Revisiting Nori's question and homotopy invariance of Euler class groups

Mrinal Kanti Das
Stat-Math Unit, Indian Statistical Institute
203 B. T. Road, Kolkata 700108 India
e-mail : mrinal@isical.ac.in

August 6, 2010

Dedicated to the memory of Amit Roy

Abstract. This paper examines the relation between the Euler class group of a Noetherian ring and the Euler class group of its polynomial extension. When the ring is a smooth affine domain, the two groups are canonically isomorphic. This is a consequence of a theorem of Bhatwadekar-Sridharan, which they proved in order to answer a question of Nori on sections of projective modules over such rings. If the smoothness assumption is removed, the result of Bhatwadekar-Sridharan is no longer valid and also the Euler class groups above are not in general isomorphic. In this paper we investigate a variant of Nori's question for arbitrary Noetherian rings and derive several consequences to understand the relation between various groups in the theory of Euler classes.

Keywords : Projective modules; Efficient generation of ideals; Euler class groups.

Mathematics Subject Classification (2000). 13C10, 13B25, 19A15.

1 Introduction

Motivated by a topological result, which he proved in an appendix to a paper of Mandal [M 2], Nori asked the following question on sections of projective modules (see [M 2], Introduction or [B-RS 1]).

Question 1.1 (*Nori*) Let A be a smooth affine domain of (Krull) dimension d over a field k and let $I \subset A[T]$ be an ideal of height n where $2n \geq d+3$. Let P be a projective A-module of rank n. Assume that there is a surjection $\varphi : P[T] \twoheadrightarrow I/(I^2T)$. Then, can φ be lifted to a surjection $\theta : P[T] \twoheadrightarrow I$?

Nori's question has been settled in the affirmative by Bhatwadekar-Keshari in [B-K] when k is infinite perfect. Prior to their solution, Nori's question was answered in many important cases (see [M 2, M-V, B-RS 1]). Of those, we quote below the version most relevant to this paper, namely, the case when P is free and $d = n$. This result is due to Bhatwadekar-Sridharan [B-RS 1, Theorem 3.8]. In this paper we shall call it the *"homotopy theorem"*.

Theorem 1.2 *(Homotopy theorem) Let A be a smooth affine domain of dimension $n \geq 3$ over an infinite perfect field k. Let $I \subset A[T]$ be an ideal of height n such that $I = (f_1, \cdots, f_n) + (I^2T)$. Then, there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ with $g_i - f_i \in (I^2T)$.*

However, the *homotopy theorem* is no longer true in general if we drop the smoothness assumption. There is an example by Bhatwadekar, Mohan Kumar and Srinivas [B-RS 1, Example 6.4], where they consider a normal affine domain A of dimension 3 over $\mathbb{C}$, construct an ideal I of $A[T]$ of height 3 such that $I = (f_1, f_2, f_3) + (I^2T)$, and show that this set of generators of $I/(I^2T)$ cannot be lifted to a set of 3 generators of I. But we note that in their example, the ideal I is not generated by 3 elements. Therefore, we may ask the following natural question (in a more general set up).

Question 1.3 Let A be a Noetherian ring of dimension $n \geq 3$. Let $I \subset A[T]$ be an ideal of height n such that I *is generated by* n *elements*. Suppose that $I = (f_1, \cdots, f_n) + (I^2T)$. Do there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ and $g_i = f_i$ mod (I^2T)?

In this paper we investigate the above question (1.3). Both the *homotopy theorem* and Question 1.3 are closely related to the theory of Euler class groups. Before describing our results, let us briefly illustrate the development of the theory of Euler class groups. For the moment assume that A is a smooth affine domain of dimension n over an infinite perfect field k. To detect obstruction to splitting off a free summand of rank one from a projective A-module P of rank n (with trivial determinant), Nori introduced the nth *Euler class group* $E^n(A)$ of A and attached to P an element in this group, called the *Euler class* of P, and asked whether the vanishing of the Euler class of P is both necessary and sufficient for P having a rank one free summand. Bhatwadekar-Sridharan came up with an affirmative answer in [B-RS 1]. Since then understanding the Euler class group has been the object of study for several research papers.

In [B-RS 1], the *homotopy theorem* was crucially used to give an alternative description of the Euler class group (equivalent to the original definition of Nori) which enabled Bhatwadekar-Sridharan to prove their results on Euler classes. Moreover, their equivalent definition paved the way for generalising Euler class theory to singular varieties, a set up where the *homotopy theorem* is no longer valid. This was done by Bhatwadekar-Sridharan in [B-RS 3], where they defined $E^n(A)$ for a commutative Noetherian ring A (containing $\mathbb{Q}$) of dimension n (In 2.3 we recall their definition of the Euler class group and remark on the non-triviality of the tangent bundle of the real 2-sphere in terms of its Euler class). Following their lead, we could successfully extend in [D 1] the theory to define $E^n(A[T])$, the nth Euler class group of $A[T]$ where A is just as in the preceding sentence. From now on we write $E(A)$ instead of $E^n(A)$ and $E(A[T])$ instead of $E^n(A[T])$. We recall the definition of $E(A[T])$ in 2.10. We warn the reader that the definition of $E(A[T])$ is different from that of $E(A)$ and is not obtained by replacing A by $A[T]$ (see Remark 2.9).

One of our objectives is to understand the relation between the Euler class groups $E(A)$ and $E(A[T])$ which we did to some extent in [D 1, D 2, D-RS], following the "Quillen-Suslin model" for $K_0(A)$ and $K_0(A[T])$. There is a canonical group homomorphism $\Phi : E(A) \longrightarrow E(A[T])$ which is injective. Naturally one wonders whether Φ is surjective or not. Now, to an element (J, ω_J) of $E(A[T])$ one can associate an ideal $I \subset A[T]$ of height n and a surjection $\overline{\alpha} : A[T]^n \twoheadrightarrow I/(I^2T)$ in such a way that (J, ω_J) has a preimage in $E(A)$ if and only if $\overline{\alpha}$ can be lifted to a surjection $\alpha : A[T]^n \twoheadrightarrow I$ (for a proof see Lemma 3.1). The *homotopy theorem* tells us that such a phenomenon occurs when A is a smooth affine domain and we obtain the following "homotopy invariance" for Euler class groups : $\Phi : E(A) \longrightarrow E(A[T])$ is an isomorphism when A is a smooth affine domain [D 1, Proposition 5.7]. Putting it another way, this is an analogue of the so called Bass-Quillen conjecture and one would expect Φ to be an isomorphism when A is a regular ring (see Question 3.24).

However, as we mentioned earlier, if A is not smooth then the *homotopy theorem* is no longer true. Also, Φ is not surjective in general (see Remark 3.4). The failure of the *homotopy theorem* (and that of the surjectivity of Φ) motivates us to investigate Question 1.3.

If dim $A = n$ is even, we prove that Question 1.3 has an affirmative answer (see Theorem 3.8). The proof is not much difficult in this case. As always, the case when n is odd is more mysterious and in this paper we prove the following theorem for general n. Here $E_0(A[T])$ is the weak Euler class group of $A[T]$ (for the definition, see 2.11 or [D 1]).

Theorem 1.4 *Let A be an affine algebra of dimension $n \geq 3$. Let $I \subset A[T]$ be an ideal of height n such that I is generated by n elements. Suppose that $I = (f_1, \cdots, f_n) + (I^2T)$. Assume further that the kernel of the canonical surjection $E(A[T]) \twoheadrightarrow E_0(A[T])$ has no nontrivial 2-torsion. Then, there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ and $g_i = f_i$ mod (I^2T).*

We also derive certain variants of the above results with weaker hypotheses on I which enable us to investigate the relation between various relevant groups better. These questions are discussed in detail at the beginning of Section 3.

A few words about the assumption on 2-torsion in the kernel of $E(A[T]) \twoheadrightarrow E_0(A[T])$. We may ask if there are examples, at all, of such 2-torsion. The question of existence of such 2-torsion was asked by Bhatwadekar-Sridharan in [B-RS 3, Question 7.11] for the kernel of $E(A) \twoheadrightarrow E_0(A)$. Recently Bhatwadekar constructed an example (personal communication) of a two-dimensional ring A containing $\mathbb{Q}(\sqrt{-1})$ such that the kernel of $E(A) \twoheadrightarrow E_0(A)$ has 2-torsion (for details see Section 4). But we do not know yet what happens when dim A is odd.

Section 5 contains some results about the existence of unimodular elements in projective modules over $A[T]$. In Section 2 we recall the definitions of the Euler class groups and the weak Euler class groups and record some results for later use.

2 Preliminaries

We make the blanket assumption that all the rings in this paper are commutative, Noetherian and contain the field of rationals. By the dimension of a ring we mean its Krull dimension. All the modules are assumed to be finitely generated and the projective modules are assumed to have constant rank.

Definition 2.1 Let A be a commutative Noetherian ring and P be a projective A-module. An element $p \in P$ is said to be *unimodular* if there exists $\theta \in \text{Hom}_A(P, A)$ such that $\theta(p) = 1$.

A classical result of Serre asserts that if rank$(P) > \dim(A)$ then P has a unimodular element. That Serre's result cannot be improved can be seen from the following well-known example.

Example 2.2 Let $A = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$. Let P be the kernel of the surjection $\alpha : A^3 \twoheadrightarrow A$, which is defined by $\alpha(e_1) = x, \alpha(e_2) = y, \alpha(e_3) = z$. Let $v = [x, y, z] \in A^3$. Since $\alpha[x, y, z] = 1$, v is a unimodular element of A^3 and $A^3 = Av \oplus P$. Hence P is a projective A-module of rank 2. However, P does not have a unimodular element (in particular P is not free). We sketch a proof of this result.

Proof. We first observe that an A-linear map $\beta : P \to A$ can be extended uniquely to an A-linear map from $A^3 \to A$ (which, by abuse of notation, we still denote by β) such that $\beta(v) = 0$. Now, given a map $\beta : A^3 \to A$ with $\beta(v) = 0$, we can associate a continuous vector field on the real two sphere S^2 as follows:

Let $\beta(e_i) = f_i(x, y, z),\ 1 \le i \le 3$ and let $F_i(X, Y, Z) \in \mathbb{R}[X, Y, Z]$ be preimages of $f_i(x, y, z)$. Then, the assignment $(a, b, c) \mapsto (F_1(a, b, c), F_2(a, b, c), F_3(a, b, c))$ gives rise to a continuous map from S^2 to the Euclidean 3-space $\mathbb{R}^3$ which depends only on elements $f_i(x, y, z) \in A$ and not on the choice of preimages $F_i(X, Y, Z)$. Moreover, since $\beta[x, y, z] = 0$, it follows that the vector $(F_1(a, b, c), F_2(a, b, c), F_3(a, b, c))$ is perpendicular to the vector (a, b, c). Therefore, the assignment sending $(a, b, c) \in S^2$ to the element $(a, b, c) + (F_1(a, b, c), F_2(a, b, c), F_3(a, b, c)) \in \mathbb{R}^3$, defines a continuous vector field on S^2.

We note that this vector field vanishes at precisely those points $(a, b, c) \in S^2$ such that $F_i(a, b, c) = 0$ for $i = 1, 2, 3$. If J is the ideal $(f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$

of A, then the zeroes of the corresponding vector field are given by the maximal ideals m of A such that $J \subset m$ and $A/m \xrightarrow{\sim} \mathbb{R}$.

In particular, if P has a unimodular element, then it follows that there exists a surjection $\gamma : A^3 \twoheadrightarrow A$, $\gamma(v) = 0$. But then we obtain, by the above process, a nowhere vanishing continuous vector field on S^2. However, this contradicts a well known topological result. ∎

Examples like the above one motivated the notion of the Euler class group. For a smooth affine domain R of dimension n, Nori introduced the *Euler class group* to detect the obstruction for a projective R-module P of rank n to have a unimodular element (or, equivalently, to split a free summand of rank one). Here we recall the definition of the Euler class group of a Noetherian ring as given in [B-RS 3]. This definition is different from Nori's original definition but if the ring is a smooth affine domain, both the definitions are equivalent [B-RS 1, 4.6].

Definition 2.3 (**The Euler class group** $E(A)$) Let A be a commutative Noetherian ring (containing $\mathbb{Q}$) of dimension $n \geq 2$. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Two surjections α, β from $(A/J)^n$ to J/J^2 are said to be related if there exists $\sigma \in SL_n(A/J)$ such that $\alpha\sigma = \beta$. Clearly this is an equivalence relation on the set of surjections from $(A/J)^n$ to J/J^2. Let $[\alpha]$ denote the equivalence class of α. Such an equivalence class $[\alpha]$ is called a *local orientation* of J. By abuse of notation, we shall identify an equivalence class $[\alpha]$ with α. A local orientation α is called a *global orientation* if $\alpha : (A/J)^n \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : A^n \twoheadrightarrow J$. Let G be the free abelian group on the set of pairs $(\mathcal{N}, \omega_{\mathcal{N}})$ where $\mathcal{N}$ is an $\mathcal{M}$-primary ideal for some maximal ideal $\mathcal{M}$ of height n such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements and $\omega_{\mathcal{N}}$ is a local orientation of $\mathcal{N}$. Now let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements and ω_J be a local orientation of J. Let $J = \cap_i \mathcal{N}_i$ be the (irredundant) primary decomposition of J. We associate to the pair (J, ω_J), the element $\sum_i (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ of G where $\omega_{\mathcal{N}_i}$ is the local orientation of $\mathcal{N}_i$ induced by ω_J. By abuse of notation, we denote $\sum_i (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ by (J, ω_J). Let H be the subgroup of G generated by the set of pairs (J, ω_J), where J is an ideal of height n and ω_J is a global orientation of J. The Euler class group of A is $E(A) \stackrel{\text{def}}{=} G/H$.

Definition 2.4 (**The Euler class of a projective A-module**) Let P be a projective A-module of rank n such that $A \simeq \wedge^n(P)$ and let $\chi : A \xrightarrow{\sim} \wedge^n P$ be an isomorphism. Let $\varphi : P \twoheadrightarrow J$ be a surjection where J is an ideal of height n. Therefore we obtain an induced surjection $\overline{\varphi} : P/JP \twoheadrightarrow J/J^2$. Let $\overline{\gamma} : (A/J)^n \simeq P/JP$, be an isomorphism such that $\wedge^n(\overline{\gamma}) = \overline{\chi}$. Let ω_J be the local orientation of J given by $\overline{\varphi}\,\overline{\gamma} : (A/J)^n \twoheadrightarrow J/J^2$. Let $e(P, \chi)$ be the image in $E(A)$ of the element (J, ω_J) of G. The assignment sending the pair (P, χ) to the element $e(P, \chi)$ of $E(A)$ is well defined (see [B-RS 3]). The *Euler class* of (P, χ) is defined to be $e(P, \chi)$.

Theorem 2.5 *[B-RS 3] Let A be a commutative Noetherian ring of dimension $n \geq 2$.*

1. *An element (I, ω_I) is zero in $E(A)$ if and only if ω_I is a global orientation.*
2. *Let P be a projective A-module of rank n together with an isomorphism $\chi : A \xrightarrow{\sim} \wedge^n P$. Then $e(P, \chi) = 0$ in $E(A)$ if and only if P has a unimodular element.*

Remark 2.6 For a smooth affine variety $X = \text{Spec}(A)$ over $\mathbb{R}$, the Euler class group has been extensively studied in [B-RS 2, B-D-M]. We may mention here that if A is the coordinate ring of the 2-sphere over $\mathbb{R}$ (as in Example 2.2), and if P is the projective A-module corresponding to the tangent bundle, then $E(A) = \mathbb{Z}$ (generated by (m, ω_m) where m is any real maximal ideal and ω_m is any local orientation of m) and the Euler class of P is 2 (up to orientation). More generally, for smooth orientable affine varieties over the reals (with at least one real point), the Euler class of the tangent bundle is same as the Euler characteristic of the smooth manifold of real points of the variety.

Definition 2.7 (**The weak Euler class group $E_0(A)$**) Let S be the set of ideals $\mathcal{N}$ of A where $\mathcal{N}$ is an $\mathcal{M}$-primary ideal of height n such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements. Let G_0 be the free abelian group on S. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let $J = \cap_i \mathcal{N}_i$ be the (irredundant) primary decomposition of J. We associate to J, the element $\sum_i \mathcal{N}_i$ of G_0. By abuse of notation, we denote this element by (J). Let H_0 be the subgroup of G_0 generated by elements of the type (J), where J is an ideal of height n such that J is generated by n elements. The *weak Euler class group* of A is denoted by $E_0(A)$ and is defined as $E_0(A) \stackrel{\text{def}}{=} G_0/H_0$.

Remark 2.8 It is clear from the above definitions that there is an obvious canonical surjective group homomorphism from $E(A)$ to $E_0(A)$ which sends an element (J, ω_J) of $E(A)$ to (J) in $E_0(A)$.

Remark 2.9 We are now going to recall the definition of the Euler class group $E(A[T])$ where A is a Noetherian ring of dimension n. Here we must remark that this is actually the n-th Euler class group of $A[T]$ whereas the dimension of $A[T]$ is $n+1$. If one just follows Definition 2.3 and replaces A by $A[T]$, one obtains $E^{n+1}(A[T])$. Due to a result of Mandal [M 1, Theorem 1.2], $E^{n+1}(A[T])$ is trivial as any ideal of height $n+1$ in $A[T]$ contains a monic polynomial.

Definition 2.10 (**The Euler class group** $E(A[T])$) Let A be a Noetherian ring of dimension $n \geq 3$ containing $\mathbb{Q}$. Let $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Two surjections α and β from $(A[T]/I)^n \twoheadrightarrow I/I^2$ are said to be *related* if there exists $\sigma \in SL_n(A[T]/I)$ such that $\alpha\sigma = \beta$. This is an equivalence relation on the set of surjections from $(A[T]/I)^n$ to I/I^2. Let $[\alpha]$ denote the equivalence class of α. We call such an equivalence class $[\alpha]$ a *local orientation of* I. It was shown in [D 1, Proposition 4.4], that if $\alpha : (A[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$ then so can any β equivalent to α. We call a local orientation $[\alpha]$ of I a *global orientation of* I if the surjection $\alpha : (A[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$. Let G be the free abelian group on the set of pairs (I, ω_I) where $I \subset A[T]$ is an ideal of height n such that $\text{Spec}(A[T]/I)$ is connected, I/I^2 is generated by n elements and $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ is a local orientation of I. Let $I \subset A[T]$ be an ideal of height n and ω_I a local orientation of I. Now I can be decomposed uniquely as $I = I_1 \cap \cdots \cap I_r$, where the I_k's are ideals of $A[T]$ of height n, pairwise comaximal, such that $\text{Spec}(A[T]/I_k)$ is connected for each k. Clearly ω_I induces local orientations ω_{I_k} of I_k for $1 \leq k \leq r$. By (I, ω_I) we mean the element $\Sigma(I_k, \omega_{I_k})$ of G. Let H be the subgroup of G generated by set of pairs (I, ω_I), where I is an ideal of $A[T]$ of height n generated by n elements and ω_I is a global orientation of I given by the set of generators of I. We define the Euler class group of $A[T]$, denoted by $E(A[T])$, to be G/H.

The weak Euler class group $E_0(A[T])$ is defined in a similar way, just dropping the orientations, as follows:

Definition 2.11 (**The weak Euler class group** $E_0(A[T])$) Let F be the free abelian group on the set of ideals $\mathcal{I}$ where $\text{ht}\,\mathcal{I} = n$, $\mathcal{I}/\mathcal{I}^2$ is generated by n elements and $\text{Spec}(A[T]/\mathcal{I})$ is connected. For an ideal I of $A[T]$ of height n such that I/I^2 is generated by n elements, we take its decomposition into connected components (as above), say, $I = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_r$, and associate to I the element $(I) := \Sigma \mathcal{I}_k$ of F. Let K be the subgroup of F generated by elements of the type (I), where $I \subset A[T]$ is an ideal of height n and I is generated by n elements. We define $E_0(A[T])$ to be F/K. Clearly there is a canonical surjective group homomorphism from $E(A[T])$ to $E_0(A[T])$.

Definition 2.12 (**The Euler class of a projective** $A[T]$**-module**) Let P be a projective $A[T]$-module of rank n with trivial determinant. Fix a trivialization $\chi : A[T] \simeq \wedge^n(P)$. Let $\alpha : P \twoheadrightarrow I$ be a surjection such that I is an ideal of height n. Note that, since P has trivial determinant and $\dim A[T]/I \leq 1$, P/IP is a free $A[T]/I$-module. Composing $\alpha \otimes A[T]/I$ with an isomorphism $\gamma : (A[T]/I)^n \simeq P/IP$ with the property $\wedge^n(\gamma) = \chi \otimes A[T]/I$ we get a local orientation, say ω_I, of I. Let $e(P, \chi)$ be the image in $E(A[T])$ of the element (I, ω_I) of G. (We say that (I, ω_I) is obtained from the pair (α, χ)). It can be proved that the assignment sending the pair (P, χ) to $e(P, \chi)$ is well defined (see [D 1]). We define the *Euler class* of P to be $e(P, \chi)$.

Theorem 2.13 *[D 1] Let A be a Noetherian ring of dimension $n \geq 3$. Let $I \subset A[T]$ be an ideal of $A[T]$ of height n such that I/I^2 is generated by n elements and ω_I be a local orientation of I. Let P be a rank n projective $A[T]$-module with trivial determinant with a trivialization $\chi : A[T] \simeq \wedge^n(P)$.*
(a) Suppose that the image of (I, ω_I) is zero in $E(A[T])$. Then ω_I is a global orientation of I.
(b) Suppose that $e(P, \chi) = (I, \omega_I)$ in $E(A[T])$. Then there exists a surjection $\alpha : P \twoheadrightarrow I$ such that ω_I is induced by α and χ (as described above).
(c) P has a unimodular element if and only if $e(P, \chi) = 0$ in $E(A[T])$.

A consequence of Quillen's local global principle [Q] is that the following sequence of groups is exact

$$0 \longrightarrow \widetilde{K}_0(A) \longrightarrow \widetilde{K}_0(A[T]) \longrightarrow \Pi_m \widetilde{K}_0(A_m[T]),$$

where the direct product runs over all maximal ideals m of A. The following is a local global principle for the Euler class groups proved in [D 1, Theorem 5.4].

Theorem 2.14 *Let A be a Noetherian ring of dimension $n \geq 3$. Then the following sequence of groups is exact.*

$$0 \longrightarrow E(A) \longrightarrow E(A[T]) \longrightarrow \Pi_m E(A_m[T]),$$

where the direct product runs over all maximal ideals m of A such that $ht(m) = n$.

Let A be a ring and A_{red} be its quotient by the nilradical $\mathcal{N}$. In [B-RS 3, Corollary 4.6] it has been proved that $E(A) \simeq E(A_{\text{red}})$. The following proposition is an analogue for polynomial algebras.

Proposition 2.15 *Let A be a Noetherian ring of dimension $n \geq 3$. Then $E(A[T]) \simeq E(A_{red}[T])$.*

Proof Let the "bar" denote reduction modulo $\mathcal{N}[T]$ where $\mathcal{N}$ is the nilradical of A (so We shall write $\bar{A}$ for A_{red}). Let I be an ideal of $A[T]$ of height n such that Spec$(A[T]/I)$ is connected and I/I^2 is generated by n elements. We have $\bar{I} = (I + \mathcal{N}[T])/\mathcal{N}[T]$. We note that Spec$(\bar{A}[T]/\bar{I})$ is also connected. Further, if $I = (f_1, \cdots, f_n) + I^2$, then $\bar{I}/\bar{I}^2$ is generated by the images of $f_1, \cdots, f_n$.

Let $J \subset A[T]$ be an ideal of height n and ω_J be a local orientation of J. Assume that $J = I_1 \cap \cdots \cap I_k$ be its decomposition into "connected components" (see definition of $E(A[T])$ above). We note that

$$\bar{J} = \bar{I}_1 \cap \cdots \cap \bar{I}_k$$

is a similar decomposition for $\bar{J}$. It is not hard to see that ω_J would induce local orientation $\omega_{\bar{J}}$ in a natural way. Therefore. We shall have a group homomorphism $\phi : E(A[T]) \longrightarrow E(\bar{A}[T])$ which takes (J, ω_J) to $(\bar{J}, \omega_{\bar{J}})$.

Now one can just follow the proof of [B-RS 3, Corollary 4.6] to see that ϕ is an isomorphism. The proof in [B-RS 3] is a bit sketchy. For a detailed proof one may look at [K, Corollary 4.13]. ∎

Now we state two very useful lemmas. The proofs of these lemmas can be found in [B-RS 1].

Lemma 2.16 *Let A be a Noetherian ring containing $\mathbb{Q}$ and let $I \subset A[T]$ be an ideal of height n. Then there exists $\lambda \in \mathbb{Q}$ such that either $I(\lambda) = A$ or $I(\lambda)$ is an ideal of height n in A, where $I(\lambda) = \{f(\lambda) : f(T) \in I\}$.*

Lemma 2.17 *Let A be a Noetherian ring and let $I \subset A[T]$ be an ideal. Suppose that $I = (f_1, \cdots, f_n) + I^2$ and $I(0) = (a_1, \cdots, a_n)$ where $f_i(0) = a_i$ modulo $I(0)^2$ for $0 \leq i \leq n$. Then there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n) + (I^2T)$ where $g_i = f_i$ modulo I^2 and $g_i(0) = a_i$ for $0 \leq i \leq n$.*

We end this section with the following "moving lemma" from [D 1], which We shall need in the next section. In Euler class theory we frequently use various forms of moving lemmas (for example, see [B-RS 3, Corollary 2.14], [D 2, Lemma 2.11]). Roughly, the idea is to find for a local complete intersection ideal I, another local complete intersection ideal I' such that II' is a complete intersection. Depending on the context one would like to obtain I' in such a way that it has some additional properties, as can be seen in the lemma below. These lemmas have their origins in the classical moving lemmas for the ideals of a Dedekind domain : For an ideal I in a Dedekind domain D there exists an ideal J such that IJ is principal.

Lemma 2.18 *[D 1, Lemma 3.9] Let A be a Noetherian ring of dimension $n \geq 3$, let $I \subset A[T]$ an ideal of height n and J be any ideal contained in $I \cap A$ such that ht $J \geq 2$. Let P be a projective $A[T]$-module of rank n. Suppose that $\psi : P \twoheadrightarrow I/(I^2T)$ is a surjection. Then we can find a lift $\phi \in Hom_{A[T]}(P, I)$ of ψ, such that the ideal $\phi(P) = I''$ satisfies the following properties:*

1. $I'' + (J^2T) = I$;
2. $I'' = I \cap I'$, *where* $ht(I') \geq n$;
3. $I' + (J^2T) = A[T]$

3 Main theorem

Consider the following commutative diagram of abelian groups with exact rows where K_1 is the kernel of the canonical surjection $E(A) \twoheadrightarrow E_0(A)$ and K_2 is that of $E(A[T]) \twoheadrightarrow E_0(A[T])$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1 & \longrightarrow & E(A) & \longrightarrow & E_0(A) & \longrightarrow & 0 \\
& & \downarrow \varphi & & \downarrow \Phi & & \downarrow \Phi_0 & & \\
0 & \longrightarrow & K_2 & \longrightarrow & E(A[T]) & \longrightarrow & E_0(A[T]) & \longrightarrow & 0
\end{array}$$

The canonical maps Φ and Φ_0 are both injective (see [D 1, D-RS]). Consequently, so is φ.

In [D 2, Theorem 3.3] we defined a group homomorphism $\Psi : E(A[T]) \longrightarrow E(A)$ and proved that it is surjective and has the property that $\Psi\Phi$ is the identity map on $E(A)$. Let us briefly recall how this non-canonical map Ψ works. Let (I, ω_I) be an element of $E(A[T])$. Let ω_I be given by : $I = (f_1, \cdots, f_n) + I^2$. Then $I(0) = (f_1(0), \cdots, f_n(0)) + I(0)^2$. Applying a 'moving lemma' [D 2, 2.11], we can find an ideal $K \subset A$ of height $\geq n$ such that $K + (I \cap A) = A$ and $K \cap I(0) = (a_1, \cdots, a_n)$ where $a_i \equiv f_i(0)$ modulo $I(0)^2$ (note that $I(0)$ may not be of height n). Then $K = (a_1, \cdots, a_n) + K^2$ and let ω_K be the corresponding local orientation. Then $\Psi((I, \omega_I)) = -(K, \omega_K)$. In the cases when $I(0) = A$ or $K = A$ we have $\Psi((I, \omega_I)) = 0$. If $I(0)$ is of height n then $\Psi((I, \omega_I)) = (I(0), \omega_{I(0)})$, where $\omega_{I(0)}$ is induced by ω_I.

We shall now discuss the question of surjectivity of the canonical map $\Phi : E(A) \longrightarrow E(A[T])$. We start with the following lemma.

Lemma 3.1 *Let A be a Noetherian ring of dimension $n \geq 3$. Let $(J, \omega_J) \in E(A[T])$. Then, there is an ideal $I \subset A[T]$ of height n and a surjection $\overline{\alpha} : A[T]^n \twoheadrightarrow I/(I^2T)$ such that (J, ω_J) is in the image of $\Phi : E(A) \longrightarrow E(A[T])$ if and only if $\overline{\alpha}$ can be lifted to a surjection $\alpha : A[T]^n \twoheadrightarrow I$.*

Proof Since A contains $\mathbb{Q}$, by Lemma 2.16 we can make a change of variable and assume that $J(0) = A$ or the height of $J(0)$ is n. If either $J(0) = A$ or $(J(0), \omega_{J(0)}) = 0$ in $E(A)$, applying Lemma 2.17 we can lift ω_J to a surjection $\overline{\alpha} : A[T]^n \twoheadrightarrow J/(J^2T)$. We show that in this case, J and $\overline{\alpha}$ will work for us. Suppose that (J, ω_J) has a preimage in $E(A)$. So there is an element $(K, \omega_K) \in E(A)$ such that $(J, \omega_J) = (KA[T], \omega_{KA[T]})$ in $E(A[T])$, where the local orientation $\omega_{KA[T]}$ of $KA[T]$ is induced by ω_K. Now we apply Ψ to both sides of the above equation. As $J(0) = A$, we have $\Psi((J, \omega_J)) = 0$ in $E(A)$, whereas $\Psi((KA[T], \omega_{KA[T]})) = (K, \omega_K)$. Therefore,

$(K,\omega_K)=0$ in $E(A)$ and consequently $(J,\omega_J)=0$ in $E(A[T])$. Conversely, assume that $\overline{\alpha}: A[T]^n \twoheadrightarrow J/(J^2T)$ has a lift to a surjection $\alpha: A[T]^n \twoheadrightarrow J$. As $\overline{\alpha}$ is a lift of ω_J, it follows that α lifts ω_J and therefore, $(J,\omega_J)=0$ in $E(A[T])$.

Now we assume that the height of $J(0)$ is n and $(J(0),\omega_{J(0)}) \neq 0$ in $E(A)$. Applying [B-RS 3, Lemma 2.14] we can find an ideal $K \subset A$ of height n which is comaximal with $J\cap A$ and a local orientation ω_K of K such that $(J(0),\omega_{J(0)}) + (K,\omega_K) = 0$ in $E(A)$. Let $I = J \cap KA[T]$. As $J + KA[T] = A[T]$, the local orientations ω_J and ω_K will induce a local orientation ω_I of I. Then we have the following equation in $E(A[T])$:

$$(I,\omega_I) = (J,\omega_J) + (KA[T],\omega_{KA[T]}).$$

Applying the homomorphism $\Psi : E(A[T]) \twoheadrightarrow E(A)$ to both sides, we have

$$(I(0),\omega_{I(0)}) = (J(0),\omega_{J(0)}) + (K,\omega_K) = 0.$$

Therefore, $\omega_{I(0)}$ is a global orientation. We can apply Lemma 2.17 to see that ω_I can be lifted to a surjection $\overline{\alpha}: A[T]^n \twoheadrightarrow I/(I^2T)$.

Suppose that there is a preimage of (J,ω_J) in $E(A)$. It then follows that (I,ω_I) has a preimage in $E(A)$. Since $\text{ht}\,(I(0)) = n$ and since $\Psi\Phi$ is the identity map on $E(A)$, we see that the preimage of (I,ω_I) is actually $(I(0),\omega_{I(0)})$. As $(I(0),\omega_{I(0)}) = 0$, we have $(I,\omega_I) = 0$ in $E(A[T])$. Therefore ω_I can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$. Since $\overline{\alpha}: A[T]^n \twoheadrightarrow I/(I^2T)$ is a lift of ω_I, it now follows from [D 1, Theorem 3.10] that $\overline{\alpha}$ can be lifted to a surjection $\alpha : A[T]^n \twoheadrightarrow I$.

Conversely, if $\overline{\alpha}$ has a lift to a surjection $\alpha : A[T]^n \twoheadrightarrow I$, then α lifts ω_I as well and therefore, $(I,\omega_I)=0$ in $E(A[T])$. Then we have $(J,\omega_J) = -(KA[T],\omega_{KA[T]})$ in $E(A[T])$, which shows that (J,ω_J) has a preimage in $E(A)$. ∎

Remark 3.2 If A is a smooth affine domain then we have the 'homotopy invariance' for the Euler class groups, i.e., Φ is an isomorphism. This was proved in [D 1]. Anyway, it clearly follows from Lemma 3.1 and the *homotopy theorem* (Theorem 1.2).

Remark 3.3 In [D-RS] we proved that the homotopy invariance holds for the weak Euler class groups as well when A is a smooth affine domain. Consequently, φ is an isomorphism when A is a smooth affine domain.

Remark 3.4 There is an example by Bhatwadekar, Mohan Kumar and Srinivas [B-RS 1, Example 6.4], where they consider a normal affine domain A of dimension 3 over $\mathbb{C}$, construct an ideal I of $A[T]$ of height 3 such that $I(0) = A$ and $I = (f_1, f_2, f_3) + (I^2T)$. They show that this set of generators of $I/(I^2T)$ cannot be lifted to a set of 3 generators of I. Let ω_I be the local orientation of I induced by f_1, f_2, f_3. As $I(0) = A$, it is now easy to see from the proof of Lemma 3.1 that (I, ω_I) cannot have a preimage in $E(A)$. Therefore, $\Phi : E(A) \longrightarrow E(A[T])$ is not surjective in general.

In the example quoted in Remark 3.4, A is a normal (but not regular) affine domain of dimension 3 over $\mathbb{C}$. We may however note that A is an affine domain over an *algebraically closed field* and for such rings we know that $E(A) \simeq E_0(A)$ (can be deduced easily from [B-RS 3, Lemma 5.4]) and $E(A[T]) \simeq E_0(A[T])$ ([D 2, Corollary 5.4]). Therefore, the kernels K_1, K_2 are both trivial for their example. We may further note that in that set up, an element, say, (J, ω_J) of $E(A[T])$, has a preimage in $E(A)$ if and only if the corresponding weak Euler class (J) ($\in E_0(A[T])$) has a preimage in $E_0(A)$. These observations prompt us to ask the following questions.

Question 3.5 Let A be a Noetherian ring of dimension $n \geq 3$.

(a) Are the kernels K_1 and K_2 isomorphic?

(b) Let $(J, \omega_J) \in E(A[T])$ be such that its image (J) in $E_0(A[T])$ has a preimage in $E_0(A)$. Then, does (J, ω_J) has a preimage in $E(A)$? In other words, are the cokernels $\text{coker}(\Phi)$ and $\text{coker}(\Phi_0)$ isomorphic?

Applying the snake lemma to the commutative diagram presented at the beginning of this section, one obtains the exact sequence

$$0 \longrightarrow \text{coker}(\varphi) \longrightarrow \text{coker}(\Phi) \longrightarrow \text{coker}(\Phi_0) \longrightarrow 0$$

which implies that (a) and (b) in Question 3.5 are equivalent. Question 3.5 is related to the main theorem in this section and We shall come back to this question once we prove the main theorem.

Let H denote the kernel of $\Psi : E(A[T]) \twoheadrightarrow E(A)$. It can be easily derived, using Lemma 2.17, that H consists of precisely those elements (I, ω_I) of $E(A[T])$ such that ω_I is induced by a set of generators of $I/(I^2T)$. The *homotopy theorem* tells us that the kernel H is trivial if A is a smooth affine domain. Again the same example [B-RS 1, 6.4] shows that H can be nontrivial if A is not smooth. However, both in the smooth case and in the set up of that example, we observe that the subgroup $H \cap K_2$ of $E(A[T])$ is trivial. We may therefore ask the following natural question.

Question 3.6 Let A be a Noetherian ring of dimension $n \geq 3$. Is the subgroup $H \cap K_2$ of $E(A[T])$ trivial?

The above question is also related to the main theorem and We shall discuss this question later. Let us now turn our attention to the main theorem.

Let A be a Noetherian ring of dimension $n \geq 3$ and $I \subset A[T]$ be an ideal of height n such that there is a surjection $\overline{\alpha} : A[T]^n \twoheadrightarrow I/(I^2T)$. In view of the *homotopy theorem* and the example where it fails, one wonders when can one lift $\overline{\alpha}$ to a surjection $\alpha : A[T]^n \twoheadrightarrow I$. Note that a necessary condition for such a phenomenon is that I should be generated by n elements. Question 1.3 raised at the introduction asks whether the same condition is sufficient.

Remark 3.7 If I is a maximal ideal of $A[T]$, then it follows easily from [B-RS 4] that Question 1.3 has an affirmative answer. On the other hand, if I (not necessarily maximal) contains a monic polynomial then Mandal [M 2] answered Question 1.3 in the affirmative where he does not need the assumption that I is n-generated.

We now proceed to prove the main theorem. There are two cases. The case when n is even is not much difficult and we prove it first. The main idea of this proof is implicit in the proof of [D-RS, Proposition 4.8].

Theorem 3.8 *Let A be a commutative Noetherian ring of dimension n where n is **even**. Let $I \subset A[T]$ be an ideal of height n such that $(I) = 0$ in $E_0(A[T])$. Suppose that $I = (f_1, \cdots, f_n) + (I^2T)$. Then there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ and $g_i = f_i$ mod (I^2T).*

Proof We first claim that we can assume that $I(0) = A$.
Proof of the claim: Applying Lemma 2.18 we can find an ideal $I' \subset A[T]$ with the following properties :

1. $I \cap I' = (h_1, \cdots, h_n)$ where $h_i - f_i \in (I^2T)$.

2. $I = (h_1, \cdots, h_n) + (J^2T)$, where $J = I \cap A$.

3. ht$(I') \geq n$ and $I' + (J^2T) = A[T]$.

Note that if $I' = A[T]$, we are done. Therefore we assume that I' is a proper ideal of $A[T]$ of height n. From (3) above it follows that $I'(0) = A$. Further, from (1) we have $(I') = 0$ in $E_0(A[T])$ and $I' = (h_1, \cdots, h_n) + {I'}^2$. Since $I'(0) = A$, by Lemma 2.17 we can lift $h_1, \cdots, h_n$ to a set of generators of $I'/({I'}^2T)$. Now we note that it is enough to prove the theorem for I' because then we can adapt Step 3 and 4 of the proof of [B-RS 1, Theorem 3.8] to obtain the desired set of generators of I. So, replacing I by I' if necessary, we can assume that $I(0) = A$ to start with. This proves the claim.

We have $I = (f_1, \cdots, f_n) + (I^2T)$. Let ω_I denote the local orientation of I induced by $f_1, \cdots, f_n$. Since $(I) = 0$ in $E_0(A[T])$, by [D 1, Proposition 6.7] there exists a stably free projective $A[T]$-module P of rank n and a trivialization $\chi : A[T] \simeq \wedge^n(P)$ such that $e(P, \chi) = (I, \omega_I)$ in $E(A[T])$. This implies, by Theorem 2.13 (b), that there is a surjection $\alpha : P \twoheadrightarrow I$.

A result of Rao [Ra 2, Corollary 2.5] asserts that if R is a local ring of dimension d such that $d!$ is invertible in R, then every stably free $R[T]$-module of rank d is free.

Since A contains $\mathbb{Q}$, by the result of Rao mentioned above, it follows that the stably free module P_m is a free $A_m[T]$-module for every maximal ideal m of A and therefore by Quillen's local-global principle (stated before Theorem 2.14), P is extended from A. Therefore, there exists a projective A-module Q such that $Q[T] = P$ where $Q[T]$ denotes $Q \otimes A[T]$. So $\alpha : Q[T] \twoheadrightarrow I$ is a surjection. Restricting α at $T = 0$ we have $\alpha : Q \twoheadrightarrow I(0)(= A)$ which means that Q has a unimodular element. Therefore, P has a unimodular element and hence by Theorem 2.13 (c), $e(P, \chi) = 0$. Consequently, $(I, \omega_I) = 0$ in $E(A[T])$. This means that we can find a set of generators, $k_1, \cdots, k_n$, of I such that $k_i = f_i$ mod I^2. Since $I = (f_1, \cdots, f_n) + (I^2T)$, moving to the ring $A(T)$ (where $A(T)$ is the ring obtained from $A[T]$ by inverting all the monic polynomials), and applying [D 1, Theorem 3.10] it follows that there exist generators $g_1, \cdots, g_n$ of I such that $g_i = f_i$ mod (I^2T). This proves the theorem. ■

Corollary 3.9 *Let A be a commutative Noetherian ring of dimension n where n is **even**. Let $I \subset A[T]$ be an ideal of height n such that I is generated by n elements. Suppose that*

$I = (f_1, \cdots, f_n) + (I^2T)$. Then there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ and $g_i = f_i$ mod (I^2T).

Proof As I is generated by n elements, we have $(I) = 0$ in $E_0(A[T])$. The result is now immediate from the above theorem. ■

Remark 3.10 Let H be the kernel of the map $\Psi : E(A[T]) \twoheadrightarrow E(A)$ and let $(I, \omega_I) \in H \cap K_2$. Now $(I, \omega_I) \in H$ implies that ω_I is actually induced by a set of generators of $I/(I^2T)$ whereas $(I, \omega_I) \in K_2$ implies that $(I) = 0$ in $E_0(A[T])$. From Theorem 3.8 it is now obvious that $H \cap K_2$ is trivial when the dimension of A is even, which answers Question 3.6 in this case.

Now we proceed to investigate Question 1.3 in the case when n is odd. In the proof we do not really make use of the assumption that n is odd. So We shall mention all the results below for general n. Now that we do not have the advantage of [D 1, Proposition 6.7], we need several preparatory results.

The following proposition is crucial to this paper. It is an improvement of [B-RS 4, Lemma 4.1].

Proposition 3.11 *Let A be a Noetherian ring of dimension $n \geq 3$ and $I \subset A[T]$ be an ideal of height n such that $I = (a_1, \cdots, a_{n-1}, f(T))$ where $a_1, \cdots, a_{n-1} \in A$ and ht $(a_1, \cdots, a_{n-2}) = n-2$. Let ω_I be any local orientation of I. Then the element $(I, \omega_I) \in E(A[T])$ is in the image of the canonical map $\Phi : E(A) \longrightarrow E(A[T])$. In particular, if the Jacobson radical of A has height at least one, then any local orientation of I is a global one (i.e., any set of n generators of I/I^2 can be lifted to a set of n generators of I).*

Proof We prove the proposition in two steps. In Step 1 we prove the proposition when A is semilocal. In Step 2 we complete the proof using the case covered in Step 1 and the local-global principle for the Euler class groups (Theorem 2.14).

Step 1. Let A be semilocal. In this case we prove that $(I, \omega_I) = 0$ in $E(A[T])$ by proving that ω_I can be lifted to a surjection $\beta : A[T]^n \twoheadrightarrow I$. We mainly follow the proof of [B-RS 4, Lemma 4.1]. Note that unlike [B-RS 4, Lemma 4.1], we do not assume that $\dim(A[T]/I) = 0$.

Let $\alpha : A[T]^n \twoheadrightarrow I$ denote the surjection which corresponds to the generators $(a_1, \cdots, a_{n-1}, f(T))$ of I. Let the bar denote reduction modulo I. Now since ht $I =$

n, $\overline{\alpha}$ and ω_I differ by an element of $GL_n(A[T]/I)$. Therefore, there exists $\delta \in GL_n(A[T]/I)$ such that $\overline{\alpha}\delta = \omega_I$. Let $u(T) \in A[T]$ be such that $\overline{u(T)} = \det(\delta)^{-1}$. Then $(u(T), f(T), a_1, \cdots, a_{n-1}) \in Um_{n+1}(A[T])$. Let $B = A/(a_1, \cdots, a_{n-2})$ and let the tilde denote reduction modulo $(a_1, \cdots, a_{n-2})$. Then B is a semilocal ring of dimension ≤ 2.

Now a result of Murthy (see [Ra 1, Corollary 2.5]) asserts that if R is a local ring of dimension 2 and 2 is invertible in R, then any unimodular row over $R[T]$ is completable. Since $\frac{1}{2} \in B$, by this result of Murthy the unimodular row $(\widetilde{u(T)}, \widetilde{a_{n-1}}, \widetilde{f(T)}) \in Um_3(B[T])$ is completable over $B_m[T]$ for every maximal ideal m of B and therefore by Quillen's local-global principle (stated before Theorem 2.14 in this paper), it is extended from B. Since B is semilocal, it follows that the unimodular row $(\widetilde{u(T)}, \widetilde{a_{n-1}}, \widetilde{f(T)})$ is completable. Therefore, by [RS, Lemma 2.4], there exists a surjection $\theta : A[T]^n \twoheadrightarrow I$ and an element $\delta' \in SL_n(A[T]/I)$ such that $\overline{\theta}\delta' = \omega_I$. Now it follows from [D 1, Proposition 4.4] that ω_I has a lift to a surjection from $A[T]^n$ to I. In other words, ω_I is a global orientation and hence $(I, \omega_I) = 0$ in $E(A[T])$.

Step 2. We now consider the case when A is not necessarily semilocal. Let m be any maximal ideal of A of height n. Consider $A_m[T]$. By Step 1 we have, $(I, \omega_I) = 0$ in $E(A_m[T])$. Since it happens for every maximal ideal m of A of height n, we have, by the local global principle for the Euler class groups (Theorem 2.14) that (I, ω_I) is in the image of the canonical map $\Phi : E(A) \longrightarrow E(A[T])$.

For the last part of the proposition, let $\operatorname{ht} J(A) \geq 1$ where $J(A)$ denotes the Jacobson radical of A., It follows from [Mo1, Corollary 3] that $E(A) = 0$. Therefore, $(I, \omega_I) = 0$ in $E(A[T])$. ∎

Now we prove an interesting result which says that a maximal ideal of $A[T]$ of height n has no local orientation.

Proposition 3.12 *Let A be a Noetherian ring of dimension $n \geq 3$ and M be a maximal ideal of $A[T]$ of height n such that M/M^2 is generated by n elements. Then, for any two local orientations ω_1 and ω_2 of M, $(M, \omega_1) = (M, \omega_2)$ in $E(A[T])$.*

Proof We prove the proposition in two steps. First, we prove it in the case when A is local. In second step we use the local-global principle for the Euler class groups (Theorem 2.14) to tackle the general case.

Step 1. Let us assume that A is a local ring. Using [B, Proposition 3.2] we see that there exists an ideal L of $A[T]$ of height n such that L is comaximal with M and

$$L \cap M = (a_1, \cdots, a_{n-1}, f(T)),$$

where $a_1, \cdots, a_{n-1} \in A$ and $f(T) \in A[T]$. Further, $\text{ht}\,(a_1, \cdots, a_{n-1}) = n - 1$.

Let ω_1, ω_2 be two local orientations of M. Let us fix a local orientation of L, say ω_L. Since L and M are comaximal, ω_1 and ω_L together induce a local orientation of $M \cap L$, say, $\omega_{M\cap L}$. Therefore, in $E(A[T])$ we have, $(M, \omega_1) + (L, \omega_L) = (M \cap L, \omega_{M\cap L})$. By Proposition 3.11 it follows that $\omega_{M\cap L}$ is a global orientation and hence we have,

$$(M, \omega_1) + (L, \omega_L) = (M \cap L, \omega_{M\cap L}) = 0.$$

Again ω_2 and ω_L together induce a local orientation of $M \cap L$, say, $\omega^*_{M\cap L}$. Similar arguments as above will show that

$$(M, \omega_2) + (L, \omega_L) = (M \cap L, \omega^*_{M\cap L}) = 0.$$

Therefore, $(M, \omega_1) = (M, \omega_2)$ in $E(A[T])$. This proves the case when A is local.

Step 2. Now suppose that A is not necessarily local. We first note that if M contains a monic polynomial then by a result of Mandal [M 1, Theorem 1.2] we would have $(M, \omega_1) = (M, \omega_2) = 0$ in $E(A[T])$. Therefore, we assume that M does not contain a monic polynomial. In particular, (T) is comaximal with M and therefore, $M(0) = A$.

Now consider the element $(M, \omega_1) - (M, \omega_2)$ of $E(A[T])$. Let $\mathcal{M}$ be any maximal ideal of A of height n. Then by the above case, $(M, \omega_1) - (M, \omega_2) = 0$ in $E(A_{\mathcal{M}}[T])$. Since this happens for every maximal ideal $\mathcal{M}$ of A of height n, we can use the local global principle for the Euler class groups (2.14) and conclude that there is an ideal J of A of height n and a local orientation ω_J of J such that

$$\Phi((J, \omega_J)) = (M, \omega_1) - (M, \omega_2)$$

in $E(A[T])$. Now we can apply the group homomorphism $\Psi : E(A[T]) \longrightarrow E(A)$ on both sides of the above equation. Since $M(0) = A$, Ψ takes both the terms on the right hand side to zero whereas $\Psi\Phi((J, \omega_J)) = (J, \omega_J)$. This implies that $(J, \omega_J) = 0$. Now it follows that $(M, \omega_1) = (M, \omega_2)$ in $E(A[T])$. This proves the proposition. ■

The following result is an easy corollary to the above proposition.

Corollary 3.13 *Let A be a Noetherian ring of dimension $n \geq 3$ and I be an ideal of $A[T]$ of height n such that : (i) I/I^2 is generated by n elements, (ii) $I = M_1 \cap \cdots \cap M_k$, where each M_i is a maximal ideal of $A[T]$ of height n. Then, for any two local orientations ω_1 and ω_2 of I, $(I, \omega_1) = (I, \omega_2)$ in $E(A[T])$.*

Proof By the Chinese remainder theorem we have

$$I/I^2 = M_1/M_1^2 \oplus \cdots \oplus M_k/M_k^2$$

and therefore ω_1 will induce local orientations $\omega_{M_1}, \cdots, \omega_{M_k}$ of $M_1, \cdots, M_k$ respectively. In the Euler class group $E(A[T])$ We shall have the equation

$$(I, \omega_1) = (M_1, \omega_{M_1}) + \cdots + (M_k, \omega_{M_k}).$$

Similarly, ω_2 will induce local orientations $\omega^*_{M_1}, \cdots, \omega^*_{M_k}$ of $M_1, \cdots, M_k$ respectively and we have,

$$(I, \omega_2) = (M_1, \omega^*_{M_1}) + \cdots + (M_k, \omega^*_{M_k}).$$

Now using the above proposition we are through. ■

Lemma 3.14 *Let A be a Noetherian ring of dimension $n \geq 3$ and (I, ω_I) be an element of $E(A[T])$. Suppose that $(I) = 0$ in $E_0(A[T])$. Then, $(I, \omega_I) + (I, -\omega_I)$ is in the image of the canonical map $\Phi : E(A) \longrightarrow E(A[T])$.*

Proof As usual we first prove the lemma in the case when A is local and then apply the local global principle.

So assume that A is local. Now since $(I) = 0$ in $E_0(A[T])$, we have $(I \otimes A(T)) = 0$ in $E_0(A(T))$, where $A(T)$ denotes the ring obtained from $A[T]$ by inverting all monic polynomials. Applying [B-RS 3, Corollary 7.9] we have

$$(I \otimes A(T), \omega_I \otimes A(T)) + (I \otimes A(T), -\omega_I \otimes A(T)) = 0$$

in $E(A(T))$. Since A is local, by [D 1, Proposition 5.8] the canonical map from $E(A[T])$ to $E(A(T))$ is injective. Therefore, $(I, \omega_I) + (I, -\omega_I) = 0$ in $E(A[T])$. Therefore, the lemma is proved in this case.

Now suppose that A is not necessarily local. Applying the above case and the local global principle (Theorem 2.14) we conclude that the element $(I, \omega_I)+(I, -\omega_I)$ of $E(A[T])$ is in the image of the canonical map $\Phi : E(A) \longrightarrow E(A[T])$. ■

Now we are ready to answer Question 1.3 for general n. We have the following theorem.

Theorem 3.15 *Let A be an affine algebra of dimension $n \geq 3$. Let $I \subset A[T]$ be an ideal of height n such that $I = (f_1, \cdots, f_n) + (I^2T)$. Assume that $(I) = 0$ in $E_0(A[T])$. Assume further that the kernel of the canonical surjection $E(A[T]) \twoheadrightarrow E_0(A[T])$ has no nontrivial 2-torsion. Then we can find a set of generators $g_1, \cdots, g_n$ of I such that $f_i = g_i$ modulo (I^2T).*

Proof We give the proof in several steps.

Step 1. We first claim that we can assume that $I(0) = A$.

Proof of the claim: Proof of this claim is same as that of the claim in Theorem 3.8 above and hence omitted.

Step 2. Let us denote the local orientation induced by the given set of generators of $I/(I^2T)$ by ω_I. Using the facts that $(I) = 0$ in $E_0(A[T])$ and $I(0) = A$, we prove in this step that $(I, \omega_I) + (I, -\omega_I) = 0$ in $E(A[T])$.

Applying Lemma 3.14 we see that $(I, \omega_I) + (I, -\omega_I)$ is in the image of the canonical map $\Phi : E(A) \longrightarrow E(A[T])$. Therefore, there is $(K, \omega_K) \in E(A)$ such that

$$(I, \omega_I) + (I, -\omega_I) = \Phi((K, \omega_K))$$

in $E(A[T])$. Now we may apply the group homomorphism $\Psi : E(A[T]) \longrightarrow E(A)$ on both sides of the above equation. Since $I(0) = A$, both the terms on the left hand side are mapped to zero whereas we have $\Psi\Phi((K, \omega_K)) = (K, \omega_K)$. This implies that $(K, \omega_K) = 0$ and therefore $(I, \omega_I) + (I, -\omega_I) = \Phi((K, \omega_K)) = 0$, as desired.

Step 3. Aim of this step is to prove that $(I, \omega_I) = (I, -\omega_I)$ in $E(A[T])$.

Applying the "moving lemma" (Lemma 2.18) we can find an ideal I_1 of $A[T]$ of height n which is comaximal with I, and a local orientation ω_{I_1} such that $I_1(0) = A$ and $(I, -\omega_I) + (I_1, \omega_{I_1}) = 0$ in $E(A[T])$. Let $I_2 = I \cap I_1$. Let ω_{I_2} be the local orientation of I_2 induced by ω_I and ω_{I_1}. Then we have $(I, \omega_I) + (I_1, \omega_{I_1}) = (I_2, \omega_{I_2})$ in $E(A[T])$. Therefore, in order to prove that $(I, \omega_I) = (I, -\omega_I)$ it is enough to show that $(I_2, \omega_{I_2}) = 0$. In what follows we prove this.

Note that in view of Proposition 2.15, we may assume A to be reduced. Since A is an affine algebra containing $\mathbb{Q}$, it will be geometrically reduced. We need this reduction to apply Swan's Bertini theorem in the next paragraph.

Let $J = I \cap A$ and let $J_2 = I_2 \cap A$. Let $b \in J^2 \cap J_2$ be a non-zerodivisor. Applying [B-RS 5, Proposition 2.5] it is easy to see that we can find $h_1, \cdots, h_n$ such that $I = (h_1, \cdots, h_n, b)$ where $h_i = f_i$ modulo I^2. As A is geometrically reduced, by adding suitable multiples of b to $h_1, \cdots, h_n$, we may assume by Swan's Bertini theorem (see [B-RS 2, Theorem 2.11]) that (i) $(h_1, \cdots, h_n) = I \cap I'$, (ii) $I + I' = A[T]$ and (iii) I' is a reduced ideal of height n.

Let $B = A_{1+bA}$. We first show that if $I'B[T] = B[T]$, we can lift the given set of generators of $I/(I^2T)$ to a set of generators of I. If $I'B[T] = B[T]$, then $IB[T] = (h_1, \cdots, h_n)$. This implies $IB(T) = (h_1, \cdots, h_n)$, where $B(T)$ is the ring obtained from $B[T]$ by inverting all monic polynomials. Since $h_i = f_i$ modulo I^2, applying [D 1, Theorem 3.10] it follows that $IB[T] = (g_1, \cdots, g_n)$ where $g_i = f_i$ modulo $(I^2T)B[T]$. Now since $b \in J$ and $B = A_{1+bA}$, applying [D 1, Lemma 3.8] we can find a set of generators of I lifting $f_1, \cdots f_n$. So the theorem is proved in this case.

We now assume that $I'B[T]$ is a proper ideal of $B[T]$. In this case, we prove that $(I_2, \omega_{I_2}) = 0$ in $E(A[T])$. Note that $I_2(0) = A$. Therefore, by Lemma 2.17 we can lift ω_{I_2} to a set of generators of $I_2/(I_2^2T)$. Since $b \in J_2 = I_2 \cap A$ and $B = A_{1+bA}$, the same argument as in the previous paragraph shows that it is enough to prove that $(I_2, \omega_{I_2}) = 0$ in $E(B[T])$. Rest of this step is devoted to proving this.

Let $I'B[T]$ be a proper ideal of $B[T]$. Since $I'B[T]$ is comaximal with $bB[T]$ and b is contained in the Jacobson radical of B, it is easy to see that $I'B[T]$ is a finite intersection of maximal ideals, each of height n, in $B[T]$. In what follows we write $I'B[T] = M$.

Since $(I) = 0$ in $E_0(A[T])$, it follows that $(M) = 0$ in $E_0(B[T])$. Further, since M is comaximal with the Jacobson radical of B, we have $M(0) = B$. Now following the same arguments as in Step 2 it can be easily deduced from Lemma 3.14 that for any local orientation ω_M of M, $(M, \omega_M) + (M, -\omega_M) = 0$ in $E(B[T])$.

On the other hand, since M is a finite intersection of maximal ideals, by Corollary 3.13 it follows that for any two local orientations ω_1 and ω_2 of M, we have, $(M, \omega_1) = (M, \omega_2)$ in $E(B[T])$. In particular, for any local orientation ω_M of M,

$(M, \omega_M) = (M, -\omega_M)$ in $E(B[T])$.

Now take ω_M to be the local orientation induced by $h_1, \cdots, h_n$, for which $(I, \omega_I) + (M, \omega_M) = 0$ in $E(B[T]$. Since $(M, \omega_M) = (M, -\omega_M)$, it follows that $(I, \omega_I) = (I, -\omega_I)$ in $E(B[T])$. Recall that we have $(I, \omega_I) + (I_1, \omega_{I_1}) = (I_2, \omega_{I_2})$ where $(I_1, \omega_{I_1}) + (I, -\omega_I) = 0$ and I_2 is $I \cap I_1$. Consequently, $(I_2, \omega_{I_2}) = 0$ in $E(B[T])$. This is what we wanted to prove.

Therefore, $(I, \omega_I) = (I, -\omega_I)$ in $E(A[T]$.

Step 4. Since we already have $(I, \omega_I) + (I, -\omega_I) = 0$, it follows that $2(I, \omega_I) = 0$ and since there is no nontrivial 2-torsion in the kernel of the map from $E(A[T])$ to $E_0(A[T])$, we have $(I, \omega_I) = 0$. Note that ω_I is induced by the given set of generators of $I/(I^2T)$. Moving to the ring $A(T)$ and applying [D 1, Theorem 3.10], the result follows. ∎

Remark 3.16 In the above theorem we can in fact make even weaker assumption that *the weak Euler class of I has a preimage in $E_0(A)$*. Let us present this in the following form.

Theorem 3.17 *Let A be an affine algebra of dimension $n \geq 3$. Let $I \subset A[T]$ be an ideal of height n such that $I = (f_1, \cdots, f_n) + (I^2T)$. Assume that $(I) \in E_0(A[T])$ has a preimage in $E_0(A)$. Assume further that the kernel of the canonical surjection $E(A[T]) \twoheadrightarrow E_0(A[T])$ has no nontrivial 2-torsion. Then we can find a set of generators $g_1, \cdots, g_n$ of I such that $f_i = g_i$ modulo (I^2T).*

Proof The proof is essentially the same as that of Theorem 3.15. We shall only highlight the necessary modifications. The hypothesis on I we have here is that $(I) = 0$ in coker(Φ_0). We make the general observation that if there is any other ideal N of height n such that $I \cap N$ is n-generated, then $(I) + (N) = 0$ in $E_0(A[T])$ and therefore (N) is also trivial in coker(Φ_0). We shall keep this in mind.

Now as in Step 1 of 3.15 we can make the same reduction and assume that $I(0) = A$ (as $I \cap I'$ is n-generated, $(I') = 0$ in coker(Φ_0)). Step 2 of 3.15 will also go through. If we mimic the proof of Step 3 of 3.15 all we need to observe is that the weak Euler class of M is trivial in the cokernel of the induced map $E_0(B) \hookrightarrow E_0(B[T])$. Since the Jacobson radical of B has height at least one, by [Mo1, Corollary

3], $E_0(B) = 0$. Combining these two facts we obtain that $(M) = 0$ in $E_0(B[T])$. The rest of the proof is now obvious from the proof of 3.15. ■

Now we can apply the above theorems to answer Questions 3.5 and 3.6 for general n. We already discussed the case when n is even.

Corollary 3.18 *Let A be an affine algebra of dimension $n \geq 3$. Let K_1 be the kernel of the canonical surjection $E(A) \twoheadrightarrow E_0(A)$ and K_2 be that of $E(A[T]) \twoheadrightarrow E_0(A[T])$. Assume that K_2 does not have 2-torsion. Then K_1 and K_2 are isomorphic.*

Proof We know that the map $\varphi : K_1 \to K_2$ is always injective. To prove surjectivity, let $(J, \omega_J) \in K_2 \subset E(A[T])$. By Lemma 3.1 there is an ideal $I \subset A[T]$ of height n and a surjection $\overline{\alpha} : A[T]^n \twoheadrightarrow I/(I^2T)$ such that (J, ω_J) has a preimage in $E(A)$ if and only if $\overline{\alpha}$ can be lifted to a surjection $\alpha : A[T]^n \twoheadrightarrow I$. As we have $(J) = 0$ in $E_0(A[T])$, from the proof of Lemma 3.1 it is clear that the element $(I)(\in E_0(A[T]))$ has a preimage in $E_0(A)$. Therefore, by Theorem 3.17, $\overline{\alpha}$ can be lifted to a surjection $\alpha : A[T]^n \twoheadrightarrow I$. As a consequence, (J, ω_J) has a preimage in $E(A)$, say, (L, ω_L). Therefore, $(J, \omega_J) = (LA[T], \omega_{LA[T]})$ in $E(A[T])$. It is easy to see that $(L) = 0$ in $E_0(A)$ which shows that $(L, \omega_L) \in K_1$. This proves the surjectivity of φ. ■

Corollary 3.19 *Let A be an affine algebra of dimension $n \geq 3$. Let C be the cokernel of $\Phi : E(A) \longrightarrow E(A[T])$ and C_0 be that of $\Phi_0 : E_0(A) \longrightarrow E_0(A[T])$. If K_2 does not have 2-torsion, then C and C_0 are isomorphic.*

Proof The proof is immediate from Corollary 3.18 and the discussion following Question 3.5.

■

Remark 3.20 Let H be the kernel of the map $\Psi : E(A[T]) \twoheadrightarrow E(A)$ and let $(I, \omega_I) \in H \cap K_2$. In 3.6 we raised the question whether $H \cap K_2$ is trivial. Recall that $(I, \omega_I) \in H$ implies that ω_I is actually induced by a set of generators of $I/(I^2T)$ whereas $(I, \omega_I) \in K_2$ implies that $(I) = 0$ in $E_0(A[T])$. Now if one looks at the proof of Theorem 3.15 one would realise that we essentially proved there that $H \cap K_2$ is 2-torsion.

In the next corollary we generalise the main theorem by taking a projective module in place of the free module. Before stating it let us recall a definition from [D 2].

Definition 3.21 Let A be a Noetherian ring of dimension $n \geq 2$ and P be a projective $A[T]$-module of rank n with trivial determinant. Suppose that $\alpha : P \twoheadrightarrow I$ is a surjection such that $\operatorname{ht} I = n$. Then the *weak Euler class* of P is denoted by $e(P)$ and is defined as $e(P) = (I)$ in $E_0(A[T])$. It is proved in [D 2, Proposition 6.1] that the weak Euler class of P is well defined.

Corollary 3.22 *Let A be an affine algebra of dimension $n \geq 3$ and P be a projective $A[T]$-module of rank n having trivial determinant. Suppose that $e(P) = (I)$ in $E_0(A[T])$, where I is an ideal of $A[T]$ of height n such that I/I^2 is generated by n elements. Suppose that we are given a surjection $\phi : P \twoheadrightarrow I/(I^2T)$. If the kernel of the surjection from $E(A[T])$ to $E_0(A[T])$ has no nontrivial 2-torsion, then ϕ can be lifted to a surjection $\theta : P \twoheadrightarrow I$. (In particular, if there is a surjection from P to I, then any surjection from P to $I/(I^2T)$ can be lifted to a surjection from P to I.)*

Proof We fix an isomorphism $\chi : A[T] \simeq \wedge^n P$.

Applying Lemma 2.18 we can find a lift $\psi \in \operatorname{Hom}_{A[T]}(P, I)$ of ϕ and an ideal $I' \subset A[T]$ of height $\geq n$ such that (i) $I' + (I^2T) = A[T]$, (ii) $\psi : P \twoheadrightarrow I \cap I'$ is a surjection and (iii) $\psi(P) + (I^2T) = I$.

If $I' = A[T]$, we are done. Therefore we assume that I' is an ideal of height n.

Since the determinant of P is trivial, $P/(I \cap I')P$ is a free $A[T]/(I \cap I')$-module of rank n. We choose an isomorphism $\overline{\lambda} : (A[T]/I \cap I')^n \simeq P/(I \cap I')P$ such that $\wedge^n \overline{\lambda} = \chi \otimes A[T]/(I \cap I')$. Composing $\psi \otimes A[T]/(I \cap I')$ with $\overline{\lambda}$ we obtain a local orientation, say $\omega_{I \cap I'}$, of $I \cap I'$. Hence, $e(P, \chi) = (I \cap I', \omega_{I \cap I'})$ in $E(A[T])$.

Now ϕ induces a surjection from P/IP to I/I^2. Note that $\overline{\lambda}$ induces an isomorphism, say, $\overline{\lambda_I} : (A[T]/I)^n \simeq P/IP$ such that $\wedge^n \overline{\lambda_I} = \chi \otimes A[T]/I$. Composing, we get a local orientation, say ω_I, of I. On the other hand $\omega_{I \cap I'}$ induces a local orientation of I. It is easy to see that it is same as ω_I.

Therefore, $e(P, \chi) = (I, \omega_I) + (I', \omega_{I'})$ in $E(A[T])$, where $\omega_{I'}$ is the local orientation of I' induced by $\omega_{I \cap I'}$. Also we have, from the surjection $\psi : P \twoheadrightarrow I \cap I'$, that $e(P) = (I) + (I')$ in $E_0(A[T])$. Since it is given that $e(P) = (I)$, we have

$(I') = 0$ in $E_0(A[T])$. Further, since $I'(0) = A$, by Lemma 2.17 we can lift $\omega_{I'}$ to a set of generators of $I'/(I'^2T)$. Hence by the above theorem, $(I', \omega_{I'}) = 0$ in $E(A[T])$. Consequently, $e(P, \chi) = (I, \omega_I)$ and hence by Theorem 2.13 there is a surjection $\alpha : P \twoheadrightarrow I$ such that (I, ω_I) is obtained from (α, χ). Now applying [D 1, Theorem 4.8] it is easy to see that we have a surjection $\theta : P \twoheadrightarrow I$ lifting ϕ, as desired. ■

Remark 3.23 The failure of the *homotopy theorem* for non-smooth affine algebras motivated us to investigate the type of questions we addressed in this section. Our journey actually began in [D 1] where we proved a "Horrocks-inspired" version of the *homotopy theorem* for general Noetherian rings [D 1, Theorem 3.10] and used it extensively to develop the theory of $E(A[T])$. There are still many unanswered questions. For example we do not know whether the *homotopy theorem* is true if we assume the base ring to be a regular ring (by Lemma 3.1, an affirmative answer would imply $E(A) \simeq E(A[T])$ for such rings). The local global principle (Theorem 2.14) for the Euler class groups reduces the question to the local situation. The precise question is the following, with which we conclude this section.

Question 3.24 Let A be a regular local ring of dimension $n \geq 3$. Let $I \subset A[T]$ be an ideal of height n such that $I = (f_1, \cdots, f_n) + (I^2T)$. Can we find $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $g_i - f_i \in (I^2T)$?

4 Existence of 2-torsion

In our main result in the last section we made the assumption that the kernel K_2 of the canonical surjection $E(A[T]) \twoheadrightarrow E_0(A[T])$ does not have 2-torsion. One natural question to ask is whether the kernel can have 2-torsion. Such a question has been raised by Bhatwadekar-Sridharan in [B-RS 3, Question 7.11], regarding the kernel K_1 of the canonical surjection $E(A) \twoheadrightarrow E_0(A)$. Note that if K_2 does not have any 2-torsion then K_1 cannot have 2-torsion. Recently Bhatwadekar showed me (personal communication) that K_1 can have 2-torsion. He constructed an example of an affine domain A of dimension 2 containing $\mathbb{Q}(\sqrt{-1})$ where such a phenomenon takes place. However, we do not know what happens when the dimension of A is odd.

Results in this section are entirely due to Bhatwadekar. My sincere thanks to him for allowing me to include the results here.

We need the following crucial result which has been proved in [B-RS 3, Lemma 5.4].

Lemma 4.1 *Let A be a ring of dimension $n \geq 2$ and $J \subset A$ be an ideal of height n and ω_J be a local orientation of J. Let $\bar{u} \in A/J$ be a unit. Then $(J, \omega_J) = (J, \bar{u}^2 \omega_J)$ in $E(A)$.*

Let A be a ring of dimension n containing the field $\mathbb{Q}(\sqrt{-1})$. Then, by the above lemma, for any ideal $J \subset A$ of height n for which J/J^2 is generated by n elements, and any local orientation ω_J we have $(J, \omega_J) = (J, -\omega_J)$ in $E(A)$. On the other hand if we assume (J, ω_J) to be in the kernel K_1 of the canonical homomorphism $E(A) \twoheadrightarrow E_0(A)$, by [B-RS 3, Corollary 7.9], $(J, \omega_J) + (J, -\omega_J) = 0$ in $E(A)$ and it implies that K_1 is a 2-torsion group. Therefore, to give an example where K_1 has 2-torsion, we need to construct an example of a ring A (containing the field $\mathbb{Q}(\sqrt{-1})$) for which K_1 is not trivial, i.e., $E(A) \not\simeq E_0(A)$. Note that if $n = 2$ then $E(A) \not\simeq E_0(A)$ if and only if there exists a stably free A-module of rank 2 which is not free.

For every prime integer p, Mohan Kumar [Mo2] constructed an example of a smooth affine domain B of dimension $p+1$ over $\mathbb{C}(t)$ which contains a maximal ideal m which is not a surjective image of a projective B-module of rank $p+1$ with *trivial determinant*. Therefore, by the following lemma, for every prime p there exists a regular domain A of dimension p containing $\mathbb{C}(t)$ which has stably free, non-free modules of rank p. We can take $p = 2$ to obtain the desired example.

The lemma is true for any $n \geq 2$.

Lemma 4.2 *Let B be a ring of dimension $n+1 \geq 3$ and let I be an ideal of B such that $(f_1, f_2, \cdots, f_{n+1}) + I^2 = I$. Suppose that I is not a surjective image of a projective B-module of rank $n+1$ with trivial determinant. Then there exists a multiplicatively closed subset T of B such that B_T is a ring of dimension n having a stably free, non-free module of rank n.*

Proof It is easy to see that, under the hypothesis, there exists an element $b \in 1+I$ such that $I_b = (f_1, \cdots, f_{n+1})_b$. Let $S = 1 + bB$ and let T be the multiplicatively closed subset of B generated by S and b. Let A denote the ring B_T. It is easy to see that $\dim(A) \leq n$.

Since $1-b \in S \cap I$, $I_S = B_S$ and hence $I_T = (f_1, \cdots, f_{n+1})_T = B_T = A$. Therefore $[f_1, \cdots, f_{n+1}] = v$ is a unimodular element of the free A-module A^{n+1}. Let $P = A^{n+1}/Av$. Then P is a stably free A-module of rank n.
Claim: P is not free.
Proof of the claim. Consider the following surjections:

$$\alpha : {B_b}^{n+1} \twoheadrightarrow I_b \; e_i \mapsto f_i, \;\; \beta : {B_S}^n \twoheadrightarrow I_S(=B_S) \; e_1 \mapsto 1, \; e_i \mapsto 0, \; 2 \le i \le n+1$$

If P is free then there exists $\sigma \in SL_{n+1}(A)$ such that $[f_1, \cdots, f_{n+1}]\sigma = [1, 0, \cdots, 0]$. Therefore $\alpha_S \sigma = \beta_b$.

Let Q be the fiber product of ${B_b}^n$ and ${B_S}^n$ over σ. Since $\sigma \in SL_{n+1}(A)$ and $\alpha_S \sigma = \beta_t$, Q is a projective B-module of rank $n+1$ with *trivial* determinant mapping surjectively onto the ideal I. This is a contradiction. Therefore the claim is proved.

Since P is a stably free, non-free A-module of rank n, we should have $\dim(A) \ge n$ as otherwise, by a cancellation theorem of Bass, P would be free. Therefore, we have $\dim(A) = n$. ∎

5 Unimodular elements in projective $A[T]$-modules

Let A be a commutative Noetherian ring of dimension n. Let P be a projective $A[T]$-module of rank n with trivial determinant and $\alpha : P \twoheadrightarrow I$ be a surjection such that ht(I) = rank(P). It is proved in [D 1, Corollary 4.12] that if P has a unimodular element then I is generated by n elements. It is natural to ask whether the converse holds, i.e., if I is generated by n elements then whether P has a unimodular element. The converse is false in general. For example, let A be the coordinate ring of the 2-dimensional real sphere and P be the projective A-module considered in Example 2.2. With same notations as in that example, consider the ideal $I = (y, z) \subset A$. It is easy to see that the surjection $\psi : A^3 \twoheadrightarrow I$ defined by $\psi(e_1) = 0$, $\psi(e_2) = -z$, $\psi(e_3) = y$ induces a surjection $\alpha : P \twoheadrightarrow I$. But we have seen in Example 2.2 that P does not have a unimodular element. Tensoring with $A[T]$ we obtain the surjection $\alpha \otimes A[T] : P[T] \twoheadrightarrow IA[T]$ which shows that the converse is false. Similar example can be constructed by taking A to be the coordinate ring of any even dimensional real sphere. This leads us to the following
Question. *Let A be a Noetherian ring with dim $A = n$ and P be a projective $A[T]$-module of rank n having trivial determinant. Suppose that there exists a surjection $\alpha : P \twoheadrightarrow I$*

where $I \subset A[T]$ is an ideal of height n which is generated by n elements. Assume further that the projective A-module P/TP has a unimodular element. Does P have a unimodular element?

In this section we investigate this question.

Definition 5.1 Let R be a ring and P be a projective R-module. A surjective R-linear map $\alpha : P \twoheadrightarrow J$ is called a *generic surjection* if J is an ideal of R such that $\text{ht}(J) = \text{rank}(P)$. For a generic surjection $\alpha : P \twoheadrightarrow J$, We shall call J to be a *generic section* of P.

Remark 5.2 By a result of Eisenbud-Evans [E-E, Remark following Theorem A], generic surjections exist for a projective module.

In this context we may note that Bhatwadekar-Sridharan proved [B-RS 3, Theorem 5.9] that if R is an affine domain of dimension n over $\mathbb{R}$ and Q is a projective R-module of rank n such that every generic section of Q is generated by n elements, then Q has a unimodular element. Taking a cue from this result we prove the following theorem.

Theorem 5.3 *Let A be an affine algebra of dimension $n \geq 3$. Let P be a projective $A[T]$-module of rank n with trivial determinant such that P/TP has a unimodular element. Assume further that for every generic section I of P, I is generated by n elements. Then P has a unimodular element.*

Proof Since P/TP has a unimodular element and P is a projective $A[T]$-module, it is easy to see that applying Eisenbud-Evans theorem ([E-E] or [P]), one can obtain a generic surjection $\alpha : P \twoheadrightarrow I$ such that $I(0) = A$. Fix an isomorphism $\chi : A[T] \simeq \wedge^n(P)$. Then (α, χ) will induce a local orientation, say ω_I of I and we would have $e(P, \chi) = (I, \omega_I)$. To prove the theorem it is enough to show that $(I, \omega_I) = 0$.

Since $I(0) = A$, by Lemma 2.17 we can lift ω_I to a set of generators of $I/(I^2T)$. We further note that I is generated by n elements and hence $(I) = 0$ in $E_0(A[T])$. Using arguments as in Step 2 of Theorem 3.15 it follows that

$$(I, \omega_I) + (I, -\omega_I) = 0$$

in $E(A[T])$. On the other hand, adapting Step 3 of Theorem 3.15 we obtain

$$(I, \omega_I) = (I, -\omega_I)$$

in $E(A[T])$.

Let ω_I be given by $I = (f_1, \cdots, f_n) + I^2$. Let $J = I \cap A$ and b be a non-zerodivisor belonging to J^2. Applying [B-RS 5, Proposition 2.5] it is easy to see that we can find $h_1, \cdots, h_n$ such that $I = (h_1, \cdots, h_n, b)$ where $h_i = f_i$ modulo I^2. By adding suitable multiples of b to $h_1, \cdots, h_n$, we may assume by Swan's Bertini theorem (see [B-RS 2, Theorem 2.11]) that (i) $(h_1, \cdots, h_n) = I \cap I'$, (ii) $I + I' = A[T]$ and (iii) I' is a reduced ideal of height n.

So, in terms of Euler classes we have the equation :

$$(I, \omega_I) + (I', \omega_{I'}) = 0$$

in $E(A[T])$, where $\omega_{I'}$ is the local orientation of I' induced by $h_1, \cdots, h_n$.

From the above three displayed equations we have,

$$(I', \omega_{I'}) = -(I, \omega_I) = (I, -\omega_I) = (I, \omega_I)$$

and therefore, $e(P, \chi) = (I', \omega_{I'})$. This implies, by Theorem 2.13, that there is a surjection $\beta : P \twoheadrightarrow I'$. So I' is also a generic section of P and since we have the assumption that every generic section of P is generated by n elements, I' is generated by n elements.

Let $B = A_{1+bA}$. If $I'B[T] = B[T]$, we are through (applying Lemma 3.8 of [D 1], as $b \in J$ and ω_I actually corresponds to a set of generators of $I/(I^2T)$). Therefore we may assume that it is a proper ideal of $B[T]$ and since it is comaximal with the Jacobson radical of B, it is easy to see that $I'B[T]$ is a finite intersection of maximal ideals, each of height n, in $B[T]$.

Since $I'B[T]$ is a finite intersection of maximal ideals, by Corollary 3.13, for any two local orientations ω_1 and ω_2 of $I'B[T]$, we have, $(I'B[T], \omega_1) = (I'B[T], \omega_2)$ in $E(B[T])$. Now I' is generated by n elements. Let ω_1 be the local orientation induced by $\omega_{I'}$ and ω_2 be the global orientation induced by a set of generators of I'. This implies that $(I'B[T], \omega_{I'} \otimes B[T]) = 0$ in $E(B[T])$. Consequently $(IB[T], \omega_I \otimes B[T]) = 0$ in $E(B[T])$. Since $b \in J$ and ω_I actually corresponds to a set of generators of $I/(I^2T)$, applying [D 1, Lemma 3.8] we are done. ■

Remark 5.4 In view of [B-RS 3, Theorem 5.9] we may ask the question that if R is a Noetherian ring of dimension n and Q is a projective R-module of rank n every generic section of which is generated by n elements, then whether Q has a unimodular element. We do not know any answer to this question. Therefore, in the above theorem we made the assumption that P/TP has a unimodular element. Note that if every generic section of the $A[T]$-module P is generated by n elements, then so is every generic section of P/TP.

Remark 5.5 In this context we may recall that in [D-RS] Raja Sridharan and I proved the following: *Let A be a Noetherian ring containing $\mathbb{Q}$ with dim $A = n$ (n even) and P be a projective $A[T]$-module of rank n with trivial determinant such that P/TP has a unimodular element. Suppose P has a generic section I which is generated by n elements. Then P has a unimodular element.* The treatment was entirely different in that set up and we crucially used the fact that n is even.

The following corollary is an application of Theorem 5.3 and [B-RS 3, Theorem 5.9].

Corollary 5.6 *Let A be an affine domain over $\mathbb{R}$ of dimension $n \geq 3$ and let P be a projective $A[T]$-module of rank n with trivial determinant. Suppose that for every generic section I of P, I is generated by n elements. Then P has a unimodular element.*

Proof We first note that the condition of the corollary implies that for every generic section J of the A-module P/TP, J is generated by n elements. To see this let $\alpha : P/TP \twoheadrightarrow J$ be a surjection where $J \subset A$ is an ideal of height n. Since P is projective we can lift α and apply Eisenbud-Evans theorem ([E-E] or [P]) to find a surjection $\phi : P \twoheadrightarrow I$ where $I \subset A[T]$ is an ideal of height n and $I(0) = J$. By assumption, I is generated by n elements. Therefore, $I(0) = J$ is generated by n elements.

Now A is an affine domain over $\mathbb{R}$ and P/TP is a projective A-module every generic section of which is generated by n elements. Therefore, by [B-RS 3, Theorem 5.9] it follows that P/TP has a unimodular element. The corollary now follows from the above theorem. ∎

The following theorem shows that the question raised at the beginning of this section has an affirmative answer if the generic section is of a particular form.

Theorem 5.7 *Let A be a Noetherian ring of dimension $n \geq 3$ and $I \subset A[T]$ be an ideal of height n such that $I = (a_1, \cdots, a_{n-1}, f(T))$ where : $a_1, \cdots, a_{n-1} \in A$, and ht $(a_1, \cdots, a_{n-1}) = n - 1$. Let P be a projective $A[T]$-module of rank n with trivial determinant such that P/TP has a unimodular element and P maps onto I. Then P has a unimodular element.*

Proof We give the proof in steps. Let $J = I \cap J(A, P)$ where $J(A, P)$ denotes the Quillen ideal of P in A (for the definition of the Quillen ideal see [D 1, Definition 2.13] or [D-RS, Definition 2.8]). In Step 1 we show that the projective $A_{1+J}[T]$-module P_{1+J} has a unimodular element. In Step 2, following a patching argument of Plumstead we patch unimodular elements of P_{1+J} and P/TP to obtain a unimodular element of P.

Step 1. Let us write $B = A_{1+J}$. We note that $\text{ht}\,(I \cap A) \geq n - 1 \geq 2$ and since the determinant of P is extended, by [D-RS, Lemma 2.9], $\text{ht}\, J(A, P) \geq 2$. Therefore it follows that $\text{ht}\, J \geq 2$. Further, JB is contained in the Jacobson radical of B. Therefore, applying Proposition 3.11 we observe that $IB[T]$ has the property that any local orientation ω of $IB[T]$ is a global one, i.e., $(IB[T], \omega) = 0$ in $E(B[T])$.

Now P maps onto I. This induces a surjection $\alpha : P_{1+J} \twoheadrightarrow IB[T]$. We fix an isomorphism $\chi : B[T] \simeq \wedge^n P_{1+J}$. Then (α, χ) induces a local orientation of $IB[T]$, say ω_1, and we have

$$e(P_{1+J}, \chi) = (IB[T], \omega_1)$$

in $E(B[T])$. By our discussion in the previous paragraph, $(IB[T], \omega_1) = 0$ in $E(B[T])$.

Therefore, it follows that $e(P_{1+J}, \chi) = 0$ in $E(B[T])$ and hence by Theorem 2.13, P_{1+J} has a unimodular element.

Step 2. We just proved that P_{1+J} has a unimodular element. Let us call it p_1. It is given that P/TP has a unimodular element, say, p. We claim that there is an elementary automorphism σ of P_{1+J} such that $\bar{\sigma}\bar{p_1} = \bar{p}$, where the "bar" denotes reduction modulo T. To see this, let us consider the ring $D = B/J(B)$ where $J(B)$ denotes the Jacobson radical of B. Since $\dim D \leq n - 2$ it follows that there is an elementary automorphism τ of $P_{1+J} \otimes D$ such that $\tau\overline{p_1} = p$ over D. Since elementary automorphisms can be lifted via a surjection of rings, we have, by repeated use of this argument, an elementary automorphism $\sigma \in \mathcal{E}(P_{1+J})$ such that $\bar{\sigma}\bar{p_1} = \bar{p}$. Let q denote the unimodular element σp_1 of P_{1+J}.

Since P_{1+J} has a unimodular element, we can find $s \in J$ such that P_{1+sA} has a unimodular element. We still call it q. Since P_s is extended from A_s, it has a unimodular element, namely p. Since p and q are equal modulo T, i.e. over $A_{s(1+sA)}$, it follows using a patching argument of Plumstead [P] that P has a unimodular element. ■

Corollary 5.8 *Let A be a Noetherian ring of dimension $n \geq 3$ and $I \subset A[T]$ be an ideal of height n such that $I = (a_1, \cdots, a_{n-1}, f(T))$ where : $a_1, \cdots, a_{n-1} \in A$, and ht $(a_1, \cdots, a_{n-1}) = n - 1$. Assume further that ht$(I(0)) = n$ and the constant term of f is a square. Let P be a projective $A[T]$-module of rank n with trivial determinant such that P maps onto I. Then P has a unimodular element.*

Proof Since P maps onto I, restricting at $T = 0$ We shall have a surjection, say, $\alpha : P/TP \twoheadrightarrow I(0)$. Since $I(0)$ is generated by n elements, namely, $I(0) = (a_1, \cdots, a_{n-1}, f(0))$ and $f(0)$ is a square, it follows [B-D-M, Theorem 3.13] that P/TP has a unimodular element. Now we can apply the above Proposition to obtain the result. ■

Acknowledgements : I sincerely thank Raja Sridharan for suggesting the questions addressed in this paper and for many inspiring discussions. Thanks are due to S. M. Bhatwadekar for his comments and suggestions on an earlier version of the paper which significantly improved the exposition and for allowing me to include his results in Section 4.

References

[B] S. M. Bhatwadekar, Projective generation of maximal ideals in polynomial rings, *J. Algebra* **91** (1984), 75-81.

[B-D-M] S. M. Bhatwadekar, M. K. Das and S. Mandal, Projective modules over smooth real affine varieties, *Invent. Math.* **166** (2006), 151-184.

[B-K] S. M. Bhatwadekar and M. K. Keshari, A question of Nori : Projective generation of ideals, *K-Theory* **28** (2003), 329-351.

[B-RS 1] S. M. Bhatwadekar and Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain and a question of Nori, *Invent. Math.* **133** (1998), 161-192.

[B-RS 2] S. M. Bhatwadekar and Raja Sridharan, Zero cycles and the Euler class groups of smooth real affine varieties, *Invent. Math.* **136** (1999), 287-322.

[B-RS 3] S. M. Bhatwadekar and Raja Sridharan, The Euler class group of a Noetherian ring, *Compositio Math.* **122** (2000), 183-222.

[B-RS 4] S. M. Bhatwadekar and Raja Sridharan, On a question of Roitman, *J. Ramanujan Math. Soc.* **16** (2001), 45-61.

[B-RS 5] S. M. Bhatwadekar and Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain (II), *K-Theory* **15** (1998), 293-300.

[D 1] M. K. Das, The Euler class group of a polynomial algebra, *J. Algebra* **264** (2003), 582-612.

[D 2] M. K. Das, The Euler class group of a polynomial algebra II, *J. Algebra* **299** (2006), 94-114.

[D-RS] M. K. Das and Raja Sridharan, The Euler class groups of polynomial rings and unimodular elements in projective modules, *J. Pure and Appl. Algebra* **175** (2003), 73-86.

[E-E] D. Eisenbud and E. G. Evans, Generating modules efficiently: Theorems from algebraic K-Theory, *J. Algebra* **27** (1973), 278-305.

[K] M. Keshari, Euler Class group of a Noetherian ring, M.Phil. thesis, available at : http://www.math.iitb.ac.in/˜keshari/acad.html

[M 1] S. Mandal, On efficient generation of ideals, *Invent. Math.* **75** (1984), 59-67.

[M 2] S. Mandal, Homotopy of sections of projective modules, *J. Algebraic Geometry* **1** (1992), 639-646.

[M-V] S. Mandal and P. L. N. Varma, On a question of Nori: the local case, *Communications in Algebra* **25** (1997), 451-457.

[Mo1] N. Mohan Kumar, On two conjectures about polynomial rings, *Invent. Math.* **46** (1978), 225-236.

[Mo2] N. Mohan Kumar, Stably free modules, *Amer. J. Math.* **107** (1985), 1439-1444.

[P] B. Plumstead, The conjectures of Eisenbud and Evans, *Amer. J. Math.* **105** (1983), 1417-1433.

[Q] D. Quillen, Projective modules over polynomial rings, *Invent. Math.* **36** (1976), 167-171.

[RS] Raja Sridharan, Projective modules and complete intersections, *K-Theory* **13** (1998), 269-278.

[Ra 1] R. A. Rao, Two examples of Bass-Quillen-Suslin conjectures, *Math. Ann.* **279** (1987), 227-238.

[Ra 2] R. A. Rao, The Bass-Quillen conjecture in dimension three but characteristic $\neq 2,3$ via a question of A. Suslin, *Invent. Math.* **93** (1988), 609-618.