRS 1]. However, note that if \((J, \omega_J) \in S \) and if \( \sigma \in E_d(R/J) \), then the element \((J, \omega_J \sigma)\) is also in \( S \). For details, see [DZ, Proposition 2.2].

**Theorem 2.2.** [BRS 1, 4.11] Let \( R \) be a smooth affine domain of dimension \( d \geq 2 \) over an infinite perfect field \( k \). Let \( J \subset R \) be a reduced ideal of height \( d \) and \( \omega_J : (R/J)^d \rightarrow J/J^2 \) be a surjection. Then, the following are equivalent:

1. The image of \((J, \omega_J)\) is \( 0 \) in \( E^d(R) \)
2. \( \omega_J \) can be lifted to a surjection \( \theta : R^d \rightarrow J \).

**Remark 2.3.** We shall refer to the elements of the Euler class group as Euler cycles. An arbitrary element of \( E^d(R) \) can be represented by a single Euler cycle \((J, \omega_J)\), where \( J \) is a reduced ideal of height \( d \) and \( \omega_J : (R/J)^d \rightarrow J/J^2 \) is a surjection (see [BRS 1, Remark 4.14]).

The following notation will be used in the rest of this article.

**Notation.** Let \( \dim(R) = d \). Let \((J, \omega_J) \in E^d(R) \) and \( u \in R \) be a unit modulo \( J \). Let \( \sigma \) be any diagonal matrix in \( GL_d(R/J) \) with determinant \( \overline{u} \) (bar means modulo \( J \)). We shall denote the composite surjection

\[
(R/J)^d \xrightarrow{\sigma} (R/J)^d \xrightarrow{\omega_J} J/J^2
\]

by \( \overline{u} \omega_J \). It is easy to check that the element \((J, \overline{u} \omega_J) \in E^d(R) \) is independent of \( \sigma \) (the key fact used here is that \( SL_d(R/J) = E_d(R/J) \) as \( \dim(R/J) = 0 \)).
2.2. [The map $\delta_R : E^d(R) \to Um_{d+1}(R)/E_{d+1}(R)$]. Let $R$ be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field $k$ with $\text{Char}(k) \neq 2$. Let $(J, \omega_J) \in E^d(R)$, where $J$ is a reduced ideal of height $d$. Now $\omega_J : (R/J)^d \to J/J^2$ is given by $J = \langle a_1, \ldots, a_d \rangle + J^2$, for some $a_1, \ldots, a_d \in J$. Applying the Nakayama Lemma one obtains $s \in J^2$ such that $J = \langle a_1, \ldots, a_d, s \rangle$ with $s - s^2 = a_1b_1 + \cdots + a_db_d$ for some $b_1, \ldots, b_d \in R$ (see [Mo] for a proof). Based on the formalism in [DTZ1], in [DTZ2] we defined a set-theoretic map $\delta_R : E^d(R) \to Um_{d+1}(R)/E_{d+1}(R)$ which takes $(J, \omega_J)$ to the orbit $[2a_1, \ldots, 1 - 2s] \in Um_{d+1}(R)/E_{d+1}(R)$. A lot of technical work has gone into proving that $\delta_R$ is well-defined. See [DTZ1, Section 3] for the preparatory steps and [DTZ2, Section 2] for the details of this definition.

**Remark 2.4.** Note that $(1 - 2s)^2 \equiv 1 \pmod{(2a_1, \ldots, 2a_d)}$. Conversely, let an orbit $[v] = [x_1, \ldots, x_d, z] \in Um_{d+1}(R)/E_{d+1}(R)$ be such that the ideal $\langle x_1, \ldots, x_d \rangle$ is reduced of height $d$, and $z^2 \equiv 1 \pmod{\langle x_1, \ldots, x_d \rangle}$, then $[v]$ is in the image of $\delta_R$.

3. **SPECIAL CASE: MENNICKE-LIKE GROUP STRUCTURE**

We will say that the group structure on $Um_{d+1}(R)/E_{d+1}(R)$ is *Mennicke-like* if for two orbits $[a_1, \ldots, a_d, x], [a_1, \ldots, a_d, y] \in Um_{d+1}(R)/E_{d+1}(R)$ we have the coordinate-wise product:

$$[a_1, \ldots, a_d, x][a_1, \ldots, a_d, y] = [a_1, \ldots, a_d, xy].$$

Throughout this section, let $R$ be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field $k$ such that $2R = R$.

**Lemma 3.1.** Let the group structure on $Um_{d+1}(R)/E_{d+1}(R)$ be Mennicke-like. Let $(J, \omega_J) \in E^d(R)$ be any element. Then $\delta_R((J, \omega_J))$ is 2-torsion.

**Proof.** Let $\omega_J$ be induced by $J = \langle a_1, \ldots, a_d \rangle + J^2$. Then, there exists $s \in J^2$ such that $J = \langle a_1, \ldots, a_d, s \rangle$ with $s - s^2 \in \langle a_1, \ldots, a_d \rangle$. By definition, $\delta_R((J, \omega_J)) = [1 - 2s, 2a_1, \ldots, 2a_d]$. As the group law is Mennicke-like,

$$[1 - 2s, 2a_1, \ldots, 2a_d]^2 = [(1 - 2s)^2, 2a_1, \ldots, 2a_d] = [2a_1, \ldots, 2a_d]. \quad \square$$

**Theorem 3.2.** Let the group structure on $Um_{d+1}(R)/E_{d+1}(R)$ be Mennicke-like. Then $\delta$ is a morphism of groups.

**Proof.** Let $(J, \omega_J), (K, \omega_K) \in E^d(R)$ be such that $J + K = R$, where $J, K$ are both reduced ideals of height $d$. Then $(J, \omega_J) + (K, \omega_K) = (J \cap K, \omega_{J\cap K})$, where $\omega_{J\cap K}$ is induced by $\omega_J$ and $\omega_K$. To prove the theorem, it is enough to show that

$$\delta_R((J, \omega_J)) \ast \delta_R((K, \omega_K)) = \delta_R((J \cap K, \omega_{J\cap K})).$$

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4In literature it has been described as *nice group structure*. Ravi Rao suggested us to use the term *Mennicke-like*. 


where $*$ denotes the product in $Um_{d+1}(R)/E_{d+1}(R)$.

Let $\omega_{J\cap K}$ be induced by $J \cap K = \langle a_1, \cdots, a_d \rangle + (J \cap K)^2$. Then $J = \langle a_1, \cdots, a_d \rangle + J^2$ and $K = \langle a_1, \cdots, a_d \rangle + K^2$. Let $J = \langle a_1, \cdots, a_d, s \rangle$ with $s - s^2 \in \langle a_1, \cdots, a_d \rangle$ and $K = \langle a_1, \cdots, a_d, t \rangle$ with $t - t^2 \in \langle a_1, \cdots, a_d \rangle$, as usual. Then it follows that $J \cap K = \langle a_1, \cdots, a_d, st \rangle$ and $st - st^2 \in \langle a_1, \cdots, a_d \rangle$.

By the definition of the map $\delta$, we have:

1. $\delta((J, \omega_J)) = [1 - 2s, 2a_1, \cdots, 2a_d]$, 
2. $\delta((K, \omega_K)) = [1 - 2t, 2a_1, \cdots, 2a_d]$, 
3. $\delta((J \cap K, \omega_{J\cap K})) = [1 - 2st, 2a_1, \cdots, 2a_d]$.

As the group law in $Um_{d+1}(R)/E_{d+1}(R)$ is Mennicke-like, we have

$$[1 - 2s, 2a_1, \cdots, 2a_d][1 - 2t, 2a_1, \cdots, 2a_d] = [1 - 2s - 2t + 4st, 2a_1, \cdots, 2a_d].$$

Let us try to locate a pre-image of the element on the right hand side of the above equation. To this end, we consider the following ideal

$L = \langle a_1, \cdots, a_d, s + t - 2st \rangle = \langle a_1, \cdots, a_d, s^2 + t^2 - 2st \rangle = \langle a_1, \cdots, a_d, (s - t)^2 \rangle$

in $R$ and note that $L + J \cap K = R$ (as $s - t$ is a unit modulo $J \cap K$). Let ‘bar’ denote modulo $(a_1, \cdots, a_d)$. Then,

$$L \cap J \cap K = \langle \bar{s}l \rangle (\bar{s} + \bar{t} - 2\bar{s}t) = \langle \bar{s}^2 \bar{l} + \bar{s} \bar{t} - 2\bar{s}\bar{l} \rangle = \langle \bar{s} \bar{l} + \bar{s} \bar{t} - 2\bar{s}\bar{l} \rangle = \langle \bar{0} \rangle,$$

and we have $L \cap (J \cap K) = \langle a_1, \cdots, a_d \rangle$. Therefore, $(L, \omega_L) + (J \cap K, \omega_{J\cap K}) = 0$, where $\omega_L$ is induced by the images of $a_1, \cdots, a_d$ in $L/L^2$. It is easy to see that $\delta((L, \omega_L)) = [1 - 2s - 2t + 4st, 2a_1, \cdots, 2a_d]$. Finally, we conclude (using (3.1)) that

$$\delta((J, \omega_J)) \ast \delta((K, \omega_K)) = \delta((L, \omega_L)) = \delta((J \cap K, \omega_{J\cap K}))^{-1} = \delta((J \cap K, \omega_{J\cap K})). \quad \square$$

4. **General case**

In this section treat the general case. Our line of arguments may be termed as “Mennicke-Newman for ideals”. For the Mennicke-Newman Lemma for elementary orbits of unimodular rows, see [vdK3, Lemma 3.2].

**Lemma 4.1.** Let $I_1, I_2$ be two comaximal ideals in a ring $R$ such that $I_1 \neq I_1^2$ and $I_2 \neq I_2^2$. Then we can find $x \in I_1 \setminus I_1^2$ and $y \in I_2 \setminus I_2^2$ such that $x + y = 1$.

Proof. As $I_1^2 + I_2^2 = R$, we can find $a \in I_1^2$, $b \in I_2^2$ such that $a + b = 1$.

Claim: $I_1 \cap I_2 \not\subseteq I_1^2$. To see this note that $I_1^2 + I_2 = R$, and we have

$$I_1 = I_1 \cap R = I_1 \cap (I_1^2 + I_2) = I_1^2 + I_1 \cap I_2.$$ 

If $I_1 \cap I_2 \subseteq I_1^2$, then $I_1 = I_1^2$, contrary to the hypothesis. Similarly, $I_1 \cap I_2 \not\subseteq I_2^2$. 


Therefore, we can choose \( \alpha \in I_1 \cap I_2 \setminus (I_1^2 \cup I_2^2) \). Take \( x = a - \alpha \) and \( y = b + \alpha \) to conclude.

**Proposition 4.2.** Let \( R \) be a ring of dimension \( d \geq 2 \). Let \( J = m_1 \cap \cdots \cap m_r \) and \( K = m_{r+1} \cap \cdots \cap m_s \) be two ideals, each of height \( d \), where \( m_i \) are all distinct maximal ideals for \( i = 1, \cdots, s \). Then, there exist \( x \in J \) and \( y \in K \) such that:

1. \( x + y = 1 \),
2. \( x \notin m_1^2 \cup \cdots \cup m_s^2 \) and \( y \notin m_1^2 \cup \cdots \cup m_s^2 \).

**Proof.** Let \( J^2 + K^2 = R \), we can find \( a \in J^2 \) and \( b \in K^2 \) such that \( a + b = 1 \). We claim that there exists \( c \in J \cap K \) such that \( c \notin m_1^2 \cup \cdots \cup m_s^2 \). If we can prove the claim, we will take \( x = a - c \) and \( y = a + c \) to prove the proposition.

**Proof of the claim.** We have \( m_1^2 + m_2^2 \cdots m_s^2 = R \). Choose \( f \in m_1^2 \) and \( g \in m_2^2 \cdots m_s^2 \) so that \( f + g = 1 \).

Observe that \( m_1 \cap (m_2^2 \cdots m_s^2) \cap m_1^2 \) (to see this, use the above lemma to obtain \( z \in m_1 \cap m_2^2 \cap \cdots \cap m_s^2 \) so that \( z + w = 1 \). Assume, if possible, that \( m_1 \cap (m_2^2 \cdots m_s^2) \cap m_1^2 \). As \( z = z^2 + wz \) and \( w \in m_1 \cap (m_2^2 \cdots m_s^2) \) it would follow that \( z \in m_1^2 \). Contradiction.)

Choose \( \alpha \in m_1 \cap (m_2^2 \cdots m_s^2) \cap m_1^2 \) and take \( c_1 = f - \alpha \), \( c_1' = g + \alpha \). Then, we have:

1. \( c_1 + c_1' = 1 \),
2. \( c_1 \in m_1 \setminus m_1^2 \),
3. \( c_1 \equiv 1 \) modulo \( m_1^2 \) for all \( i \neq 1 \).

Following a similar method, for each \( i = 1, \cdots, s \), choose \( c_i \in m_i \setminus m_i^2 \) so that \( c_i \equiv 1 \) modulo \( m_1^2 \cdots m_i^2 \cdots m_{i+1}^2 \cdots m_s^2 \). Take \( c = \prod_{i=1}^s c_i \). Then \( c \in m_1 \cdots m_s \) and it is easy to check that \( c \notin m_i^2 \) for any \( i \). This completes the proof of the claim.

**Theorem 4.3.** Let \( R \) be a smooth affine domain of dimension \( d \geq 2 \) over an infinite perfect field \( k \) of characteristic unequal to 2. Then \( \delta_R : E^d(R) \rightarrow Um_{d+1}(R)/E_{d+1}(R) \) is a morphism of groups.

**Proof.** Let \( (J, \omega_J), (K, \omega_K) \in E^d(R) \) be such that \( J + K = R \), where \( J, K \) are both reduced ideals of height \( d \). Then \( (J, \omega_J) + (K, \omega_K) = (J \cap K, \omega_{J \cap K}) \), where \( \omega_{J \cap K} \) is induced by \( \omega_J \) and \( \omega_K \). To prove the theorem, it is enough to show that

\[
\delta_R((J, \omega_J)) + \delta_R((K, \omega_K)) = \delta_R((J \cap K, \omega_{J \cap K})),
\]

where * denotes the product in \( Um_{d+1}(R)/E_{d+1}(R) \).

Let \( J = m_1 \cdots m_r \) and \( K = m_{r+1} \cdots m_s \). Applying the above proposition, choose \( x \in J \) and \( y \in K \) such that \( x + y = 1 \) and \( x \notin m_1^2 \cup \cdots \cup m_s^2 \) and \( y \notin m_1^2 \cup \cdots \cup m_s^2 \). Then \( xy \in (J \cap K) \setminus (J \cap K)^2 \). As \( x + y = 1 \), it is easy to check that for each \( i \), the image of \( xy \) in \( m_i/m_i^2 \) is not trivial. Therefore, \( xy \) is a part of a basis of \( m_i/m_i^2 \), for each \( i, 1 \leq i \leq s \). Consequently, \( xy \) is a part of generators of \( (J \cap K)/(J \cap K)^2 \). Similarly, \( x \) is a part of generators of \( J/J^2 \) and \( y \) is a part of generators of \( K/K^2 \).
Let $J \cap K = \{ xy, a_1, \cdots, a_{d-1} \} + (J \cap K)^2$ for some $a_1, \cdots, a_{d-1} \in J \cap K$. Let $\omega'_{J \cap K} : (R/(J \cap K))^d \to (J \cap K)/(J \cap K)^2$ denote the corresponding surjection. By [BRS 3, 2.2 and 5.0] there is a unit $u$ modulo $J \cap K$ such that $(J \cap K, \omega_{J \cap K}) = (J \cap K, u\omega'_{J \cap K})$ in $E^d(R)$. Therefore, $(J \cap K, \omega_{J \cap K})$ is given by $J \cap K = \{ xy, ua_1, a_2, \cdots, a_{d-1} \} + (J \cap K)^2$.

Similarly, $J = \{ x, ua_1, a_2, \cdots, a_{d-1} \} + J^2$ gives $(J, \omega_J)$ and $K = \{ x, ua_1, a_2, \cdots, a_{d-1} \} + K^2$ gives $(K, \omega_K)$.

We can choose $s \in J \cap K$ such that $s - s^2 \in \{ xy, ua_1, a_2, \cdots, a_{d-1} \}$ and $J \cap K = \{ xy, ua_1, a_2, \cdots, a_{d-1} \}$. As $s - s^2 \notin \{ xy, ua_1, a_2, \cdots, a_{d-1} \}$ and $s-s^2 \notin \{ y, ua_1, a_2, \cdots, a_{d-1} \}$, it follows that $(J, \omega_J)$ corresponds to $J = \{ x, ua_1, a_2, \cdots, a_{d-1} \}$ and $(K, \omega_K)$ corresponds to $K = \{ y, ua_1, a_2, \cdots, a_{d-1} \}$ (to check this, use $x + y = 1$).

We then have,

1. $\delta_R((J, \omega_J)) = [2x, 2ua_1, 2a_2, \cdots, 2a_{d-1}, 1 - 2s] = [x, ua_1, 2a_2, \cdots, 2a_{d-1}, 1 - 2s]$,
2. $\delta_R((K, \omega_K)) = [2y, 2ua_1, 2a_2, \cdots, 2a_{d-1}, 1 - 2s] = [y, ua_1, 2a_2, \cdots, 2a_{d-1}, 1 - 2s]$,
3. $\delta_R((J \cap K, \omega_{J \cap K})) = [2xy, 2ua_1, 2a_2, \cdots, 2a_{d-1}, 1 - 2s] = [xy, ua_1, 2a_2, \cdots, 2a_{d-1}, 1 - 2s]$.

It follows that $\delta_R((J, \omega_J)) * \delta_R((K, \omega_K)) = \delta_R((J \cap K, \omega_{J \cap K}))$, as $x + y = 1$.

5. A Few Remarks

Let $X = \text{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Let $X(\mathbb{R})$ denote the set of real points of $X$. Assume that $X(\mathbb{R}) \neq \emptyset$. Then $X(\mathbb{R})$ is a smooth real manifold. Let $X(\mathbb{R})$ denote the ring obtained from $R$ by inverting all the functions which do not have any real zeros. We can apply (3.2) to obtain the following result.

**Theorem 5.1.** Let $X = \text{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$ such that $X(\mathbb{R})$ is orientable, and the number of compact connected components of $X(\mathbb{R})$ is at least one. Then the group structure on $Um_{d+1}(R)/E_{d+1}(R)$ can never be Mennicke-like.

Proof. In our earlier paper [DTZ2], we proved the following assertions:

1. $Um_{d+1}(R)/E_{d+1}(R) = Um_{d+1}(R(\mathbb{R}))/E_{d+1}(R(\mathbb{R})) \bigoplus K$,
2. $\delta_R(X) : E^d(R(\mathbb{R})) \to Um_{d+1}(R(\mathbb{R}))/E_{d+1}(R(\mathbb{R}))$ is an isomorphism,
3. $Um_{d+1}(R(\mathbb{R}))/E_{d+1}(R(\mathbb{R})) \rightarrow \bigoplus_i \mathbb{Z}$, where $t$ is the number of compact connected components of $X(\mathbb{R})$.

Now, assume that the group structure on $Um_{d+1}(R(\mathbb{R}))/E_{d+1}(R(\mathbb{R}))$ is Mennicke-like. Then, by (3.2), we shall find non-trivial orbits which are 2-torsion. But as $t \geq 1$, the group $Um_{d+1}(R(\mathbb{R}))/E_{d+1}(R(\mathbb{R}))$ is non-trivial and is free abelian. Thus we arrive at a contradiction. As $Um_{d+1}(R(\mathbb{R}))/E_{d+1}(R(\mathbb{R}))$ is a subgroup of $Um_{d+1}(R)/E_{d+1}(R)$, the theorem follows.
Remark 5.2. In [DTZ2] we also computed the universal Mennicke symbol $\text{MS}_{d+1}(R)$, where $R$ is as in the above theorem. It follows from there as well that the group structure on $Um_{d+1}(R)/E_{d+1}(R)$ can never be Mennicke-like. The arguments given above only avoids the computation of $\text{MS}_{d+1}(R)$.

We now comment on the case when $X(\mathbb{R})$ is non-orientable.

Theorem 5.3. Let $X = \text{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$ such that $X(\mathbb{R})$ is non-orientable. Then $\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \to Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is a surjective morphism. As a consequence, $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is an $\mathbb{Z}/2\mathbb{Z}$-vector space of dimension $\leq t$, where $t$ is the number of compact connected components of $X(\mathbb{R})$.

Proof. It has already been proved in [DTZ2, Theorem 3.2] that $\delta_{\mathbb{R}(X)}$ is surjective. In this article we proved that $\delta_{\mathbb{R}(X)}$ is a morphism. As $E^d(\mathbb{R}(X)) = \bigoplus_i \mathbb{Z}/2\mathbb{Z}$, the result follows. \qed

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