EULER CYCLES AND MENNICKE SYMBOLS

MRINAL KANTI DAS, SOUMI TIKADER, AND MD. ALI ZINNA

1. Introduction

Let R be a commutative Noetherian ring of (Krull) dimension $d \geq 2$. The group $E_{d+1}(R)$ (the subgroup of $SL_{d+1}(R)$ generated by the elementary matrices) acts on $Um_{d+1}(R)$, the set of unimodular rows of length $d+1$ over R. When $d = 2$, Vaserstein [SuVa, Section 5] showed that the orbit space $Um_3(R)/E_3(R)$ carries the structure of an abelian group. Later, van der Kallen [vdK 1] extended this result to show that $Um_{d+1}(R)/E_{d+1}(R)$ has an abelian group structure for all $d \geq 2$.

The group $Um_{d+1}(R)/E_{d+1}(R)$ is intimately related to the d^{th} Euler class group $E^d(R)$ studied by Bhatwadekar-Sridharan (see [BRS 3, DZ, vdK 3, vdK 4] for details on the connection between these two groups). The idea of the Euler class group was envisioned by Nori in order to detect the obstruction for a projective R-module of rank d to split off a free summand of rank one. Although this *"splitting problem"* was settled by Bhatwadekar-Sridharan quite sometime back in [BRS 1, BRS 3], surprisingly, the Euler class group has not yet lost its relevance. Very recently, in [DTZ2], the current authors have succeeded in computing the structure of $Um_{d+1}(R)/E_{d+1}(R)$ for smooth affine $\mathbb{R}$-algebras by comparing this group with the Euler class group, and appealing to the structure theorems for $E^d(R)$ available in [BRS 2] for such rings. To facilitate such a comparison, a set-theoretic map $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ was defined in [DTZ2], based on the formalism developed in [DTZ1], when R is a smooth affine domain of dimension d over an infinite perfect field k of characteristic unequal to 2. If $k = \mathbb{R}$, it was proved in [DTZ2] that δ_R is a morphism of groups but at that time it was not clear whether δ_R is a morphism in general. In this article we prove that δ_R is indeed a morphism of groups. We believe this morphism will enable us to understand these two groups better, as it did in [DTZ2] when $k = \mathbb{R}$.

In Section 2 we recall the basics. In Section 3 we treat the special case when the group law in $Um_{d+1}(R)/E_{d+1}(R)$ is Mennicke-like (as this case is simpler and the treatment is entirely different). In Section 4 we treat the general case.

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2. Generalities: the objects and the maps

Notation. To avoid confusion, we shall write an ideal generated by $f_1, \cdots, f_{d+1}$ as $\langle f_1, \cdots, f_{d+1}\rangle$, whereas unimodular rows will be written with first brackets. Let $v = (a_1, \cdots, a_{d+1}) \in Um_{d+1}(R)$. The orbit of v in $Um_{d+1}(R)/E_{d+1}(R)$ will be written as $[v] = [a_1, \cdots, a_{d+1}]$.

2.1. **The Euler class group.** Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. Let B be the set of pairs (m, ω_m) where m is a maximal ideal of R and $\omega_m : (R/m)^d \twoheadrightarrow m/m^2$. Let G be the free abelian group generated by B. Let $J = m_1 \cap \cdots \cap m_r$, where m_i are distinct maximal ideals of R. Any $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ induces surjections $\omega_i : (R/m_i)^d \twoheadrightarrow m_i/m_i^2$ for each i. We associate $(J, \omega_J) := \sum_1^r (m_i, \omega_i) \in G$. Now, Let S be the set of elements (J, ω_J) of G for which ω_J has a lift to a surjection $\theta : R^d \twoheadrightarrow J$ and H be the subgroup of G generated by S . The Euler class group $E^d(R)$ is defined as $E^d(R) := G/H$.

Remark 2.1. The above definition appears to be slightly different from the one given in [BRS 1]. However, note that if $(J, \omega_J) \in S$ and if $\overline{\sigma} \in E_d(R/J)$, then the element $(J, \omega_J\overline{\sigma})$ is also in S. For details, see [DZ, Proposition 2.2].

Theorem 2.2. [BRS 1, 4.11] *Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. Let $J \subset R$ be a reduced ideal of height d and $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ be a surjection. Then, the following are equivalent:*

(1) *The image of $(J, \omega_J) = 0$ in $E^d(R)$*
(2) *ω_J can be lifted to a surjection $\theta : R^d \twoheadrightarrow J$.*

Remark 2.3. We shall refer to the elements of the Euler class group as *Euler cycles*. An arbitrary element of $E^d(R)$ can be represented by a single Euler cycle (J, ω_J), where J is a reduced ideal of height d and $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ is a surjection (see [BRS 1, Remark 4.14]).

The following notation will be used in the rest of this article.

Notation. Let $\dim(R) = d$. Let $(J, \omega_J) \in E^d(R)$ and $u \in R$ be a unit modulo J. Let σ be any diagonal matrix in $GL_d(R/J)$ with determinant $\overline{u}$ (bar means modulo J). We shall denote the composite surjection

$$(R/J)^d \overset{\sigma}{\tilde{\to}} (R/J)^d \overset{\omega_J}{\twoheadrightarrow} J/J^2$$

by $\overline{u}\omega_J$. It is easy to check that the element $(J, \overline{u}\omega_J) \in E^d(R)$ is independent of σ (the key fact used here is that $SL_d(R/J) = E_d(R/J)$ as $\dim(R/J) = 0$).

2.2. **[The map $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$].** Let R be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k with $\mathrm{Char}(k) \neq 2$. Let $(J, \omega_J) \in E^d(R)$, where J is a reduced ideal of height d. Now $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ is given by $J = \langle a_1, \cdots, a_d \rangle + J^2$, for some $a_1, \cdots, a_d \in J$. Applying the Nakayama Lemma one obtains $s \in J^2$ such that $J = \langle a_1, \cdots, a_d, s \rangle$ with $s - s^2 = a_1 b_1 + \cdots + a_d b_d$ for some $b_1, \cdots, b_d \in R$ (see [Mo] for a proof). Based on the formalism in [DTZ1], in [DTZ2] we defined a set-theoretic map $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ which takes (J, ω_J) to the orbit $[2a_1, \cdots 2a_d, 1-2s] \in Um_{d+1}(R)/E_{d+1}(R)$. A lot of technical work has gone into proving that δ_R is well-defined. See [DTZ1, Section 3] for the preparatory steps and [DTZ2, Section 2] for the details of this definition.

Remark 2.4. Note that $(1-2s)^2 \equiv 1$ modulo the ideal $\langle 2a_1, \cdots 2a_d \rangle$. Conversely, let an orbit $[v] = [x_1, \cdots, x_d, z] \in Um_{d+1}(R)/E_{d+1}(R)$ be such that the ideal $\langle x_1, \cdots, x_d \rangle$ is reduced of height d, and $z^2 \equiv 1$ modulo $\langle x_1, \cdots, x_d \rangle$, then $[v]$ is in the image of δ_R.

3. Special case: Mennicke-like group structure

We will say that the group structure on $Um_{d+1}(R)/E_{d+1}(R)$ is *Mennicke-like*[1] if for two orbits $[a_1, \cdots, a_d, x], [a_1, \cdots, a_d, y] \in Um_{d+1}(R)/E_{d+1}(R)$ we have the *coordinate-wise product*:

$$[a_1, \cdots, a_d, x][a_1, \cdots, a_d, y] = [a_1, \cdots, a_d, xy].$$

Throughout this section, let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k such that $2R = R$.

Lemma 3.1. *Let the group structure on $Um_{d+1}(R)/E_{d+1}(R)$ be Mennicke-like. Let $(J, \omega_J) \in E^d(R)$ be any element. Then $\delta_R((J, \omega_J))$ is 2-torsion.*

Proof. Let ω_J be induced by $J = \langle a_1, \cdots, a_d \rangle + J^2$. Then, there exists $s \in J^2$ such that $J = \langle a_1, \cdots, a_d, s \rangle$ with $s - s^2 \in \langle a_1, \cdots, a_d \rangle$. By definition, $\delta_R((J, \omega_J)) = [1 - 2s, 2a_1, \cdots, 2a_d]$. As the group law is Mennicke-like,

$$[1-2s, 2a_1, \cdots, 2a_d]^2 = [(1-2s)^2, 2a_1, \cdots, 2a_d] = [1, 2a_1, \cdots, 2a_d]. \quad \square$$

Theorem 3.2. *Let the group structure on $Um_{d+1}(R)/E_{d+1}(R)$ be Mennicke-like. Then δ is a morphism of groups.*

Proof. Let $(J, \omega_J), (K, \omega_K) \in E^d(R)$ be such that $J + K = R$, where J, K are both reduced ideals of height d. Then $(J, \omega_J) + (K, \omega_K) = (J \cap K, \omega_{J \cap K})$, where $\omega_{J \cap K}$ is induced by ω_J and ω_K. To prove the theorem, it is enough to show that

$$\delta_R((J, \omega_J)) * \delta_R((K, \omega_K)) = \delta_R((J \cap K, \omega_{J \cap K})),$$

[1] In literature it has been described as *nice group structure*. Ravi Rao suggested us to use the term *Mennicke-like*.

where $*$ denotes the product in $Um_{d+1}(R)/E_{d+1}(R)$

Let $\omega_{J\cap K}$ be induced by $J\cap K = \langle a_1, \cdots, a_d\rangle + (J\cap K)^2$. Then $J = \langle a_1, \cdots, a_d\rangle + J^2$ and $K = \langle a_1, \cdots, a_d\rangle + K^2$. Let $J = \langle a_1, \cdots, a_d, s\rangle$ with $s - s^2 \in \langle a_1, \cdots, a_d\rangle$ and $K = \langle a_1, \cdots, a_d, t\rangle$ with $t - t^2 \in \langle a_1, \cdots, a_d\rangle$, as usual. Then it follows that $J \cap K = \langle a_1, \cdots, a_d, st\rangle$ and $st - s^2t^2 \in \langle a_1, \cdots, a_d\rangle$.

By the definition of the map δ, we have:

(1) $\delta((J, \omega_J)) = [1 - 2s, 2a_1, \cdots, 2a_d]$,
(2) $\delta((K, \omega_K)) = [1 - 2t, 2a_1, \cdots, 2a_d]$,
(3) $\delta((J \cap K, \omega_{J\cap K})) = [1 - 2st, 2a_1, \cdots, 2a_d]$.

As the group law in $Um_{d+1}(R)/E_{d+1}(R)$ is Mennicke-like, we have

$$[1 - 2s, 2a_1, \cdots, 2a_d][1 - 2t, 2a_1, \cdots, 2a_d] = [1 - 2s - 2t + 4st, 2a_1, \cdots, 2a_d].$$

Let us try to locate a pre-image of the element on the right hand side of the above equation. To this end, we consider the following ideal

$$L = \langle a_1, \cdots, a_d, s + t - 2st\rangle = \langle a_1, \cdots, a_d, s^2 + t^2 - 2st\rangle = \langle a_1, \cdots, a_d, (s - t)^2\rangle$$

in R and note that $L + J \cap K = R$ (as $s - t$ is a unit modulo $J \cap K$). Let 'bar' denote modulo $\langle a_1, \cdots, a_d\rangle$. Then,

$$\overline{L} \cap \overline{J \cap K} = \langle \overline{s}\overline{t}\rangle\langle \overline{s} + \overline{t} - \overline{2st}\rangle = \langle \overline{s^2t} + \overline{st^2} - 2\overline{s}\overline{t}\rangle = \langle \overline{s}\overline{t} + \overline{s}\overline{t} - 2\overline{s}\overline{t}\rangle = \langle \overline{0}\rangle,$$

and we have $L\cap(J\cap K) = \langle a_1, \cdots, a_d\rangle$. Therefore, $(L, \omega_L) + (J\cap K, \omega_{J\cap K}) = 0$, where ω_L is induced by the images of $a_1, \cdots, a_d$ in L/L^2. It is easy to see that $\delta((L, \omega_L)) = [1 - 2s - 2t + 4st, 2a_1, \cdots, 2a_d]$. Finally, we conclude (using (3.1)) that

$$\delta((J, \omega_J)) * \delta((K, \omega_K)) = \delta((L, \omega_L)) = \delta((J \cap K, \omega_{J\cap K}))^{-1} = \delta((J \cap K, \omega_{J\cap K})). \quad \square$$

4. General case

In this section treat the general case. Our line of arguments may be termed as *"Mennicke-Newman for ideals"*. For the Mennicke-Newman Lemma for elementary orbits of unimodular rows, see [vdK 3, Lemma 3.2].

Lemma 4.1. *Let I_1, I_2 be two comaximal ideals in a ring R such that $I_1 \neq I_1^2$ and $I_2 \neq I_2^2$. Then we can find $x \in I_1 \setminus I_1^2$ and $y \in I_2 \setminus I_2^2$ such that $x + y = 1$.*

Proof. As $I_1^2 + I_2^2 = R$, we can find $a \in I_1^2$, $b \in I_2^2$ such that $a + b = 1$.

Claim: $I_1 \cap I_2 \not\subseteq I_1^2$. To see this note that $I_1^2 + I_2 = R$, and we have

$$I_1 = I_1 \cap R = I_1 \cap (I_1^2 + I_2) = I_1^2 + I_1 \cap I_2.$$

If $I_1 \cap I_2 \subseteq I_1^2$, then $I_1 = I_1^2$, contrary to the hypothesis. Similarly, $I_1 \cap I_2 \not\subseteq I_2^2$.

Therefore, we can choose $\alpha \in I_1 \cap I_2 \smallsetminus (I_1^2 \cup I_2^2)$. Take $x = a - \alpha$ and $y = b + \alpha$ to conclude. □

Proposition 4.2. *Let R be a ring of dimension $d \geq 2$. Let $J = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$ and $K = \mathfrak{m}_{r+1} \cap \cdots \cap \mathfrak{m}_s$ be two ideals, each of height d, where $\mathfrak{m}_i$ are all disinct maximal ideals for $i = 1, \cdots, s$. Then, there exist $x \in J$ and $y \in K$ such that:*

(1) $x + y = 1$,
(2) $x \notin \mathfrak{m}_1^2 \cup \cdots \mathfrak{m}_s^2$ *and* $y \notin \mathfrak{m}_1^2 \cup \cdots \mathfrak{m}_s^2$.

Proof. As $J^2 + K^2 = R$, we can find $a \in J^2$ and $b \in K^2$ such that $a + b = 1$. We claim that there exists $c \in J \cap K$ such that $c \notin \mathfrak{m}_1^2 \cup \cdots \mathfrak{m}_s^2$. If we can prove the claim, we will take $x = a - c$ and $y = a + c$ to prove the proposition.

Proof of the claim. We have $\mathfrak{m}_1^2 + \mathfrak{m}_2^2 \cdots \mathfrak{m}_s^2 = R$. Choose $f \in \mathfrak{m}_1^2$ and $g \in \mathfrak{m}_2^2 \cdots \mathfrak{m}_s^2$ so that $f + g = 1$.

Observe that $\mathfrak{m}_1 \cap (\mathfrak{m}_2^2 \cdots \mathfrak{m}_s^2) \not\subseteq \mathfrak{m}_1^2$ (to see this, use the above lemma to obtain $z \in \mathfrak{m}_1 \smallsetminus \mathfrak{m}_1^2$ and $w \in \mathfrak{m}_2^2 \cdots \mathfrak{m}_s^2$ so that $z + w = 1$. Assume, if possible, that $\mathfrak{m}_1 \cap (\mathfrak{m}_2^2 \cdots \mathfrak{m}_s^2) \subseteq \mathfrak{m}_1^2$. As $z = z^2 + wz$ and $wz \in \mathfrak{m}_1 \cap (\mathfrak{m}_2^2 \cdots \mathfrak{m}_s^2)$ it would follow that $z \in \mathfrak{m}_1^2$. Contradiction.)

Choose $\alpha \in \mathfrak{m}_1 \cap (\mathfrak{m}_2^2 \cdots \mathfrak{m}_s^2) \smallsetminus \mathfrak{m}_1^2$ and take $c_1 = f - \alpha$, $c_1' = g + \alpha$. Then, we have: (1) $c_1 + c_1' = 1$, (2) $c_1 \in \mathfrak{m}_1 \smallsetminus \mathfrak{m}_1^2$, (3) $c_1 \equiv 1$ modulo $\mathfrak{m}_i^2$ for all $i \neq 1$.

Following a similar method, for each $i = 1, \cdots, s$, choose $c_i \in \mathfrak{m}_i \smallsetminus \mathfrak{m}_i^2$ so that $c_i \equiv 1$ modulo $\mathfrak{m}_1^2 \cdots \mathfrak{m}_{i-1}^2 \mathfrak{m}_{i+1}^2 \cdots \mathfrak{m}_s^2$. Take $c = \prod_{i=1}^s c_i$. Then $c \in \mathfrak{m}_1 \cdots \mathfrak{m}_s$ and it is easy to check that $c \notin \mathfrak{m}_i^2$ for any i. This completes the proof of the claim. □

Theorem 4.3. *Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k of characteristic unequal to 2. Then $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ is a morphism of groups.*

Proof. Let $(J, \omega_J), (K, \omega_K) \in E^d(R)$ be such that $J + K = R$, where J, K are both reduced ideals of height d. Then $(J, \omega_J) + (K, \omega_K) = (J \cap K, \omega_{J\cap K})$, where $\omega_{J\cap K}$ is induced by ω_J and ω_K. To prove the theorem, it is enough to show that

$$\delta_R((J, \omega_J)) * \delta_R((K, \omega_K)) = \delta_R((J \cap K, \omega_{J\cap K})),$$

where $*$ denotes the product in $Um_{d+1}(R)/E_{d+1}(R)$.

Let $J = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$ and $K = \mathfrak{m}_{r+1} \cap \cdots \cap \mathfrak{m}_s$. Applying the above proposition, choose $x \in J$ and $y \in K$ such that $x + y = 1$ and $x \notin \mathfrak{m}_1^2 \cup \cdots \mathfrak{m}_s^2$ and $y \notin \mathfrak{m}_1^2 \cup \cdots \mathfrak{m}_s^2$. Then $xy \in (J \cap K) \smallsetminus (J \cap K)^2$. As $x + y = 1$, it is easy to check that for each i, the image of xy in $\mathfrak{m}_i/\mathfrak{m}_i^2$ is not trivial. Therefore, xy is a part of a basis of $\mathfrak{m}_i/\mathfrak{m}_i^2$, for each i, $1 \leq i \leq s$. Consequently, xy is a part of generators of $(J \cap K)/(J \cap K)^2$ Similarly, x is a part of generators of J/J^2 and y is a part of generators of K/K^2.

Let $J \cap K = \langle xy, a_1, \cdots, a_{d-1}) + (J \cap K)^2$ for some $a_1, \cdots, a_{d-1} \in J \cap K$. Let $\omega'_{J\cap K} : (R/(J \cap K))^d \twoheadrightarrow (J \cap K)/(J \cap K)^2$ denote the corresponding surjection. By [BRS 3, 2.2 and 5.0] there is a unit u modulo $J \cap K$ such that $(J \cap K, \omega_{J\cap K}) = (J \cap K, u\omega'_{J\cap K})$ in $E^d(R)$. Therefore, $(J \cap K, \omega_{J\cap K})$ is given by $J \cap K = \langle xy, ua_1, a_2, \cdots, a_{d-1}\rangle + (J \cap K)^2$. Similarly, $J = \langle x, ua_1, a_2, \cdots, a_{d-1}\rangle + J^2$ gives (J, ω_J) and $K = \langle x, ua_1, a_2, \cdots, a_{d-1}\rangle + K^2$ gives (K, ω_K).

We can choose $s \in J \cap K$ such that $s - s^2 \in \langle xy, ua_1, a_2, \cdots, a_{d-1}\rangle$ and $J \cap K = \langle xy, ua_1, a_2, \cdots, a_{d-1}, s\rangle$. As $s-s^2 \in \langle x, ua_1, a_2, \cdots, a_{d-1}\rangle$ and $s-s^2 \in \langle y, ua_1, a_2, \cdots, a_{d-1}\rangle$ as well, it follows that (J, ω_J) corresponds to $J = \langle x, ua_1, a_2, \cdots, a_{d-1}, s\rangle$ and (K, ω_K) corresponds to $K = \langle y, ua_1, a_2, \cdots, a_{d-1}, s\rangle$ (to check this, use $x + y = 1$).

We then have,

(1) $\delta_R((J, \omega_J)) = [2x, 2ua_1, 2a_2, \cdots, 2a_{d-1}, 1 - 2s] = [x, ua_1, 2a_2, \cdots, 2a_{d-1}, 1 - 2s]$,
(2) $\delta_R((K, \omega_K)) = [2y, 2ua_1, 2a_2, \cdots, 2a_{d-1}, 1-2s] = [y, ua_1, 2a_2, \cdots, 2a_{d-1}, 1-2s]$,
(3) $\delta_R((J\cap K, \omega_{J\cap K})) = [2xy, 2ua_1, 2a_2, \cdots, 2a_{d-1}, 1-2s] = [xy, ua_1, 2a_2, \cdots, 2a_{d-1}, 1-2s]$.

It follows that $\delta_R((J, \omega_J)) * \delta_R((K, \omega_K)) = \delta_R((J \cap K, \omega_{J\cap K}))$, as $x + y = 1$. □

5. A FEW REMARKS

Let $X = \text{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Let $X(\mathbb{R})$ denote the set of real points of X. Assume that $X(\mathbb{R}) \neq \emptyset$. Then $X(\mathbb{R})$ is a smooth real manifold. Let $\mathbb{R}(X)$ denote the ring obtained from R by inverting all the functions which do not have any real zeros. We can apply (3.2) to obtain the following result.

Theorem 5.1. *Let $X = Spec(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$ such that $X(\mathbb{R})$ is orientable, and the number of compact connected componenets of $X(\mathbb{R})$ is at least one. Then the group structure on $Um_{d+1}(R)/E_{d+1}(R)$ can never be Mennicke-like.*

Proof. In our earlier paper [DTZ2], we proved the following assertions:

(1) $Um_{d+1}(R)/E_{d+1}(R) = Um_{d+1}(\mathbb{R}(X))/E_{d+1}((\mathbb{R}(X)) \bigoplus K$,
(2) $\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \longrightarrow Um_{d+1}(\mathbb{R}(X))/E_{d+1}((\mathbb{R}(X))$ is an isomorphism,
(3) $Um_{d+1}(\mathbb{R}(X))/E_{d+1}((\mathbb{R}(X)) \xrightarrow{\sim} \bigoplus_t \mathbb{Z}$, where t is the number of compact connected components of $X(\mathbb{R})$.

Now, assume that the group structure on $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is Mennicke-like. Then, by (3.2), we shall find non-trivial orbits which are 2-torsion. But as $t \geq 1$, the group $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is non-trivial and is free abelian. Thus we arrive at a contradiction. As $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is a subgroup of $Um_{d+1}(R)/E_{d+1}(R)$, the theorem follows. □

Remark 5.2. In [DTZ2] we also computed the universal Mennicke symbol $MS_{d+1}(R)$, where R is as in the above theorem. It follows from there as well that the group structure on $Um_{d+1}(R)/E_{d+1}(R)$ can never be Mennicke-like. The arguments given above only avoids the computation of $MS_{d+1}(R)$.

We now comment on the case when $X(\mathbb{R})$ is non-orientable.

Theorem 5.3. *Let $X = Spec(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$ such that $X(\mathbb{R})$ is non-orientable. Then $\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \longrightarrow Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is a surjective morphism. As a consequence, $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is an $\mathbb{Z}/2\mathbb{Z}$-vector space of dimension $\leq t$, where t is the number of compact connected componenets of $X(\mathbb{R})$.*

Proof. It has already been proved in [DTZ2, Theorem 3.2] that $\delta_{\mathbb{R}(X)}$ is surjective. In this article we proved that $\delta_{\mathbb{R}(X)}$ is a morphism. As $E^d(\mathbb{R}(X)) = \bigoplus_t \mathbb{Z}/2\mathbb{Z}$, the result follows. □

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STAT-MATH UNIT, INDIAN STATISTICAL INSTITUTE, 203 B. T. ROAD, KOLKATA 700108 INDIA
E-mail address: mrinal@isical.ac.in

STAT-MATH UNIT, INDIAN STATISTICAL INSTITUTE, 203 B. T. ROAD, KOLKATA 700108 INDIA
E-mail address: tsoumi_r@isical.ac.in

SCHOOL OF MATHEMATICAL SCIENCES, NISER - BHUBANESWAR, P.O.: BHIMPUR-PADANPUR, VIA - JATNI, DISTRICT - KHURDA, ODISHA 752050, INDIA.
E-mail address: zinna2012@gmail.com