A QUESTION OF NORI, SEGRE CLASSES OF IDEALS AND OTHER APPLICATIONS

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1. Introduction

Let R be a commutative, Noetherian ring and $I \subset R$ be an ideal such that I is locally generated by n elements. In general, local generators may not have a lift to a set of n global generators of I. For example, if R is the coordinate ring of a real 3-sphere and $I = \mathfrak{m}$ is a real maximal ideal, then it is well-known that $\mathfrak{m}/\mathfrak{m}^2$ is 3-generated (therefore, $\mathfrak{m}$ is locally 3-generated) but no surjection $\overline{\theta} : R^3 \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ can be lifted to a surjection $\theta : R^3 \twoheadrightarrow \mathfrak{m}$ (in fact, $\mathfrak{m}$ is not generated by 3 elements). Let us now cite a non-trivial instance where one has an affirmative conclusion. Let $R = A[T]$ and $I \subset A[T]$ be an ideal such that I/I^2 is generated by n elements, where $n \geq \dim(A[T]/I) + 2$. If I contains a monic polynomial, then it is implicit in Mandal's famous result [12] that any surjection $\overline{\theta} : A[T]^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$.

We now focus on the set up when $R = A[T]$ and consider the following (more general) situation : Let $I \subset A[T]$ be an ideal and P be a projective A-module of rank $n \geq \dim(A[T]/I) + 2$. Assume that there is a surjection $\overline{\theta} : P[T] \twoheadrightarrow I/I^2$. One may wonder, under what condition(s) $\overline{\theta}$ may be lifted to a surjection $\theta : P[T] \twoheadrightarrow I$. A necessary condition obviously would be that $I(0)$ should be image of P. But there are examples [4, 5.2] to show that this is not sufficient. In this context, an intriguing open question is the following one which is a variant of a question of Nori (see [13, 14]).

Question 1.1. Let A be a regular ring, $I \subset A[T]$ be an ideal and P be a projective A-module of rank n where $n \geq \dim(A[T]/I) + 2$. Assume that there is a surjection $\overline{\varphi} : P[T] \twoheadrightarrow I/(I^2T)$. Can $\overline{\varphi}$ be lifted to a surjective $A[T]$-linear map $\varphi : P[T] \twoheadrightarrow I$?

Note that $\overline{\varphi}(0) : P \twoheadrightarrow I(0)$. The assumption of regularity cannot be dropped (as there is an example due to Bhatwadekar, Mohan Kumar and Srinivas [4, 6.4]) unless the ideal I has some special properties (as shown by Mandal in [13] where he gave an affirmative answer when I contains a monic polynomial, without assuming the ring to be regular). The best result that we have so far is the one due to Bhatwadekar-Keshari [2, 4.13] where they assume A to be a regular domain of dimension d which is

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essentially of finite type over an infinite perfect field k and further that $\operatorname{ht} I = n$ with $2n \geq d+3$. It has been shown in [8] that one need not take the field to be perfect in [2, 4.13].

As it stands, even when A is a ring of the type as considered in [2], one does not have a complete answer to (1.1). One of our objectives in this paper is to explore whether we can improve the result of [2] by relaxing the condition on height of the ideal, i.e., allowing ideals of possibly smaller height into consideration. We prove (see (3.6) below)

Theorem 1.2. *Let A be a regular domain which is essentially of finite type over an infinite field k. Let $I \subset A[T]$ be an ideal and P be a projective A-module of rank n where $n + \operatorname{ht} I \geq \dim A[T] + 2$. Then any surjection $\phi : P[T] \twoheadrightarrow I/(I^2T)$ can be lifted to a surjection from $P[T]$ to I.*

Note that when $\operatorname{ht} I = n$, we recover [2, 4.13]. However, we are not giving a new proof of the theorem of Bhatwadekar-Keshari here. We are using their theorem to prove the above result, for which we need to have a suitable set of so called "moving lemma", "addition" and "subtraction" principles in a more general set up than the existing ones. We prove these in Section 3.

The cluster of moving lemma, addition and subtraction principles also enables us to answer another interesting question. Let R be a Noetherian ring, $J \subset R$ be an ideal with $\mu(J/J^2) = n$, where $n + \operatorname{ht} J \geq \dim R + 3$. Let $J = (a_1, \cdots, a_n) + J^2$ be given. It is natural to ask under what condition these local generators can be lifted to a set of global generators of J. In other words, when can we find $b_1, \cdots, b_n$ such that $J = (b_1, \cdots, b_n)$ where $a_i - b_i \in J^2$? We address this question in Section 4 in the following way. Given a surjection $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ (corresponding to a set of local generators) we associate an element $s^n(J, \omega_J)$ in the n^{th} Euler class group $E^n(R)$ (see (4.1) for the definition of $E^n(R)$) such that $s^n(J, \omega_J) = 0$ if and only if ω_J can be lifted to a surjective map $\theta : R^n \twoheadrightarrow J$. We call $s^n(J, \omega_J)$ the n^{th} *Segre class* of the pair (J, ω_J). For further details look at Section 4.

Any reader familiar with the theory of Euler class groups will be aware that Question 1.1 is intimately related to the Euler class groups. When Nori proposed the definition of the Euler class group (of a smooth affine domain A) some twenty years back, he also suggested Question 1.1 as an important tool (see [13, 14] for motivation). Nori's definition of the Euler class group, given in terms of "homotopy" with respect to the affine line $\mathbb{A}^1$, appeared in [4] (see (5.2) below). However, Bhatwadekar-Sridharan settled (1.1) in some special case [4, 3.8], then came up with an alternative definition of the Euler class group and proved its equivalence with Nori's definition by using [4, 3.8]. Section 5 is about revival of Nori's "homotopical" definition of the Euler class group. In this section we closely investigate Nori's definition, illustrate how it works for smooth affine domains and recover the main result of [4] on Euler classes. With an example due to Bhatwadekar (personal communication) we show why Nori's definition does not naturally extend to non-regular rings. At the end, we formulate the correct generalization to such rings.

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2. Preliminaries

All rings are assumed to be commutative Noetherian and modules are assumed to be finitely generated. We refer to [2] for any undefined term. In this section we collect some results which will be used frequently in later sections.

We begin by stating two classical results due to Serre [18] and Bass [1], respectively.

Theorem 2.1. *(Serre) Let A be a ring and P be a projective A-module. If rank$(P) > \dim A/\mathcal{J}(A)$, then $P \simeq Q \oplus A$ for some A-module Q, where $\mathcal{J}(A)$ is the Jacobson radical of A.*

Theorem 2.2. *(Bass) Let A be a ring and let P be a projective A-module of rank $> \dim A/\mathcal{J}(A)$. Then the group $\mathcal{E}(P\oplus A)$ of transvections of $P\oplus A$ acts transitively on $\mathrm{Um}(P\oplus A)$.*

The following result is about lifting of automorphisms [3, 4.1].

Proposition 2.3. *Let I be an ideal of a ring A and let P be a projective A-module. Then any transvection ϕ of P/IP (i.e. $\phi \in \mathcal{E}(P/IP)$) can be lifted to an automorphism Φ of P.*

The following result is due to Eisenbud and Evans [10].

Theorem 2.4. *Let A be a ring and let P be a projective A-module of rank n. Let $(\alpha, a) \in (P^*\oplus A)$. Then there exists $\beta \in P^*$ such that $\mathrm{ht}\, I_a \geq n$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq n$ and I is a proper ideal of A, then $\mathrm{ht}\, I = n$.*

The next two results are due to Bhatwadekar and Raja Sridharan.

Lemma 2.5. [5, 2.11] *Let A be a ring and let I be an ideal of A. Let I_1 and I_2 be ideals of A contained in I such that $I_2 \subset I^2$ and $I_1 + I_2 = I$. Then $I = I_1 + (e)$ for some $e \in I_2$ and $I_1 = I \cap I'$, where $I_2 + I' = A$.*

Lemma 2.6. [4, 3.5] *Let A be a regular domain containing a field and let I be an ideal of $A[T]$. Let P be a projective A-module and $J = I \cap A$. Let $\phi : P[T] \twoheadrightarrow I/(I^2T)$ be a surjection. Suppose $\phi \otimes A_{1+J}[T]$ can be lifted to a surjection $\psi : P_{1+J}[T] \twoheadrightarrow I_{1+J}$. Then ϕ can be lifted to a surjection $\Phi : P[T] \twoheadrightarrow I$.*

The following result is due to Mandal and Raja Sridharan [14, 2.3].

Theorem 2.7. *Let A be a ring and let I_1, I_2 be two comaximal ideals of $R = A[T]$. Assume that I_1 contains a monic polynomial and I_2 is an extended ideal from A, i.e. $I_2 = I_2(0)R$. Let P be a projective A-module of rank $r \geq \dim(R/I_1)+2$ and let $I = I_1 \cap I_2$. Let $\rho : P \twoheadrightarrow I(0)$ and $\delta : P[T]/I_1P[T] \twoheadrightarrow I_1/I_1^2$ be two surjections such that $\delta(0) = \rho \otimes A/I_1(0)$. Then there exists a surjection $\eta : P[T] \twoheadrightarrow I$ such that $\eta(0) = \rho$.*

3. Main Theorem

Unless otherwise mentioned, by a ring we mean a commutative Noetherian ring.

We begin with a lemma. When $\operatorname{ht} I = n$ and $f = 1$, it follows from [2, 5.5].

Lemma 3.1. *(Moving lemma) Let J be an ideal of a ring A and let P be a projective A-module of rank $n \geq \dim A/J + 1$. Let $\theta : P \twoheadrightarrow J/J^2f$ be a surjection for some $f \in A$. Given any ideal $K \subset A$ with $\dim A/K \leq n-1$, θ can be lifted to a surjection $\Theta : P \twoheadrightarrow J''$ such that*
(i) $J'' + (J^2 \cap K)f = J$,
(ii) $J'' = J \cap J'$, where $\operatorname{ht} J' \geq n$ and
(iii) $J' + (J^2 \cap K)f = A$.

Proof. Let $\Delta : P \to J$ be a lift of θ. Then $\Delta(P) + J^2f = J$. By (2.5), there exists $b \in J^2f$ such that $\Delta(P) + (b) = J$. Let "bar" denote reduction modulo the ideal $(J^2 \cap K)f$. Note that $\dim A/(J^2 \cap K) \leq n-1$.

Applying (2.4) on $(\overline{\Delta}, \overline{b}) \in (\overline{P^*} \oplus \overline{A})$, there exists $\Delta_1 \in P^*$ such that if $N = (\Delta + b\Delta_1)(P)$, then $\operatorname{ht} \overline{N}_{\overline{b}} \geq n$. Since $\Delta \oplus b\Delta_1$ is also a lift of θ, replacing Δ with $\Delta + b\Delta_1$, we may assume that $N = \Delta(P)$.

Now $N + (b) = J$ and $b \in J^2f$. By (2.5), $N = J \cap J_1$, where $J_1 + (b) = A$. Since $N_b = (J_1)_b$ and $\overline{N} = \overline{J} \cap \overline{J}_1$, we get $\operatorname{ht} \overline{J}_1 = \operatorname{ht}(\overline{J}_1)_{\overline{b}} = \operatorname{ht} \overline{N}_{\overline{b}} \geq n$. But $n \leq \operatorname{ht} \overline{J}_1 = \operatorname{ht}(\overline{J}_1)_{\overline{f}} \leq \dim \overline{A_f} \leq n-1$. Hence we get $\overline{J}_1 = \overline{A}$. Therefore, $\overline{N} = \overline{J}$, i.e. $\Delta(P) + (J^2 \cap K)f = J$.

By (2.5), there exists $c \in (J^2 \cap K)f$ such that $\Delta(P) + (c) = J$. By (2.4), replacing Δ by $\Delta + c\Delta_2$ for some $\Delta_2 \in P^*$, we may assume that $\Delta(P) = J \cap J'$, where $\operatorname{ht} J' \geq n$ and $J' + (c) = A$. This proves the lemma. $\square$

We now prove some "addition" and "subtraction" principles tailored to suit our needs.

Proposition 3.2. *(Subtraction Principle) Let I, J be two comaximal ideals of a ring A and let $P = Q \oplus A$ be a projective A-module of rank n. Assume that $n \geq \dim(A/J) + 2$ and $n + \operatorname{ht} I \geq \dim A + 3$. Assume that $\Phi : P \twoheadrightarrow I$ and $\Psi : P \twoheadrightarrow I \cap J$ are two surjections such that $\Phi \otimes A/I = \Psi \otimes A/I$. Then there exists a surjection $\Delta : P \twoheadrightarrow J$ such that $\Delta \otimes A/J = \Psi \otimes A/J$.*

Proof. As each of I, J is locally generated by n elements, we have $\operatorname{ht} I \leq n$ and $\operatorname{ht} J \leq n$. Note that to prove the result we can change Φ and Ψ by composing it with automorphisms of $Q \oplus A$.

Let "bar" denote reduction modulo J^2 and we write $\Phi = (\Phi_1, a_1)$, where $\Phi_1 \in Q^*$. Then $(\overline{\Phi}_1, \overline{a}_1) \in \operatorname{Um}(\overline{Q} \oplus \overline{A})$. Since $\dim \overline{A} \leq n-2$, by (2.1), $\overline{Q}$ has a unimodular element i.e. $\overline{Q} = Q_1 \oplus \overline{A}$. Write $\overline{\Phi}_1 \in \overline{Q}^*$ as $(\overline{\alpha}, \overline{b}_1)$, where $\overline{\alpha} \in Q_1^*$. By (2.2), there exists $\sigma \in \mathcal{E}(\overline{Q} \oplus \overline{A})$ such that $(\overline{\alpha}, \overline{b}_1, \overline{a}_1)\sigma = (0, 1, 0)$.

Using (2.3), let $\theta \in \operatorname{Aut}(Q \oplus A)$ be a lift of σ. If $(\Phi_1, a_1)\theta = (\Phi_2, a_2)$, then $a_2 \in J^2$ and $\overline{\Phi}_2 \in \operatorname{Um}(\overline{Q})$. By (2.4), there exists $\Gamma \in Q^*$ such that if $K = (\Phi_2 + a_2\Gamma)(Q)$, then $\operatorname{ht} K_{a_2} \geq n-1$. Since $(\Phi_2 + a_2\Gamma, a_2)$ is also a lift of $\Phi = (\Phi_1, a_1)$, replacing Φ_2 with $\Phi_2 + a_2\Gamma$, we may assume that $K = \Phi_2(Q)$ and $\operatorname{ht} K_{a_2} \geq n-1$. Note that $(K, a_2) = I$.

Case 1. Assume that $\operatorname{ht} I < n$. It is easy to see that $\operatorname{ht} K = \operatorname{ht} I$. Since $\Phi_2(Q) + J^2 = A$, replacing a_2 by $a_2 + \Phi_2(q)$ for some $q \in Q$, we may assume that $a_2 = 1$ modulo J^2.

Consider the following ideals in the ring $A[Y]$: $\mathcal{I}_1 = (K, Y + a_2)$, $\mathcal{I}_2 = JA[Y]$ and $\mathcal{I}_3 = \mathcal{I}_1 \cap \mathcal{I}_2$. Note that $\mathcal{I}_1(0) = I$ and $\mathcal{I}_2(0) = J$. We have two surjections

$$\Psi : P \twoheadrightarrow \mathcal{I}_3(0) \text{ and } \delta := (\Phi_2 \otimes A[Y], Y + a_2) : P[Y] \twoheadrightarrow \mathcal{I}_1$$

such that $\Psi \otimes A/\mathcal{I}_1(0) = \Phi \otimes A/\mathcal{I}_1(0) = \delta \otimes A/\mathcal{I}_1(0)$. Further $\dim A[Y]/\mathcal{I}_1 = \dim A/K$.

Since $K \subset I$ have the same height and $n + \operatorname{ht} I \geq \dim A + 3$, we have $\dim A/K \leq n-3$. Hence, applying (2.7), we get a surjection $\eta : P[Y] \twoheadrightarrow \mathcal{I}_3$ such that $\eta(0) = \Psi : Q \oplus A \twoheadrightarrow J$. Since $1 - a_2 \in J^2$, putting $Y = 1 - a_2$, we get a surjection $\eta(1 - a_2) := \Delta : P \twoheadrightarrow J$ with $\Delta \otimes A/J = \Psi \otimes A/J$. This proves the result in this case.

Case 2. Assume that $\operatorname{ht} I = n$. Then height of $K_{a_2} \geq n-1$ and $I = (K, a_2)$ implies that $\operatorname{ht} K \geq n-1$. Since $n + \operatorname{ht} I \geq \dim A + 3$, we get $\dim A/K \leq n-2$. Now we can complete the proof as in case 1. $\square$

Proposition 3.3. *(Addition principle) Let I, J be two comaximal ideals of a ring A and let $P = Q \oplus A$ be a projective A-module of rank n, where $n + \operatorname{ht}(I \cap J) \geq \dim A + 3$. Let $\Phi : P \twoheadrightarrow I$ and $\Psi : P \twoheadrightarrow J$ be two surjections. Then there exists a surjection $\Delta : P \twoheadrightarrow I \cap J$ such that $\Phi \otimes A/I = \Delta \otimes A/I$ and $\Psi \otimes A/J = \Delta \otimes A/J$.*

Proof. As each of I, J is locally generated by n elements, we have $\operatorname{ht} I \leq n$ and $\operatorname{ht} J \leq n$. Further, we can change Φ and Ψ by composing it with automorphisms of $Q \oplus A$.

Let "bar" denote reduction modulo J^2 and write $B = A/J^2$. Since $n + \operatorname{ht} J \geq \dim A + 3$, we get $\dim B \leq n-3$, by (2.1), $\overline{Q} = Q_1 \oplus B$. Further, $I + J = A$ and hence $\overline{I} = B$. Write $\overline{\Phi} = (\phi_1, b_1, b_2) : Q_1 \oplus B^2 \twoheadrightarrow B$ for the natural surjection induced from Φ. Since $\overline{\Phi} = (\phi_1, b_1, b_2) \in \operatorname{Um}(Q_1 \oplus B^2)$, applying (2.2), we get $\Theta \in \mathcal{E}(Q_1 \oplus B^2)$ such that $(\phi_1, b_1, b_2)\Theta = (0, 1, 0)$. Using (2.3), let $\theta \in \mathcal{E}(Q \oplus A)$ be a lift of Θ. If $\Phi\theta = (\Phi_1, b)$, then $\Phi_1(Q) + J^2 = A$ and $b \in J^2$. Applying (2.4), and replacing Φ_1 by $\Phi_1 + b\Gamma$ for some $\Gamma \in$

Q^*, we may assume that $\operatorname{ht}(H_b) \geq n-1$, where $H = \Phi_1(Q)$. Since $n+\operatorname{ht} I \geq \dim A+3$, as in the proof of (3.2), we can conclude that (i) if $\operatorname{ht} I < n$, then $\operatorname{ht} I = \operatorname{ht} H$ and (ii) if $\operatorname{ht} I = n$, then $\operatorname{ht} H \geq n-1$. In both the cases, we get that $\dim A/H \leq n-2$.

Let $C = A/H$ and let "tilde" denote reduction modulo H. Since $H + J^2 = A$, we get $\widetilde{\Psi} \in \operatorname{Um}(\widetilde{Q}\oplus C)$. Further, $\dim C \leq n-2$, hence by (2.1), $\widetilde{Q} = Q_2 \oplus C$. Write $\widetilde{\Psi} = (\Psi_1, c_1, c_2) \in \operatorname{Um}(Q_2\oplus C^2)$. Applying (2.2) on $\widetilde{\Psi} = (\Psi_1, c_1, c_2)$, we get $\Sigma \in \mathcal{E}(Q_2\oplus C^2)$ such that $(\Psi_1, c_1, c_2)\Sigma = (0, 1, 0)$. Using (2.3), let $\sigma \in \mathcal{E}(Q\oplus A)$ be a lift of Σ. If $\Psi\sigma = (\Psi_2, c)$, then $\Psi_2(Q) + H = A$ and $c \in H$. Applying (2.4), and replacing Ψ_2 by $\Psi_2 + c\Gamma'$ for some $\Gamma' \in Q^*$, we may assume that $\operatorname{ht}(K_c) \geq n-1$, where $K = \Psi_2(Q)$. Once again it is easy to see that (i) if $\operatorname{ht} J < n$, then $\operatorname{ht} K = \operatorname{ht} J$ and (ii) if $\operatorname{ht} J = n$, then $\operatorname{ht} K \geq n-1$. From this, we can conclude that $\dim A/K \leq n-2$. We have $H + K = A$ and $\dim A/(H \cap K) \leq n-2$.

Consider the following ideals of $A[T]$:

$$\mathcal{I}_1 = (H, T^2 + (b-2)T + 1),\ \mathcal{I}_2 = (K, T^2 + (c-2)T + 1) \text{ and } \mathcal{I}_3 = \mathcal{I}_1 \cap \mathcal{I}_2.$$

If we write $\theta_1 = (\Phi_1, T^2 + (b-2)T + 1)$ and $\theta_2 = (\Psi_2, T^2 + (c-2)T + 1)$, then θ_1, θ_2 are surjections from $Q[T]\oplus A[T]$ to $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. Note that $\mathcal{I}_1(1) = I$ and $\mathcal{I}_2(1) = J$.

Since $\mathcal{I}_1 + \mathcal{I}_2 = A[T]$, we have $\mathcal{I}_3/\mathcal{I}_3^2 = \mathcal{I}_1/\mathcal{I}_1^2 \oplus \mathcal{I}_2/\mathcal{I}_2^2$. Hence, using surjections θ_1 and θ_2, we get a surjection $\Delta : Q[T]\oplus A[T] \twoheadrightarrow \mathcal{I}_3/\mathcal{I}_3^2$ such that $\Delta \otimes A[T]/\mathcal{I}_1 = \theta_1 \otimes A[T]/\mathcal{I}_1$ and $\Delta \otimes A[T]/\mathcal{I}_2 = \theta_2 \otimes A[T]/\mathcal{I}_2$. Since $\mathcal{I}_3(0) = A$, it is easy to see that the surjection Δ can be lifted to a surjection $\Delta_1 : Q[T]\oplus A[T] \twoheadrightarrow \mathcal{I}_3/(\mathcal{I}_3^2 T)$. Since $\mathcal{I}_3$ contains a monic polynomial and $\dim A[T]/\mathcal{I}_3 = \dim A[T]/(\mathcal{I}_1 \cap \mathcal{I}_2) = \dim A/(H \cap K) \leq n-2$, applying (2.7), we can lift Δ_1 to a surjection $\Delta_2 : Q[T]\oplus A[T] \twoheadrightarrow \mathcal{I}_3$.

Write $\Delta_2(1) := \Delta_3$. Then $\Delta_3 : Q\oplus A \twoheadrightarrow I \cap J$ is a surjection. Further, we have $\Delta_3 \otimes A/I = \Delta(1) \otimes A/I = \theta_1(1) \otimes A/I = (\Phi_1, b) \otimes A/I = \Phi \otimes A/I$. Similarly, $\Delta_3 \otimes A/J = \Psi \otimes A/J$. Hence Δ_3 is the required surjection. This completes the proof. $\square$

The following result generalises [2, 4.6]. Recall that $A(T)$ denotes the ring obtained from $A[T]$ by inverting all monic polynomials.

Lemma 3.4. *Let A be a ring, I an ideal of $A[T]$ with $I + \mathcal{J}A[T] = A[T]$ and let n be a positive integer such that $\dim A/\mathcal{J} \leq n-2$, where $\mathcal{J}$ denotes the Jacobson radical of A. Let P be a projective A-module of rank $n \geq \dim A[T] - \operatorname{ht} I + 2$. Let $\phi : P[T] \twoheadrightarrow I/I^2$ be a surjection. If the surjection $\phi \otimes A(T) : P(T) \twoheadrightarrow IA(T)/I^2A(T)$ can be lifted to a surjection $\Phi_1 : P(T) \twoheadrightarrow IA(T)$, then ϕ can be lifted to a surjection $\Phi : P[T] \twoheadrightarrow I$.*

Proof. As I is locally generated by n elements, we have $\operatorname{ht} I \leq n$. Further, if $\operatorname{ht} I = n$, then the result follows from ([2], Lemma 4.6). Hence, we assume that $\operatorname{ht} I < n$.

As $\dim A[T]/\mathcal{J}A[T] \leq n-1$, by (3.1), ϕ has a lift $\Psi : P[T] \twoheadrightarrow J'$ such that

(i) $J' + (I^2 \cap \mathcal{J}A[T]) = I$,
(ii) $J' = I \cap J$, where J is an ideal of height $\geq n$, and
(iii) $J + (I^2 \cap \mathcal{J}A[T]) = A[T]$.

We assume that $\operatorname{ht} J = n$ (if $\operatorname{ht} J > n$ then $J = A$ and we are done). We get a surjection $\psi : P[T] \twoheadrightarrow J/J^2$ induced from Ψ. Note that $\Phi_1 \otimes A(T)/IA(T) = \Psi \otimes A(T)/IA(T)$. Hence Φ_1 is a lift of $\Psi \otimes A(T)/IA(T)$.

We observe that $\dim A(T) - \operatorname{ht} IA(T) \leq \dim A[T] - 1 - \operatorname{ht} I \leq n - 3$ and

$$\begin{aligned}\dim (A(T)/JA(T)) &\leq \dim A(T) - \operatorname{ht} JA(T) \leq \dim A[T] - 1 - \operatorname{ht} J \\ &= \dim A[T] - 1 - n \leq \dim A[T] - 1 - \operatorname{ht} I \leq n - 3.\end{aligned}$$

Applying (3.2) to the surjections Φ_1 and $\Psi \otimes A(T)$, we get a surjection $\Psi_1 : P(T) \twoheadrightarrow JA(T)$ such that $\Psi_1 \otimes A(T)/JA(T) = \Psi \otimes A(T)/JA(T) = \psi \otimes A(T)/JA(T)$.

Since $\operatorname{ht} J = n$ and $J + \mathcal{J}A[T] = A[T]$, applying [2, 4.6], we conclude that $\Psi \otimes A[T]/J$ can be lifted to a surjection $\Delta : P[T] \twoheadrightarrow J$. Note that $\dim A[T]/I \leq n - 2$ and $\dim A[T] - \operatorname{ht} J < \dim A[T] - \operatorname{ht} I \leq n - 2$ (since $\operatorname{ht} J = n > \operatorname{ht} I$). Hence $\dim A[T] - \operatorname{ht} J \leq n - 3$. Applying (3.2) to Δ and Ψ, we get a surjection $\Phi : P[T] \twoheadrightarrow I$ such that $\Phi \otimes A[T]/I = \phi$. This proves the result. $\square$

The following result generalises ([2], Proposition 4.9).

Proposition 3.5. *Let A be a regular domain containing a field and let I be an ideal of $A[T]$. Let P be a projective A-module of rank $n \geq \dim A[T] - \operatorname{ht} I + 2$. Let $\psi : P[T] \twoheadrightarrow I/I^2T$ be a surjection. If there exists a surjection $\Psi' : P(T) \twoheadrightarrow IA(T)$ which is a lift of $\psi \otimes A(T)$, then we can lift ψ to a surjection from $P[T]$ to I.*

Proof. As I is locally generated by n elements, we have $\operatorname{ht} I \leq n$. Further, if $\operatorname{ht} I = n$, then the result follows from [2, 4.9]. Hence, we assume that $\operatorname{ht} I < n$.

Step 1. By [4, 3.5], we may assume that $J = I \cap A \subset \mathcal{J}(A)$. Note that $\dim (A/J) \leq \dim A - \operatorname{ht} J = \dim A[T] - 1 - \operatorname{ht} J \leq \dim A[T] - \operatorname{ht} I \leq n - 2$. Therefore, by (2.1), we may assume that $P = Q \oplus A^2$.

By (3.1), we can lift ψ to a surjection $\Psi : P[T] \twoheadrightarrow I \cap I'$, where $I' \subset A[T]$ is of height n with $\Psi(P[T]) + (J^2T) = I$ and $I' + (J^2T) = A[T]$. (If $\operatorname{ht} I' > n$, then $I' = A[T]$ and we are done.)

Let $\psi_1 : P[T] \twoheadrightarrow I'/I'^2$ be induced from Ψ. Since $I'(0) = A$ and P has a unimodular element, ψ_1 can be lifted to a surjection $\psi_2 : P[T] \twoheadrightarrow I'/I'^2T$ by [4, 3.9].

Observe that $\dim A(T) - \operatorname{ht} IA(T) \leq n - 3$ and $\dim A(T)/I'A(T) \leq n - 2$. Further, we have $\Psi \otimes A(T)/IA(T) = \Psi' \otimes A(T)/IA(T)$. Hence, applying (3.2), we get a surjection $\Delta : P(T) \twoheadrightarrow I'A(T)$ such that $\Delta \otimes A(T)/I'A(T) = \psi_2 \otimes A(T)/I'A(T) =$

$\psi_1 \otimes A(T)/I'A(T)$. Since $\operatorname{ht} I' = n$, by [2, 4.9], ψ_2 can be lifted to a surjection $\Delta_1 : P[T] \twoheadrightarrow I'$.

Step 2. Write $B = A[T]/(J^2T)A[T]$. Since $I' + (J^2T) = A[T]$, we get $(\Delta_1 \otimes B) \in \operatorname{Um}(P[T]^* \otimes B)$. Since $P = Q \oplus A^2$, we write $\Delta_1 \otimes B = (\Delta_2, a_1, a_2)$, where $\Delta_2 \in Q[T]^* \otimes B$ and $a_1, a_2 \in B$. Note that $B/JB = (A/J)[T]$ and $\dim A/J \le n-2$.

Let "bar" denote reduction modulo JB and write $\overline{B} := B/JB$. By a result of Plumstead [17], there exists $\Theta \in \mathcal{E}(\overline{P[T]^* \otimes B})$ such that $\Theta(\overline{\Delta}_2, \overline{a}_1, \overline{a}_2) = (0,1,0)$. Since JB is contained in the Jacobson radical of B, we can lift Θ to $\Theta_1 \in \mathcal{E}(P[T]^* \otimes B)$ such that $\Theta_1(\Delta_2, a_1, a_2) = (0,1,0)$. Let $\Theta_2 \in \operatorname{Aut}(P[T]^*)$ be a lift of Θ_1. If $\Theta_2(\Delta_1) = (\Delta_3, b_1, b_2)$, then we get that $\Delta_3(Q[T]) \subset (J^2T)$, $b_1 = 1$ modulo (J^2T) and $b_2 \in (J^2T)$.

By (2.4), replacing (Δ_3, b_1, b_2) by $(\Delta_3 + b_2\delta_1, b_1 + cb_2, b_2)$ for some $\delta_1 \in Q[T]^*$ and $c \in A[T]$, we may assume that $\operatorname{ht}(\Delta_3(Q[T]), b_1) = n-1$. Note that we still have $(\Delta_3(Q[T]), b_1) + (J^2T) = A[T]$. Further, replacing b_2 by $b_1 + b_2$, we may assume that $b_2 = 1$ modulo (J^2T). As $(\Delta_3(Q), b_1)$ is comaximal with $\mathcal{J}(A)A[T]$, we have

$$\begin{aligned}\dim A[T]/(\Delta_3(Q[T]), b_1) &\le \dim A[T] - \operatorname{ht}(\Delta_3(Q), b_1) - 1\\ &= \dim A[T] - n \le \dim A[T] - \operatorname{ht} I \le n-2.\end{aligned}$$

Write $C := A[T]$, $\widetilde{P} = P[T]$ and consider the following ideals of $C[Y]$:

$$\mathcal{K}_1 = (\Delta_3(Q[T]), b_1, Y + b_2),\ \mathcal{K}_2 = IC[Y] \text{ and } \mathcal{K}_3 = \mathcal{K}_1 \cap \mathcal{K}_2.$$

Note that $\mathcal{K}_1(0) = I'$ and $\mathcal{K}_2(0) = I$. We have two surjections

$$\Psi : \widetilde{P} \twoheadrightarrow \mathcal{K}_3(0) \text{ and } \Gamma = (\Delta_3, b_1, Y + b_2) : \widetilde{P}[Y] \twoheadrightarrow \mathcal{K}_1$$

such that $\Gamma(0) \otimes C/\mathcal{K}_1(0) = \Psi \otimes C/\mathcal{K}_1(0)$. Further, $\dim C[Y]/\mathcal{K}_1 = \dim C/(\Delta_3(Q), b_1) \le n-2$. Applying (2.7), we get a surjection $\eta : \widetilde{P}[Y] \twoheadrightarrow K_3$ such that $\eta(0) = \Psi$. Since $1 - b_2 \in (J^2T)$, we get a surjection $\eta_1 := \eta(1-b_2) : \widetilde{P} \twoheadrightarrow I$ with $\eta_1 = \Psi$ modulo (J^2T). This proves the result. $\square$

We now prove our main theorem which is a generalization of [2, 4.13].

Theorem 3.6. *Let k be an infinite field and let A be a regular domain which is essentially of finite type over k. Let I be an ideal of $A[T]$ and let P be a projective A-module of rank $n \ge \dim A[T] - \operatorname{ht} I + 2$. Then any surjection $\phi : P[T] \twoheadrightarrow I/(I^2T)$ can be lifted to a surjection from $P[T]$ to I.*

Proof. As I is locally generated by n elements, we have $\text{ht}\, I \leq n$. Further, if $\text{ht}\, I = n$, then the result follows from ([2], Theorem 4.13). Hence, we assume that $\text{ht}\, I < n$.

By (2.6), we may assume that $J = I \cap A \subset \mathcal{J}(A)$. Since

$$\dim A/J \leq \dim A - \text{ht}\, J \leq \dim A - \text{ht}\, I + 1 = \dim A[T] - \text{ht}\, I \leq n - 2,$$

we may assume that P has a unimodular element.

By (3.1), we can lift ϕ to a surjection $\Phi : P[T] \twoheadrightarrow I \cap I'$, where I' is an ideal of $A[T]$ of height n with $\Phi(P[T]) + (J^2T) = I$ and $I' + (J^2T) = A[T]$. Let $\psi : P[T] \twoheadrightarrow I'/I'^2$ be induced from Φ. Since $I'(0) = A$ and P has a unimodular element, ψ can be lifted to a surjection $\psi_1 : P[T] \twoheadrightarrow I'/(I'^2T)$. By [2, 4.13] and [8], ψ_1 can be lifted to a surjection $\Psi : P[T] \twoheadrightarrow I'$.

Applying (3.2), to $\Psi \otimes A(T)$ and $\Phi \otimes A(T)$, we get a surjection $\Delta : P(T) \twoheadrightarrow IA(T)$ such that $\Delta \otimes A(T)/IA(T) = \Phi \otimes A(T)/IA(T) = \phi \otimes A(T)/IA(T)$. By (3.5), ψ can be lifted to a surjection $\Theta : P[T] \twoheadrightarrow I$. This proves the result. $\square$

In case of regular domain, we have the following subtraction and addition principles. We give a proof of the Subtraction principle. The Addition principle can be proved similarly.

Proposition 3.7. *(Subtraction principle) Let A be a regular domain containing an infinite field k and let I, J be two comaximal ideals of $A[T]$. Let $P = Q \oplus A$ be a projective A-module of rank $n \geq \dim A[T] - \text{ht}\,(I \cap J) + 2$. Assume that $\Phi : P[T] \twoheadrightarrow I$ and $\Psi : P[T] \twoheadrightarrow I \cap J$ are two surjections such that $\Phi \otimes A[T]/I = \Psi \otimes A[T]/I$. Then there exists a surjection $\Delta : P[T] \twoheadrightarrow J$ such that $\Delta \otimes A[T]/J = \Psi \otimes A[T]/J$.*

Proof. Let $P_1, \cdots, P_r$ be the associated prime ideals of I and $Q_1, \cdots, Q_s$ be the associated prime ideals of J. As k is infinite, we can choose $\lambda \in k$ such that $T - \lambda \notin (\cup_1^r P_i) \cup (\cup_1^s Q_j)$. If $T - \lambda$ is a unit modulo the ideal I, then $I(\lambda) = A$; a similar conclusion holds for $J(\lambda)$. In the case when $T - \lambda$ is not a unit modulo I or J, we see that $T - \lambda$ is a non-zerodivisor modulo I as well as J. In this case, $\text{ht}\,(I, T - \lambda) = \text{ht}\, I + 1$ and $\text{ht}\,(J, T - \lambda) = \text{ht}\, J + 1$. As a consequence, $\text{ht}\, I(\lambda) \geq \text{ht}\, I$ and $\text{ht}\, J(\lambda) \geq \text{ht}\, J$. Replacing T by $T - \lambda$, we can take λ to be 0.

Note that Ψ induces a surjection $\overline{\psi} : P[T] \twoheadrightarrow J/J^2$. We have induced surjections $\Phi(0) : P \twoheadrightarrow I(0)$ and $\Psi(0) : P \twoheadrightarrow I(0) \cap J(0)$.

If $J(0) = A$, then $\overline{\psi}$ can be lifted to a surjection from $P[T]$ to $J/(J^2T)$. We now assume $J(0)$ to be a proper ideal. If $I(0) = A$, then $\Psi(0)$ is a surjection from P to $J(0)$. Since $\overline{\psi}(0) = \Psi(0) \otimes A/J(0)$, by ([4], Remark 3.9) $\overline{\psi}$ can be lifted to a surjection from $P[T]$ to $J/(J^2T)$.

Now assume that $I(0), J(0)$ both are proper ideals of A. We have, $\dim A - \text{ht}\, I(0) \leq \dim A[T] - 1 - \text{ht}\, I \leq n - 3$. Similarly, $\dim\,(A/J(0)) \leq n - 3$. Applying Subtraction principle (3.2) to surjections $\Phi(0)$ and $\Psi(0)$, we get a surjection $\alpha : A \twoheadrightarrow J(0)$ such that $\alpha \otimes A/J(0) = \Psi(0) \otimes A/J(0) = \overline{\psi}(0)$. Consequently, by ([4], Remark 3.9), $\overline{\psi}$ can be lifted to a surjection from $P[T]$ to $J/(J^2T)$.

Therefore, in any case, $\overline{\psi}$ can be lifted to a surjection, say, $\theta : P[T] \twoheadrightarrow J/(J^2T)$.

We now go to the ring $A(T)$ and consider surjections $\Phi \otimes A(T)$ and $\Psi \otimes A(T)$. Again, applying (3.2), we can find a surjection $\beta : P \otimes A(T) \twoheadrightarrow JA(T)$ such that $\beta \otimes A(T)/JA(T) = \Psi \otimes A(T)/JA(T)$. Clearly, β lifts θ. Therefore, by (3.5), we get a map $\Delta : P[T] \twoheadrightarrow J$ such that Δ lifts θ. It is easy to see that $\Delta \otimes A[T]/J = \Psi \otimes A[T]/J$. This proves the result. $\square$

Proposition 3.8. *(Addition principle) Let A be a regular domain containing an infinite field k. Let I, J be two comaximal ideals of $A[T]$ and let $P = Q \oplus R$ be a projective A-module of rank $n \geq \dim A[T] - \operatorname{ht}(I \cap J) + 2$. Let $\Phi : P[T] \twoheadrightarrow I$ and $\Psi : P[T] \twoheadrightarrow J$ be two surjections. Then there exists a surjection $\Delta : P[T] \twoheadrightarrow I \cap J$ such that $\Phi \otimes A[T]/I = \Delta \otimes A[T]/I$ and $\Psi \otimes A[T]/J = \Delta \otimes A[T]/J$.*

4. Segre classes

Let A be a commutative Noetherian ring of dimension d and let $I \subset A$ be an ideal such that $\mu(I/I^2) = n$ where $n + \operatorname{ht} I \geq d + 3$. Let $I = (a_1, \cdots, a_n) + I^2$ be given. It is natural to ask under what condition these local generators can be lifted to a set of global generators of I. In other words, when can we find $b_1, \cdots, b_n$ such that $I = (b_1, \cdots, b_n)$ where $a_i - b_i \in I^2$?

When $\operatorname{ht} I = n$, this has been accomplished in [6], where an abelian group $E^n(A)$ (called the n-th Euler class group of A) is defined and corresponding to the local data for I an element in this group (called the Euler class) is attached and it is shown that a desired set of global generators exists for I if the corresponding Euler class in $E^n(A)$ is zero.

In this section we consider the case when $\operatorname{ht} I$ is not necessarily equal to n. Given I and $\omega_I : (A/I)^n \twoheadrightarrow I/I^2$ (local data) we shall associate an element $s^n(I, \omega_I)$ in the Euler class group $E^n(A)$. We call this the *n-th Segre class* of the pair (I, ω_I). It will be shown that $s^n(I, \omega_I) = 0$ in $E^n(A)$ if and only if ω_I can be lifted to a surjection $\theta : A^n \twoheadrightarrow I$ (global generators). Further, when $\operatorname{ht} I = n$, the Segre class coincides with the Euler class of (I, ω_I).

We may note that the above question was considered in [9] under the hypotheses : $d = n = \mu(I/I^2) \geq 3$ and $\operatorname{ht} I \geq 2$. For further motivation the reader may look at [9], which in turn is inspired by Murthy's definition of Segre classes [16] .

Before proceeding to define the Segre class, we first quickly recall the definition of the n-th Euler class group $E^n(A)$ from [6].

Definition 4.1. Let A be a Noetherian ring of dimension d and let n be an integer with $2n \geq d+3$. A *local orientation* ω_I of an ideal $I \subset A$ of height n is a surjective homomorphisms from $(A/I)^n$ to I/I^2, up to an $\mathcal{E}_n(A/I)$-equivalence (here $\mathcal{E}_n$ stands for the group of elementary matrices). Let $L^n(A)$ denote the set of all pairs (I, ω_I), where I is an ideal of height n such that Spec (A/I) is connected and $\omega_I : (A/I)^n \twoheadrightarrow I/I^2$ is a local orientation. Let $G^n(A)$ denote the free abelian group generated by $L^n(A)$. Suppose I is an ideal of height n and $\omega_I : (A/I)^n \twoheadrightarrow I/I^2$ is a local orientation. By ([6], Lemma 4.1), there is a unique decomposition $I = \cap_1^r I_i$, such that I_i's are pairwise comaximal ideals of height n and Spec (A/I_i) is connected. Then ω_I naturally induces local orientations $\omega_{I_i} : (A/I_i)^n \twoheadrightarrow I_i/I_i^2$. Denote $(I, \omega) := \sum (I_i, \omega_i) \in G^n(A)$. We say a local orientation $\omega_I : (A/I)^n \twoheadrightarrow I/I^2$ is *global* if ω_I can be lifted to a surjection $\Omega : A^n \twoheadrightarrow I$. Let $H^n(A)$ be the subgroup of $G^n(A)$ generated by all (I, ω_I) where ω_I is a global orientation. The Euler class group of codimension n cycles is defined as $E^n(A) := G^n(A)/H^n(A)$.

Now let J be an ideal of A such that J/J^2 is generated by n elements, where $n + \operatorname{ht} J \geq \dim A + 3$. Given a surjection $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$, we will define the n^{th} Segre class of (J, ω_J), denoted by $s^n(J, \omega_J)$, as an element of the n^{th} Euler class group $E^n(A)$, as follows:

Definition 4.2. By (3.1), ω_J can be lifted to a surjection $\alpha : A^n \twoheadrightarrow J \cap J_1$, where J_1 is an ideal of height $\geq n$ with $J + J_1 = A$. If $J_1 = A$, then we define the n^{th} Segre class $s^n(J, \omega_J) = 0$ in $E^n(A)$. If J_1 is a proper ideal of height n, then α induces a surjection $\omega_{J_1} : (A/J_1)^n \twoheadrightarrow J_1/J_1^2$. We define the n^{th} Segre class $s^n(J, \omega_J) = -(J_1, \omega_{J_1})$ in $E^n(A)$.

We need to show that $s^n(J, \omega_J)$ is well defined as an element of $E^n(A)$. The argument is along the same line as in [9]. We give a sketch for the convenience of the reader.

Let J_2 be another ideal of A of height $\geq n$ such that $J + J_2 = A$ and ω_J has a lift to $\beta : A^n \twoheadrightarrow J \cap J_1$. If $J_2 = A$ then it is easy to check using addition and subtraction principles in the last section that $(J_1, \omega_{J_1}) = 0$ in $E(A)$. Therefore assume that J_2 is a proper ideal. Let $\omega_{J_2} : (A/J_2)^n \twoheadrightarrow J_2/J_2^2$ be the local orientation induced by β. Using Moving lemma 3.1 we can find an ideal J_3 of A of height n and a local orientation ω_{J_3} such that : (i) J_3 is comaximal with each of J, J_1 and J_2, (ii) $(J_1, \omega_{J_1}) + (J_3, \omega_{J_3}) = 0$ in $E(A)$. Again applying Lemma 3.1 we can find an ideal J_4 of A of height n such that $J \cap J_4$ is generated by n elements and J_4 is comaximal with each of J, J_1, J_2 and J_3.

Now addition principle implies that the ideal $J_1 \cap J_3 \cap J \cap J_4$ is generated by n elements. Since $J_1 \cap J$ is generated by n elements, by the subtraction principle (3.2) it follows that $J_3 \cap J_4$ is generated by n elements with appropriate set of generators. Now consider $J_2 \cap J_3 \cap J \cap J_4$. A similar chain of arguments will show that $J_2 \cap J_3$ is

n-generated by the appropriate set of generators. Keeping track of the generators, it is easy to see that this implies $(J_2, \omega_{J_2}) + (J_3, \omega_{J_3}) = 0$ in $E(A)$.

Therefore, $(J_1, \omega_{J_1}) = (J_2, \omega_{J_2})$ in $E(A)$ and $s(J, \omega_J)$ is well defined.

Remark 4.3. It is clear from the definition of the Segre class that $s^n(J, \omega_J) = (J, \omega_J)$ in $E^n(A)$ if $\operatorname{ht} J = n$.

Theorem 4.4. *Let J be an ideal of a ring A and let $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ be a surjection, where $n \geq \dim A - \operatorname{ht} J + 3$. Suppose $s^n(J, \omega_J) = 0$ in $E^n(A)$. Then ω_J can be lifted to a surjection $\theta : A^n \twoheadrightarrow J$.*

Proof. Let $s^n(J, w_J) = (J_1, \omega_{J_1})$ in $E^n(A)$ where ω_J has a lift $\alpha : A^n \twoheadrightarrow J \cap J_1$ and $\omega_{J_1} = \alpha \otimes A/J_1$. Now $s^n(J, \omega_J) = 0$ implies $(J_1, \omega_{J_1}) = 0$ in $E(A)$. Therefore, by [6], ω_{J_1} is a global orientation of J_1. This means that there exist a lift $\phi : A^n \twoheadrightarrow J_1$ of ω_{J_1}. Now we can apply the subtraction principle (3.2) to see that ω_J has the desired lift to a surjection $\theta : A^n \twoheadrightarrow J$. This proves the theorem. $\square$

The following result, on additivity of the Segre classes, is easy and we leave the proof to the reader.

Theorem 4.5. *(Addition) Let J_1, J_2 be two comaximal ideals of a ring A and let $\omega_{J_1} : (A/J_1)^n \twoheadrightarrow J_1/J_1^2$ and $\omega_{J_2} : (A/J_2)^n \twoheadrightarrow J_2/J_2^2$ be two surjections, where $n \geq \dim A - \operatorname{ht}(J_1 \cap J_2) + 3$. Then $s^n(J_1 \cap J_2, \omega_{J_1 \cap J_2}) = s^n(J_1, \omega_{J_1}) + s^n(J_2, \omega_{J_2})$ in $E^n(A)$, where $\omega_{J_1 \cap J_2} : (A/(J_1 \cap J_2))^n \twoheadrightarrow (J_1 \cap J_2)/(J_1 \cap J_2)^2$ is the surjection induced by ω_{J_1} and ω_{J_2}.*

Let A be a ring of dimension d and $J \subset A$ be an ideal such that $\mu(J/J^2) = n$ where $n + \operatorname{ht} J \geq d + 3$. It is almost a trivial application of the Nakayama lemma to see that if A is semilocal, then any $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : A^n \twoheadrightarrow J$. One may wonder if there exists any non-trivial example of a ring for which such a phenomenon holds. We give one such below.

Example 4.6. Let $(A, \mathfrak{m}, k)$ be a regular local ring which is either (i) essentially of finite type over an infinite field; or (ii) essentially of finite type and smooth over an excellent DVR (V, π) such that k is infinite and is separably generated over $V/\pi V$. Let $\dim A = d + 1$ and $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ be a regular parameter. Let $J \subset A_f$ be an ideal such that $\mu(J/J^2) = n$ where $n + \operatorname{ht} J \geq d + 3$. Then any $\omega_J : (A_f/J)^n \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : A_f^n \twoheadrightarrow J$. This follows from [8, 4.2, 5.2], since it is proved there that $E^n(A_f) = 0$.

Remark 4.7. Let A be a regular domain containing an infinite field k and let $I \subset A[T]$ be an ideal such that $\mu(I/I^2) = n$, where $n + \operatorname{ht} I = \dim A[T] + 2$. Let $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ be a given surjection. Following the same method as in (4.2), one can define the n^{th} Segre class $s^n(I, \omega_I)$ as an element of $E^n(A[T])$ and prove results similar as above using (3.7, 3.8).

5. HOMOTOPY RETURNS

In the final decade of the last century, Nori suggested a definition of the (nth) Euler class group of a smooth affine domain R of dimension n and associated for a projective R-module P of rank n (with trivial determinant) an element in this group, called the Euler class of P, and asked whether the vanishing of the Euler class is the precise obstruction for P to decompose as $P \simeq Q \oplus R$, for some R-module Q. In [4], Bhatwadekar-Sridharan settled Nori's question in the affirmative. They achieved this with a different (but equivalent to the one proposed by Nori) definition of the Euler class group which seems to be a bit easier to work with. (For the record, we may note that [4, Theorem 3.8], stated below as Theorem 5.1, turned out to be crucial to establish the equivalence). Moreover, their definition paved the way for further generalization to the Euler class group of a Noetherian ring R. All the papers written after [4] in this area are based on the definition of Bhatwadekar-Sridharan and within all the development Nori's definition has been lost. We believe that Nori's definition has its own intrinsic appeal and our aim in this section is to investigate it closely. We first recall how this definition works for a smooth affine domain (for which it was formulated). We then show why Nori's definition does not naturally extend to singular varieties. Finally, we provide a reformulation of Nori's definition which extends to general Noetherian rings.

We first quote the following result from [4] which is a special case of Question 1.1. This theorem will be crucially used to recover results from [4] using Nori's definition of the Euler class group.

Theorem 5.1. [4, 3.8] *Let R be a smooth affine domain of dimension $n \geq 3$ over an infinite perfect field k. Let $I \subset R[T]$ be an ideal of height n and P be a projective R-module of rank n. Assume that we are given a surjection $\varphi : P[T] \twoheadrightarrow I/(I^2T)$. Then there exists a surjection $\Phi : P[T] \twoheadrightarrow I$ such that Φ lifts φ.*

We now start with both the definitions of Euler class groups as given in [4]. For simplicity, in this section we consider rings with dimension $n \geq 3$. Further, as we shall only talk about the n^{th} Euler class group of a ring R of dimension n, we shall write $E(R)$ instead of $E^n(R)$.

Euler class group : Let R be a smooth affine domain of dimension n over an infinite perfect field k. Let B be the set of pairs (m, ω_m) where m is a maximal ideal of R and $\omega_m : (R/m)^n \twoheadrightarrow m/m^2$. Let G be the free abelian group generated by B. Let $J = m_1 \cap \cdots \cap m_r$, where m_i are maximal ideals of R. Any $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ induces surjections $\omega_i : (R/m_i)^n \twoheadrightarrow m_i/m_i^2$ for each i. We associate $(J, \omega_J) := \sum_1^r (m_i, \omega_i) \in G$.
Definition 5.2. (Nori) Let S be the set of elements $(I(1), \omega(1)) - (I(0), \omega(0))$ of G where (i) $I \subset R[T]$ is a local complete intersection ideal of height n; (ii) Both $I(0)$ and $I(1)$ are

reduced ideals of height n; (iii) $\omega(0)$ and $\omega(1)$ are induced by $\omega : (R[T]/I)^n \twoheadrightarrow I/I^2$. Let H be the subgroup generated by S. The Euler class group $E(R)$ is defined as $E(R) := G/H$.

Definition 5.3. (Bhatwadekar-Sridharan) Let H_1 be the subgroup of G generated by those elements (J, ω_J) of G for which ω_J has a lift to a surjection $\theta : R^n \twoheadrightarrow J$. The Euler class group $E(R)$ is defined as $E(R) := G/H_1$.

Remark 5.4. Let $(J, \omega_J) = (I(0), \omega(0)) - (I(1), \omega(1)) \in S$ and let $\overline{\sigma} \in SL_n(R/J)$. Then we have $\omega_J \overline{\sigma} : (R/J)^n \twoheadrightarrow J/J^2$. Since $\dim R/J = 0$, we have $SL_n(R/J) = \mathcal{E}_n(R/J)$ where $\mathcal{E}_n(R/J)$ is the elementary subgroup of $SL_n(R/J)$. As $\mathcal{E}_n(R) \longrightarrow \mathcal{E}_n(R/J)$ is surjective, there exists a preimage σ of $\overline{\sigma}$. Since σ is elementary, there exists $\Delta \in GL_n(R[T])$ such that $\Delta(0) = I_n$ (the identity matrix) and $\Delta(1) = \sigma$. Let $\omega' = \omega \circ (\Delta \otimes R[T]/I) : (R[T]/I)^n \twoheadrightarrow I/I^2$. We note that $J \cap I(0) = I(1)$ and since all the ideals are reduced, it follows that $J + I(0) = R$. It is now easy to check that $(J, \omega_J \overline{\sigma}) = (I(1), \omega'(1)) - (I(0), \omega'(0))$. Therefore, $(J, \omega_J \overline{\sigma}) \in S$.

Remark 5.5. In Nori's definition, the relations are given by homotopy with respect to the affine line $\mathbb{A}^1$. In this section we shall focus mainly on relations given by homotopy (hence the title of this section). We shall revisit some of the results from [4].

Lemma 5.6. [4, 4.5] *Let R be a smooth affine domain of dimension n and $J \subset R$ be a reduced ideal of height n. Assume that $J = (a_1, \cdots, a_n)$ and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ is induced by $a_1, \cdots, a_n$. Then there exists a local complete intersection ideal $I \subset R[T]$ and a surjection $\omega : (R[T]/I)^n \twoheadrightarrow I/I^2$ such that $I(0), I(1)$ are both reduced ideals of height n in R and $(J, \omega_J) = (I(0), \omega(0)) - (I(1), \omega(1))$ in G.*

Remark 5.7. For a proof, see [4] and note that the above lemma also works for a Noetherian ring R if the terms "reduced" and "local complete intersection" are dropped.

In the following important proposition, (5.1) is crucially used. Note that (5.1) does not hold for affine domains which are not smooth (for an example, see [4, 6.4]).

Proposition 5.8. *Let $(I(1), \omega(1)) - (I(0), \omega(0)) \in S$. Then there exists an ideal $I' \subset R[T]$ of height n such that $I' = (g_1, \cdots, g_n)$ and $(I(1), \omega(1)) - (I(0), \omega(0)) = (I'(1), \omega'(1)) - (I'(0), \omega'(0)) \in S$ where $\omega' : (R[T]/I')^n \twoheadrightarrow I'/I'^2$ is induced by $g_1, \cdots, g_n$.*

Proof. The proposition essentially is a restatement of [4, 4.3] and we shall urge the reader to see [4] for the proof. Retaining the same notations, we only identify the terms. Let $\omega : (R[T]/I)^n \twoheadrightarrow I/I^2$ be induced by $I = (f_1, \cdots, f_n) + I^2$. Using Swan's Bertini theorem instead of Eisenbud-Evans theorem in Moving lemma, there exists a reduced ideal $K \subset R$ of height n such that : (1) $K = (b_1, \cdots, b_n) + K^2$; (2) $I + KR[T] = R[T]$; (3) $I' = I \cap K[T] = (g_1, \cdots, g_n)$ where $g_i - f_i \in I^2$ and $g_i - b_i \in K^2 R[T]$.

Let $\omega' : (R[T]/I')^n \twoheadrightarrow I'/I'^2$ be induced by $g_1, \cdots, g_n$. It is now easy to see that the proposition follows from (1)-(3) above. $\square$

Proposition 5.9. *The set $S \subset G$ has the following properties :*

(1) *If $x \in S$, then $-x \in S$.*
(2) *If $(J_1, \omega_{J_1}), (J_2, \omega_{J_2}) \in S$, then $(J_1, \omega_{J_1}) - (J_2, \omega_{J_2}) \in S$.*
(3) *Let $(J_1, \omega_{J_1}), (J_2, \omega_{J_2}) \in G$ where $J_1 + J_2 = R$. If any two of the elements (J_1, ω_{J_1}), (J_2, ω_{J_2}), $(J_1 \cap J_2, \omega_{J_1 \cap J_2})$ belong to S, then so does the third.*

Proof. (1) Let $x = (I(1), \omega(1)) - (I(0), \omega(0)) \in S$ where $I \subset R[T]$ is a local complete intersection ideal of height n and $\omega : (R[T]/I)^n \twoheadrightarrow I/I^2$ a surjection. Also $I(0), I(1)$ are both reduced ideals of height n. Consider the automorphism $\phi : R[T] \longrightarrow R[Y]$ given by $T \mapsto Y = T - 1$. Let $\mathcal{I} = \phi(I)$ and $\omega' : R[Y]/\mathcal{I} \twoheadrightarrow \mathcal{I}/\mathcal{I}^2$ be the surjection corresponding to ω. Then it is easy to see that $-x = (I(0), \omega(0)) - (I(1), \omega(1)) = (\mathcal{I}(1), \omega'(1)) - (\mathcal{I}(0), \omega'(0)) \in S$.

(2) Let $(J_1, \omega_{J_1}), (J_2, \omega_{J_2}) \in S$. Then, there exists local complete intersection ideals $I_1, I_2 \subset R[T]$, each of height n, and surjections $\omega_i : (R[T]/I_i)^n \twoheadrightarrow I_i/I_i^2$, $i = 1, 2$ such that $I_1(0), I_1(1), I_2(0), I_2(1)$ are all reduced ideals of height n and $(J_i, \omega_{J_i}) = (I_i(1), \omega_i(1)) - (I_i(0), \omega_i(0))$ in G for $i = 1, 2$.

In view of (5.8), we may assume that $I_1 = (f_1, \cdots, f_n)$ and ω_1 is induced by $f_1, \cdots, f_n$. Similarly, $I_2 = (g_1, \cdots, g_n)$ and ω_2 is induced by $g_1, \cdots, g_n$. Therefore, we have $I_1(0) = (f_1(0), \cdots, f_n(0))$ and $\omega_1(0)$ is induced by $f_1(0), \cdots, f_n(0)$. Similar conclusion holds for $(I_1(1), \omega_1(1))$, $(I_2(0), \omega_2(0))$, $(I_2(1), \omega_2(1))$.

We have $J_1 \cap I_1(0) = I_1(1)$, $J_2 \cap I_2(0) = I_2(1)$. Note that $J_1 + I_1(0) = R = J_2 + I_2(0)$.

Since we have $(J_i, \omega_{J_i}) + (I_i(0), \omega_i(0)) = (I_i(1), \omega_i(1))$, it follows that $f_i(0) - f_i(1) \in I_1(0)^2$. Applying subtraction principle (3.2), we conclude that $J_1 = (a_1, \cdots, a_n)$ and ω_{J_1} is induced by $a_1, \cdots, a_n$. By a similar argument it follows that $J_2 = (b_1, \cdots, b_n)$ such that ω_{J_2} is induced by $b_1, \cdots, b_n$.

Take $\mathcal{I} = (a_1 T + b_1(1-T), \cdots, a_n T + b_n(1-T)) \subset R[T]$. Then $\mathcal{I}(0) = J_2$ and $\mathcal{I}(1) = J_1$. Let $\omega_{\mathcal{I}} : (R[T]/\mathcal{I})^n \twoheadrightarrow \mathcal{I}$ be induced by $a_1 T + b_1(1-T), \cdots, a_n T + b_n(1-T)$. Therefore, $(J_1, \omega_{J_1}) = (\mathcal{I}(1), \omega(1))$ and $(J_2, \omega_{J_2}) = (\mathcal{I}(0), \omega(0))$. This proves the result.

(3) First let $(J_1, \omega_{J_1}), (J_2, \omega_{J_2}) \in S$. The proof goes verbatim as in (2) above, except for the last paragraph. So, we have $J_1 = (a_1, \cdots, a_n)$, $J_2 = (b_1, \cdots, b_n)$ and $J_1 + J_2 = R$. By addition principle, $J_1 \cap J_2 = (c_1, \cdots, c_n)$ where $c_i - a_i \in J_1^2$ and $c_i - b_i \in J_2^2$. Further, $\omega_{J_1 \cap J_2}$ is induced by $c_1, \cdots, c_n$. It now follows from (5.6) that $(J_1 \cap J_2, \omega_{J_1 \cap J_2}) \in S$.

The other case follows from (2). $\square$

Note that an element of the form $(I(1), \omega(1)) - (I(0), \omega(0))$ *a priori* need not be of the form (J, ω_J), where $J \subset R$ is a reduced ideal. But the above proposition inspires us to consider the following subset of G :

$$S' = \{(J, \omega_J) \in G \,|, \exists I \subset R[T] \text{ and } \omega \text{ such that } (J, \omega_J) = (I(1), \omega(1)) - (I(0), \omega(0))\},$$

where the terms have usual meaning. We show that it is enough to work with S'.

Lemma 5.10. *The subgroup of G generated by S is the same as that generated by S'.*

Proof. Clearly, from the definition, we have $S' \subset S$. Therefore, it is enough to show that any $(I(1), \omega(1)) - (I(0), \omega(0)) \in S$ belongs to the subgroup generated by S'.

We may assume by (5.8) that $I \subset R[T]$ is a complete intersection, say, $I = (f_1, \cdots, f_n)$ and ω is induced by $f_1, \cdots, f_n$. Therefore, we have $I(1) = (f_1(1), \cdots, f_n(1))$, $I(0) = (f_1(0), \cdots, f_n(0))$ and $\omega(1)$, $\omega(0)$ are induced by these generators, respectively.

Applying (5.6) to the element $(I(1), \omega(1))$ of G we can see that there exists an ideal $\mathcal{I} \subset R[T]$ of height n and $\mu : (R[T]/\mathcal{I})^n \twoheadrightarrow \mathcal{I}/\mathcal{I}^2$ such that $(I(1), \omega(1)) = (\mathcal{I}(1), \mu(1)) - (\mathcal{I}(0), \mu(0))$ in G. Similarly, we can conclude that $(I(0), \omega(0)) = (\mathcal{J}(1), \eta(1)) - (\mathcal{J}(0), \eta(0))$ in G, where $\mathcal{J} \subset R[T]$. Therefore, $(I(1), \omega(1)), (I(0), \omega(0)) \in S'$ and their difference is in the subgroup generated by S'. $\square$

Lemma 5.11. *Let R be a smooth affine domain of dimension n over an infinite perfect field k. Let $(J, \omega_J) \in G \setminus S$. Then, there exists $(K, \omega_K) \in G$ such that K is reduced, $J + K = R$ and $(J, \omega_J) + (K, \omega_K) \in S$. Further, given finitely many ideals $J_1, \cdots, J_l$ of R, each of which has height n, K can be chosen to be comaximal with $J_1 \cap \cdots \cap J_l$.*

Proof. Follows from [5, 2.14] (using Swan's Bertini theorem instead of the theorem of Eisenbud-Evans and) and Lemma 5.6 above.

We now state a lemma from [11, 4.13].

Lemma 5.12. *Let G be a free abelian group with basis $B = (e_i)_{i \in \mathcal{I}}$. Let $\sim$ be an equivalence relation on B. Define $x \in G$ to be "reduced" if $x = e_1 + \cdots + e_r$ and $e_i \neq e_j$ for $i \neq j$. Define $x \in G$ to be "nicely reduced" if $x = e_1 + \cdots + e_r$ is such that $e_i \not\sim e_j$ for $i \neq j$. Let $S \subset G$ be such that*

1. *Every element of S is nicely reduced.*
2. *Let $x, y \in G$ be such that each of $x, y, x + y$ is nicely reduced. If two of $x, y, x + y$ are in S, then so is the third.*
3. *Let $x \in G \setminus S$ be nicely reduced and let $\mathcal{J} \subset \mathcal{I}$ be finite. Then there exists $y \in G$ with the following properties : (i) y is nicely reduced; (ii) $x + y \in S$; (iii) $y + e_j$ is nicely reduced $\forall j \in \mathcal{J}$.*

Let H be the subgroup of G generated by S. If $x \in H$ is nicely reduced, then $x \in S$.

We have the following theorem.

Theorem 5.13. *Let R be a smooth affine domain of dimension n over an infinite perfect field k. Let $(J, \omega_J) = 0$ in $E(R)$. Then $(J, \omega_J) = (I(0), \omega(0)) - (I(1), \omega(1))$ in G where (i) $I \subset R[T]$ is a local complete intersection ideal of height n; (ii) Both $I(0)$ and $I(1)$ are reduced ideals of height n; (iii) $\omega(0)$ and $\omega(1)$ are induced by $\omega : (R[T]/I)^n \twoheadrightarrow I/I^2$.*

Proof. The theorem is a direct application of Lemma 5.12. We take G to be the free abelian group generated by the set B of pairs (m, ω_m), as in the definition of the Euler class group at the beginning of this section. The equivalence relation on B is simply $(m_1, \omega_{m_1}) \sim (m_2, \omega_{m_2})$ if $m_1 = m_2$.

We have $E(R) = G/H$, where H is the subgroup generated by S (see 5.2). By (5.10) above, H is also generated by S'. We prove this theorem by showing that S' satisfies properties (1)-(3) of the above lemma.

Let $(J, \omega_J) \in S'$. As J is a reduced ideal, it follows that (J, ω_J) is nicely reduced.

Let $(J_1, \omega_{J_1}), (J_2, \omega_{J_2}) \in S'$ be nicely reduced such that $(J_1, \omega_{J_1}) + (J_2, \omega_{J_2})$ is also nicely reduced (i.e., $J_1 + J_2 = R$). By (5.9), if any two of (J_1, ω_{J_1}), (J_2, ω_{J_2}), $(J_1, \omega_{J_1}) + (J_2, \omega_{J_2})$ belong to S', then so does the third. Property (3) follows from (5.11). □

We now quickly recall the following definition from [4].

Definition 5.14. **Euler class of a projective module :** Let P be a projective R-module of rank n with trivial determinant. Fix $\chi : R \simeq \wedge^n(P)$. By Swan's Bertini theorem, there is a surjection $\alpha : P \twoheadrightarrow J$ such that J is a reduced ideal of height n. Choose an isomorphism $\sigma : (R/J)^n \simeq P/JP$ (note that P/JP is free by (2.1)) such that $\wedge^n \sigma = \chi \otimes R/J$. Let $\omega_J : (R/J)^n \overset{\sigma}{\simeq} P/JP \overset{\overline{\alpha}}{\twoheadrightarrow} J/J^2$. The Euler class of (P, χ) is defined to be the image of (J, ω_J) in $E(R)$ and is denoted as $e(P, \chi)$ (see [4] for further details).

We reprove the following theorem from [4] which shows that the Euler class is the precise obstruction for a projective module to split off a free summand.

Theorem 5.15. [4, 4.13] *Let R be as in (5.13) and let P be a projective R-module of rank n with trivial determinant. Fix $\chi : R \simeq \wedge^n(P)$. Then $e(P, \chi) = 0$ in $E(R)$ if and only if $P \simeq Q \oplus R$ for some R-module Q.*

Proof. We first assume that $e(P, \chi) = 0$ in $E(R)$. Let us choose a surjection $\alpha : P \twoheadrightarrow J$, where J is a reduced ideal of height n. Let $\sigma : (R/J)^n \simeq P/JP$ be an isomorphism such that $\wedge^n \sigma = \chi \otimes R/J$. Composing with $\alpha \otimes R/J$ we obtain a surjection $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$. Then $e(P, \chi) = (J, \omega_J)$.

From the given condition, $(J, \omega_J) = 0$ in $E(R)$. Therefore, by (5.13), we have $(J, \omega_J) = (I(1), \omega(1)) - (I(0), \omega(0))$ in G where (i) $I \subset R[T] = (f_1, \cdots, f_n)$ is a complete intersection ideal of height n; (ii) Both $I(0)$ and $I(1)$ are reduced ideals of height n; (iii) $\omega(0)$ and $\omega(1)$ are induced by $\{f_1(0), \cdots, f_n(0)\}$ and $\{f_1(1), \cdots, f_n(1)\}$, respectively.

For simplicity, let us write $f_i(1) = a_i$ for $i = 1, \cdots, n$ and $f_i(0) = b_i$ for $i = 1, \cdots, n$. As $(J, \omega_J) + (I(0), \omega(0)) = (I(1), \omega(1))$ in G and all the ideals are reduced, it follows that $J \cap I(0) = I(1)$ and $J + I(0) = R$. Now, $I(0) = (b_1, \cdots, b_n)$. Using a standard general position argument we may assume that ht$(b_1, \cdots, b_{n-1}) = n - 1$ and $(b_1, \cdots, b_{n-1}) + J = R$. We consider $I' = (b_1, \cdots, b_{n-1}, (1 - b_n)T + b_n) \subset R[T]$ and $I'' = I' \cap J[T]$. Let $\omega'' : (R[T]/I'')^n \twoheadrightarrow I''/I''^2$ be induced by ω_J and $\{b_1, \cdots, b_{n-1}, (1 - b_n)T + b_n\}$. Then, $I''(1) = J$, $I''(0) = J \cap I(0) = I(1)$ and moreover, $(I''(1), \omega''(1)) = (J, \omega_J)$ and $(I''(0), \omega''(0)) = (I(1), \omega(1))$ in G.

We have $e(P, \chi) = (J, \omega_J) = (I''(1), \omega''(1))$. Therefore, by (5.1) and [14, Theorem, pp 457] it follows that there exists a surjection $\beta : P \twoheadrightarrow I(1)$ such that $e(P, \chi) = (I(1), \omega(1))$. Combination of (5.1) and [14, Theorem, pp 457] essentially implies that if two Euler cycles are homotopic and one of them is the Euler class of a projective module, then the other one is also the Euler class of the same projective module.

Note that, we have $I(1) = (a_1, \cdots, a_n)$ and $\omega(1)$ is induced by $\{a_1, \cdots, a_n\}$. By a general position argument we may assume that ht$\,(a_1, \cdots, a_{n-1}) = n - 1$. Let $K = (a_1, \cdots, a_{n-1}, T^2 - Ta_n + a_n)$. Let $\gamma : R[T]^n \twoheadrightarrow K$ be the map which is given by $e_i \mapsto a_i$ for $1 \leq i \leq n - 1$ and $e_n \mapsto T^2 - Ta_n + a_n$.

We choose $\Delta : P[T]/KP[T] \simeq (R[T]/K)^n$ such that $\wedge^n \Delta = (\chi \otimes R[T]/K)^{-1}$. Composing, we get a surjection $\overline{\phi} : P[T] \twoheadrightarrow K/K^2$. It is easy to check that $\overline{\phi}(0) = \beta \otimes R/I(1)$. As $K(0) = I(1)$, it follows from [4, 3.9] that $\overline{\phi}$ has a lift to $\overline{\varphi} : P[T] \twoheadrightarrow K/(K^2T)$. By (5.1) $\overline{\varphi}$ can be lifted to $\varphi : P[T] \twoheadrightarrow K$. But then $\varphi(1) : P \twoheadrightarrow K(1) = R$.

Conversely, if $P \simeq Q \oplus R$, by a result of Mohan Kumar [15, Theorem 1], P maps onto an ideal J of height n such that J is generated by n elements. It is easy to check using (5.8) that $e(P, \chi) = 0$ in $E(R)$. $\square$

From now on we assume R to be a Noetherian ring of dimension n containing $\mathbb{Q}$. For such a ring the (n-th) Euler class group was defined in [5]. Let us quickly recall.

Definition 5.16. Let G be the free abelian group on the set B of pairs $(\mathfrak{n}, \omega_\mathfrak{n})$, where $\mathfrak{n}$ is an $\mathfrak{m}$-primary ideal of height n such that $\mu(\mathfrak{n}/\mathfrak{n}^2) = n$ and $\omega_\mathfrak{n} : (R/\mathfrak{n})^n \twoheadrightarrow \mathfrak{n}/\mathfrak{n}^2$ is a surjection. Let $J \subset R$ be an ideal of height n such that $\mu(J/J^2) = n$ and let $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. Let $J = \cap_1^r \mathfrak{n}_i$ be the (irredundant) primary decomposition of J. Therefore, $\mathfrak{n}_i$ is an $\mathfrak{m}_i$-primary ideal for some maximal ideal $\mathfrak{m}_i$ of height n. Then, ω_J induces $\omega_{\mathfrak{n}_i} : (R/\mathfrak{n}_i)^n \twoheadrightarrow \mathfrak{n}_i/\mathfrak{n}_i^2$. One associates and writes $(J, \omega_J) := \sum_1^r (\mathfrak{n}_i, \omega_{\mathfrak{n}_i}) \in G$. Let H_1 be the subgroup of G generated by elements (J, ω_J) for which ω_J has a lift to a surjection $\theta : R^n \twoheadrightarrow J$. The Euler class group is defined as $E(R) := G/H_1$.

Compare this definition with (5.3). Now, if we want to extend Nori's definition (5.2) to a Noetherian ring, we should first ask, obviously, the following natural question :

Question 5.17. Let R be a commutative Noetherian ring of dimension n containing $\mathbb{Q}$. Let $I \subset R[T]$ be an ideal of height n such that there is a surjection $\omega : (R[T]/I)^n \twoheadrightarrow I/I^2$. Assume further that $I(0)$ and $I(1)$ are both of height n. Then, do we have $(I(1), \omega(1)) = (I(0), \omega(0))$ in $E(R)$?

To put the above question in proper perspective, we proceed in the following way. Let K be the subset of $E(R)$ which consists of elements (J, ω_J) for which $(J, \omega_J) = (I(1), \omega(1)) - (I(0), \omega(0))$ in $E(R)$ for some ideal $I \subset R[T]$ and surjection $\omega : (R[T]/I)^n \twoheadrightarrow I/I^2$. We now prove the following proposition.

Proposition 5.18. *The set K, as described above, is a subgroup of $E(R)$.*

Proof. It is easy to check that if $x \in K$, then $-x \in K$.

Let $(J, \omega_J) = (I(1), \omega(1)) - (I(0), \omega(0))$ in $E(R)$ and $(J', \omega'_J) = (I'(1), \omega'(1)) - (I'(0), \omega'(0))$ in $E(R)$. We need only show that $(J, \omega_J) + (J', \omega'_J) \in K$. For this proof we need to invoke a result from [7]. Applying the moving lemma (3.1) twice, we can find $(\mathcal{I}, \omega_{\mathcal{I}}) \in E(R[T])$ such that $(I, \omega) = (\mathcal{I}, \omega_{\mathcal{I}})$ in $E(R[T])$ and $\mathcal{I} + I' = R[T]$. Now, there exists a group homomorphism $\Phi_0 : E(R[T]) \twoheadrightarrow E(R)$ which is induced by specialization at $T = 0$. It takes (I, ω) to $(I(0), \omega(0))$ and $(\mathcal{I}, \omega_{\mathcal{I}})$ to $(\mathcal{I}(0), \omega_{\mathcal{I}(0)})$. Similarly, one can define a group homomorphism at $T = 1$ with similar properties. Therefore, $(I(1), \omega(1)) - (I(0), \omega(0)) = (\mathcal{I}(1), \omega_{\mathcal{I}(1)}) - (\mathcal{I}(0), \omega_{\mathcal{I}(0)})$ in $E(R)$. Now,

$$(J, \omega_J) + (J', \omega'_J) = (\mathcal{I}(1), \omega_{\mathcal{I}(1)}) - (\mathcal{I}(0), \omega_{\mathcal{I}(0)}) + (I'(1), \omega'(1)) - (I'(0), \omega'(0))$$

and as $\mathcal{I} + I' = R[T]$, writing $I'' = I \cap \mathcal{I}$ we have,

$$(J, \omega_J) + (J', \omega'_J) = (I''(1), \tilde{\omega}(1)) - (I''(0), \tilde{\omega}(0)),$$

where $\tilde{\omega} : (R[T]/I'')^n \twoheadrightarrow I''/I''^2$ is induced by $\omega_{\mathcal{I}}$ and ω'. $\square$

We may now rephrase Question 5.17 in the following way.

Question 5.19. Is K a non-trivial subgroup of $E(R)$?

Remark 5.20. It is clear from the first half of this section that if R is a smooth affine domain over an infinite perfect field, then K is trivial.

Bhatwadekar, through personal communication, pointed out to us that if R is not smooth, then K could be non-trivial, even for a normal affine algebra over an algebraically closed field. His example is as follows. We sincerely thank him for allowing us to include the example here.

Example 5.21. (Bhatwadekar) We consider the same affine algebra as in [4, Example 6.4]. We shall freely use facts and details from that example. Let

$$B = \frac{\mathbb{C}[X, Y, Z, W]}{(X^5 + Y^5 + Z^5 + W^5)}$$

Then B is a graded normal affine domain over $\mathbb{C}$ of dimension 3, having an isolated singularity at the origin. Let $F(B)$ be the subgroup of $\widetilde{K}_0(B)$ generated by all elements of the type $[P]-[P^*]$, where P is a finitely generated projective B-module. As B is graded, Pic$(B)=0$. Therefore, by [4, 6.1] $F(B)=F^3K_0(B)$. Since Proj(B) is a smooth surface of degree 5 in $\mathbb{P}^3$, it follows from a result of Srinivas that $F(B)=F^3K_0(B)\neq 0$. Therefore, there exists a projective B-module P of rank 3 with trivial determinant such that $[P]-[P^*]$ is a nonzero element of $F(B)$. This implies that P does not have a unimodular element. We now consider the ring homomorphism $f: B\to B[T]$ given by $f(x)=xT, f(y)=yT, f(z)=zT, f(w)=wT$. We regard $B[T]$ as a B-module through this map and $Q=P\otimes_B B[T]$. Then it is easy to see that Q/TQ is free and $Q/(T-1)Q=P$. Therefore, Q is a projective $B[T]$-module which is not extended from B. Now consider a surjection $\alpha: Q\twoheadrightarrow I$ where $I\subset B[T]$ is an ideal of height 3. Fix an isomorphism $\chi: B[T]\simeq \wedge^3(Q)$. Note that, by (2.1), Q/IQ is a free $B[T]/I$-module. We choose an isomorphism $\sigma: (B[T]/I)^3\simeq Q/IQ$ such that $\wedge^3\sigma=\chi\otimes B[T]/I$. Composing σ and $\alpha\otimes B[T]/I$, we obtain a surjection $\omega: (B[T]/I)^3\twoheadrightarrow I/I^2$. It is now easy to see that $(I(0),\omega(0))=0$ in $E(B)$ (as Q/TQ is free), whereas $(I(1),\omega(1))=e(Q/(T-1)Q,\chi(1))=e(P,\chi(1))$ in $E(B)$ cannot be trivial (as P does not have a unimodular element). $\square$

We now extend Nori's definition to Noetherian rings, as follows.

Definition 5.22. Let R be a commutative Noetherian ring of dimension n containing $\mathbb{Q}$. Let G be the free abelian group as considered in (5.16). Now let S be the set of elements $(J,\omega_J)=(I(0),\omega(0))-(I(1),\omega(1))$ of G where (i) $I\subset R[T]$ is an ideal of height n such that $I=(f_1,\cdots,f_n)$; (ii) Both $I(0)$ and $I(1)$ are ideals of height n; (iii) $\omega(0)$ and $\omega(1)$ are induced by $\{f_1(0),\cdots,f_n(0)\}$ and $\{f_1(1),\cdots,f_n(1)\}$, respectively. Let H be the subgroup generated by S. We define the group $\widetilde{E}(R)$ as $\widetilde{E}(R):=G/H$.

Clearly, as an easy consequence of Lemma 5.8 above, one can see that when R is a smooth affine domain, this definition is equivalent to the one given in 5.2. We now show that this definition is equivalent to the definition given by Bhatwadekar-Sridharan for Noetherian rings in [5] ((5.16) here).

Proposition 5.23. *Let $E(R)$ be the Euler class group defined as in (5.16). Then $\widetilde{E}(R)\simeq E(R)$.*

Proof. It is obvious from definitions (5.16, 5.22) that $H\subset H_1$. To see that $H_1\subset H$, apply (5.6, 5.7). $\square$

Remark 5.24. In a similar manner, one can also define the weak Euler class group $E_0(R)$ in terms of homotopy.

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