Euler Class Construction

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1 Introduction

The proof of the following theorem of Boratynski ([B] or see [Ma1]) gave rise to a method for explicit construction of projective modules in complete intersections theory.

Theorem 1.1 ([**B**]) *Suppose* A *is a commutative ring and* I *is an ideal of* A*. Let* $I = (f_1, \ldots, f_r) + I^2$ *and* $J = (f_1, \ldots, f_{r-1}) + I^{(r-1)!}$*. Then* J *is image of a projective* $A-$*module* P *with rank*$(P) = r$*.*

This method of construction of Boratynski was used later by various authors ([MK], [Sw], [Mu], [Ma2], [DM]) to achieve a variety of objectives. In this paper, we do the same in the context of Euler class groups and Euler classes of projective modules.

The work of Boratynski was considered for the universal ring

$$A_n = \mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z]/(\sum_{i=1}^{n} X_iY_i - Z(1 + Z))$$

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by N. Mohan Kumar and M. V. Nori ([MK], [Sw]). They proved that *the ideal* $I = (X_1^{r_1}, \dots, X_n^{r_n}, Z^N)A_n$, *for* N *large enough, is the image of a projective* A_n-*module* Q *of rank* n *if and only if* $r_1 r_2 \cdots r_n$ *is divisible by* $(n-1)!$. The Grothendieck group $K_0(A_n) = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}[A] \oplus \mathbb{Z}[A/I]$ was also computed (see [Sw]).

Murthy used the description of $K_0(A_n)$ of the universal ring to prove the following theorem.

Theorem 1.2 (**[Mu]**) *Let* A *be a noetherian commutative ring and* $I \subset A$ *be a local complete intersection ideal of height* r. *Suppose* $I = (f_1, \cdots, f_r) + I^2$ *and* $J = (f_1, \cdots, f_{r-1}) + I^{(r-1)!}$. *Assume* $f_1, \dots, f_r$ *is a regular sequence. Then there is a projective* A*-module* P *of rank* r *and a surjective homomorphism* $P \to J$, *such that* $[P] - [A^r] = -[A/I] \in K_0(A)$.

This theorem of Murthy ([Mu]) was a key ingredient that led to the proof of his theorem that *for a projective* $A-$*module* P *of with* $rank(P) = n = \dim(A)$, *over an affine algebra* A *over an algebraically closed field* k, *we have* $P \approx Q \oplus A$ *if an only if the top Chern class* $C_n(P) = 0$. These techniques were also used in [Ma2] in a similar way to obtain results for arbitrary noetherian commutative rings. Following is a result from ([Ma2]).

Theorem 1.3 (**[Ma2]**) *Let* A *be a commutative noetherain ring. Suppose* $f_1, \dots, f_r$ *is a regular sequecnce of length* $r \leq \dim A$. *Suppose* P *is a projective* $A-$*module of rank* r *that maps onto the ideal* $J = (f_1, \dots, f_{r-1}, f_r^{(r-1)!})$. *Then* $[P] = [Q \oplus A]$ *in* $K_0(A)$, *for some projective* $A-$*module of* Q.

In a recent paper, Das and Mandal ([DM]) used similar techniques for some constuction of projective modules in the context of weak Euler classes.

Euler class of a ring was originally defined by Nori (see [MS], [BRS1]) for smooth affine algebras. We will use the definition of the Euler class group that was given by S. M. Bhatwadekar and Raja Sridharan ([BRS2]).

Given a commutative ring A with $\dim A = n$ and a line bundle L on $Spec(A)$, we write $F = A^{n-1} \oplus L$. An Euler $L-$cycle is a pair (I, ω) where I is an ideal of height n and $\omega : F/IF \to I/I^2$ is an equivalence class of surjective maps. An Euler $L-$cycle (I, ω) is said to be a global Euler $L-$cycle, if ω lifts to a surjection $F \to I$. The Euler class group, relative to L, was defined as the quotient group $E(A, L) = G(L)/H(L)$, where $G(L)$ is the free abelian group generated by the set

$$S = \{(N, \omega) : (N, \omega) \text{ } is \text{ } an \text{ } Euler \text{ } L - cylce \text{ } and \text{ } N \text{ } is \text{ } a \text{ } primary \text{ } ideal\}$$

and $H(L)$ is the subgroup of $G(L)$ generated by global $L-$cycles.

For a projective $A-$module P with $rank(P) = r \leq \dim A$ and determinant $\det(P) = L$, a pair (P, χ) will be called an $L-$oriented projective $A-$module, where $\chi : L \xrightarrow{\sim} \wedge^r P$ is an isomorphism. Given such an $L-$oriented projective $A-$module (P, χ) of rank $n = \dim A$ the Euler class $e(P, \chi) \in E(A, L)$ of (P, χ) has been defined in [BRS2] (see the section on preliminaries for details).

In this paper we construct, following the work of Boratynski ([B]), oriented projective modules with a given Euler class as follows.

Theorem 1.4 *Let A be a commutative noetherian ring of dimension n. Let L be a line bundle on $Spec(A)$ and $F = A^{n-1} \oplus L$. Let J be a local complete intersection ideal of height n with J/J^2 free and $J = (f_1, \ldots, f_{n-1}, f_n) + J^2$. Let*

$$I = (f_1, \ldots, f_{n-1}) + J^{(n-1)!}.$$

Let (I, ω) be any $L-$cycle in $E(A, L)$.

Then, there is an $L-$oriented projective $A-$module (P, χ) of rank n such that

1. *$[P] - [F] = -[A/J]$ in $K_0(A)$,*
2. *P maps onto I,*
3. *the Euler class $e(P, \chi) = (I, \omega) \in E(A, L)$, if A contains the field of rationals $\mathbb{Q}$.*

In this paper, all rings we consider are noetherian and commutative and modules are finitely generated. The dimension of all the rings we consider is at least 2.

In section 2, we discuss some of the preliminaries and in section 3 we discusss our main results.

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2 Preliminaries

In this section on preliminaries, we recall the definition of the Euler class group and include some basic facts about patching techniques and complete intersections.

2.1 Definition of the Euler Class Group

First, we define oriented projective modules.

Definition 2.1 *Let A be a noetherian commutative ring and L be a line bundle on Spec(A). An $L-$oriented projective $A-$module of rank r is a pair (P, χ) where P is a projective $A-$module of rank r and $\chi : L \xrightarrow{\sim} \wedge^r P$ is an isomorphism. Such an isomorphism χ will be called an orientation of P.*

Now we will recall the definition of the Euler class group ([BRS2]).

Definition 2.2 *Let A be a noetherian commutative ring with* $\dim A = n$ *and let L be a rank one projective $A-$module. Write $F = A^{n-1} \oplus L$.*

1. *For an ideal I of height n, two surjective homomorphisms $\omega_1, \omega_2 : F/IF \to I/I^2$ are said to be equivalent if $\omega_1 \sigma = \omega_2$ for some automorphism $\sigma \in SL(F/IF)$. An equivalence class of surjective homomorphisms $\omega : F/IF \to I/I^2$ is called a local $L-$orientation.*

2. *A local $L-$orientation $\omega : F/IF \to I/I^2$ of an ideal I of height n is said to be a global Euler $L-$orientation, if ω lifts to a surjection $F \to I$.*

3. *Let $G(L)$ be the free abelian group generated by the set S of all pairs (N, ω) where N is a primary ideal of height n and ω is a local $L-$orientation of N.* Elements of $G(L)$ will be called Euler $L-$cycles.

4. *Let J be an ideal of height n and $\omega : F/IF \to J/J^2$ be a local $L-$orientation of J. Let $J = N_1 \cap N_2 \cap \cdots \cap N_k$ be a primary decomposition of J. Then ω induces a local $L-$orientations $\omega_i : F/N_iF \to N_i/N_i^2$ for $i = 1, \ldots, k$. Let (J, ω) denote the element $\sum_{i=1}^{r} (N_i, \omega_i)$ in $G(L)$. We say that (J, ω) is the Euler $L-$cycle determined by (J, ω).*

5. *Let $H(L)$ be the subgroup of $G(L)$ generated by by the set of all pairs (J, ω) where ω is a global Euler $L-$orientation.*

6. *Define $E(A,L) = G(L)/H(L)$. This group is called the* **Euler class group** *of A (relative to L).*

7. **Notation:** The image of an Euler $L-$cycle $(J,\omega) \in G(L)$, in $E(A,L)$ will be denoted by the same notation (J,ω) and also be called an Euler $L-$cycle. It will be clear from the context, whether we mean in $G(L)$ or in $E(A,L)$.

8. *Let $\chi_0 : L \to \wedge^n F$ be the isomorphism defined by $\chi_0(l) = e_1 \wedge \cdots \wedge e_{n-1} \wedge l$, where $e_1, \ldots, e_{n-1}$ is the standard basis of $A^{n-1} \subsetneq F$. This orientation χ_0 will be called the* **standard orientation** *of F.*

9. *Now we assume that the field of rationals $\mathbb{Q}$ is contained in A. Let (P,χ) be an $L-$oriented projective $A-$module with $rank(P) = n$.* (So *$det(P) = L$.) Let $f : P \to I$ be a surjective homomorphism, where I is an ideal of height n. Suppose $\gamma : F/IF \to P/IP$ is an isomprphism such that $\wedge^n \gamma \overline{\chi_0} = \overline{\chi}$ where "overline" denotes "modulo I". Let $\omega = \overline{f}\gamma$. Define the Euler class of (P,χ) as $e(P,\chi) = (I,\omega) \in E(A,L)$. In fact, this association $e(P,\chi) = (I,\omega)$ is well defined.*

2.2 Patching Techniques

For our later discussions, it will be convenient to have the following notation regarding fiber product of modules.

Notation 2.1 *Let A be a commutative ring and $As + At = A$ for some $s,t \in A$. Let M be an A_t-module, N be an A_s-module and $\alpha : M_s \xrightarrow{\sim} N_t$ be an isomorphism. The $A-$module obtained by patching M and N via α will be denoted by $\mathcal{P}(M,N,\alpha)$. So, following*

$$\begin{array}{ccccc} \mathcal{P}(M,N,\alpha) & \longrightarrow & & & N \\ \downarrow & & & & \downarrow \\ M & \longrightarrow & M_s & \xrightarrow{\alpha} & N_t \end{array}$$

is a fiber product diagram.

Before we proceed any further, we want to recall the definition of isotopy, due to Plumstead ([P] or see [Ma1]).

Definition 2.3 ([**P**]) *Let A be a commutative ring and $f, g : M \xrightarrow{\sim} N$ be two isomorphisms of $A-$modules. We say that f is isotopic to g, if there is an isomorphism $H[T] : M \otimes A[T] \xrightarrow{\sim} N \otimes A[T]$ such that $H(0) = f$ and $H(1) = g$, where T is a variable.*

The following is a result of Plumstead ([P]) on patching techniques.

Lemma 2.1 ([**P**]) *Suppose A is a commutative ring and M, N are two $A-$modules. Let $s, t \in A$ be such that $As + At = A$. Suppose $f_1 : M_t \to N_t$ and $f_2 : M_s \to N_s$ are two isomorphisms. Assume that $(f_2)_t^{-1}(f_1)_s$ is isotopic to $Id_{M_{st}}$. Then $M \approx N$.*

Proof. See ([P] or [Ma1]).

The following lemma on isotopy of isomorphisms will be useful later.

Lemma 2.2 *Let A be a commutative ring and M and N be two $A-$modules. Let $\varphi : M \to M$ be an automorhism that is isotopic to Id_M. Then $\sigma^{-1}\varphi\sigma$ is isotopic to Id_N, for any isomprorphism $\sigma : N \xrightarrow{\sim} M$.*

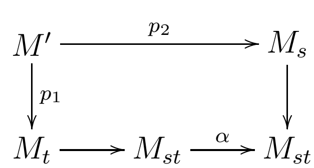
Proof. Proof is obvious.

The following is another lemma on patching that will be needed later in this paper.

Lemma 2.3 *Let A be a commutative ring and M be an $A-$module. Let $s, t \in A$ be two elements such that $As + At = A$. Let $\alpha : M_{st} \to M_{st}$ be an automorphism. Let $\beta = \epsilon\alpha$, where $\epsilon : M_{st} \to M_{st}$ is an automorphism that is isotopic to identity. Let M', M'' be the $A-$module obtained by patching M_t and M_s, respectively, via α and β. Then $M' \approx M$". Notationally,*

$$\mathcal{P}(M_t, M_s, \alpha) \approx \mathcal{P}(M_t, M_s, \epsilon\alpha).$$

Proof. We have, M' and M'' are given by the following



$$\begin{array}{ccccc} M'' & & \xrightarrow{q_2} & & M_s \\ \downarrow{\scriptstyle q_1} & & & & \downarrow \\ M_t & \longrightarrow & M_{st} & \xrightarrow{\beta} & M_{st} \end{array}$$

two fiber product diagrams. Here p_1, p_2 are the natural maps for the first diagram and q_1, q_2 are the same for the second diagram.

Let $f_1 = ((q_1)_t)^{-1}(p_1)_t : M'_t \to M''_t$ and $f_2 = ((q_2)_s)^{-1}(p_2)_s : M'_s \to M''_s$. Then

$$(f_2^{-1})_t(f_1)_s = ((p_2)_s)^{-1}(q_2)_s((q_1)_t)^{-1}(p_1)_t = ((p_2)_s)^{-1}\beta(p_1)_t$$

$$= ((p_2)_s)^{-1}\epsilon\alpha(p_1)_t = ((p_2)_s)^{-1}\epsilon(p_2)_s.$$

Since ϵ is isotopic to identity, by Lemma 2.2, we have $(f_2^{-1})_t(f_1)_s$ is also isotopic to identity. By lemma 2.1, f_1 and f_2 will induce an isomorphism $f : M' \to M''$. This completes the proof of this lemma.

For our later discussions, it will be convenient to have the following formulation of the Murthy's ([Mu]) version of the theorem of Boratynski ([B]).

Theorem 2.1 (**[Mu]**) *Let A be a noetherian commutative ring. Suppose $f_1, f_2, \ldots, f_r$ is a regular sequence and $s(1+s) = f_1g_1 + \cdots + f_rg_r$ for some $s \in A$ and $g_1, \ldots, g_r$ in A. Write $I = (f_1, \ldots, f_r, s)$.*

Let

$$A_r = \mathbb{Z}[X_1, \ldots, X_r, Y_1, \ldots, Y_r, Z]/(\sum X_iY_i - Z(1+Z)).$$

Consider the map

$$A_r \longrightarrow A$$

that sends X_i, Y_i, respectively, to f_i, g_i and Z to s.

Let $\Gamma \in GL_r((A_r)_{z(1+z)})$ be an invertible matrix whose first row is $(x_1, \ldots, x_{r-1}, x_r^{(r-1)!})$, where x_i denotes the image of X_i in A_r. Let α be the image of Γ in $GL_r(A_{s(1+s)})$.

Let $P = \mathcal{P}(A^r_{1+s}, A^r_s, \alpha)$ be the projective $A-$module obtained by patching A^r_{1+s} and A^r_s via α.

Then

1. *$[P] - [A^r] = -[A/I] \in K_0(A)$ and*

2. *$det(P) = A$.*

3. *Further, if* $J = (f_1, \ldots, f_{r-1}) + I^{(r-1)!}$*, then* P *maps onto* J*.*

Proof. See ([Mu]) or [DM]).

3 Main Results

Suppose A is a commutative noetherian ring of dimension n and L be a line bundle on $Spec(A)$. Also suppose I is a local complete intersection ideal of height n and I/I^2 is n generated. Following the theorem (1.1) of Boratynski ([B]), our main result constructs an $L-$oriented projective $A-$module (P, χ) of rank n with a given Euler class $e(P, \chi) \in E(A, L)$. Our main theorem is a consequence of the following theorem.

Theorem 3.1 *Let* A *be a commutative noetherian ring of dimension* $n \geq 2$*. Let* L *be a line bundle on* $Spec(A)$ *and* $F = A^{n-1} \oplus L$*. Suppose* J *is a local complete intersection ideal of height* n *such that* J/J^2 *is generated by* n *elements and* $J = (f_1, \ldots, f_{n-1}, f_n) + J^2$*. Write* $N = (n-1)!$*. Let*

$$I = (f_1, \ldots, f_{n-1}) + J^N.$$

Let e_i *be standard basis of* $A^{n-1} \subseteq F$ *and* $e_n = (0, \ldots, 0, e) \in F$ *be such that* e *generates* L/IL*.* (Here "overline" will denote modulo I.) *Define a local* $L-$*orientation* $\omega : F/IF \to I/I^2$ *such that* $\omega(\overline{e_i}) = \overline{f_i}$ *for* $i = 1, \ldots, n-1$ *and* $\omega(\overline{e_n}) = \overline{f_n^N}$*.*

There is a projective $A-$*module* P *of rank* n *such that*

1. $det(P) = L$*, and* $[P] - [F] = -[A/J]$ *in* $K_0(A)$*,*

2. P *maps onto* I*,*

3. *an orientation* $\chi : L \to \wedge^n P$*, will also be defined such that*

4. *the Euler class* $e(P, \chi) = (I, \omega) \in E(A, L)$*, if* A *contains the field of rationals* $\mathbb{Q}$*.*

Proof. If necessary, we replace f_i by $f_i + \lambda_i$ for some $\lambda_i \in J^{2(n-1)!}$, and assume that $f_1, \ldots, f_n$, is a regular sequence. This can be done because for all prime ideals $\wp$ containing I, we have $depth_I A_\wp = n$.

There is an isomorphism $\zeta_0 : L/IL \to A/I$ such that $\zeta_0(\bar{e}) = 1$. Let $\zeta_1 : L \to A$ be a lift of ζ_0. Then $\zeta_1(e) = 1 + s_0$ for some $s_0 \in I$. Note that (ζ_1, s_0)

is an unimodular element in $Hom(L,A) \oplus A$. For some $\zeta_2 \in Hom(L,A)$, $\zeta = \zeta_1 + s_0\zeta_2$ is basic on $V(I) \cup Ass(A)$ (see [EE]). Then ζ is injective and $t_1 = \zeta(e) = 1 + s_1$ for some $s_1 \in I$. So, identifying L with $\zeta(L)$, we can assume that

1. L is an invertible ideal,

2. $L + I = A$,

3. we have, $1 = \bar{e}$, in this identification $L/IL \equiv A/I$.

There is an $s \in J$ such that $(1+s)J \subseteq (f_1, \ldots, f_{n-1}, f_n)$. Write $t = 1 + s$. Replacing t by tt_1, we can assume that $t \in L$. So, $L_t = A_t$ and $I_t = (f_1, \ldots, f_{n-1}, f_n^{(n-1)!})$.

We have $F = A^{n-1} \oplus L$. Let $\chi_0 : L \to \wedge^n F$ be the standard orientation given by $\chi_0(l) = e_1 \wedge \ldots \wedge e_{n-1} \wedge l$, for $l \in L$.

Let

$$A_n = \mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z]/(\sum X_iY_i - Z(1+Z)).$$

We have $\sum_{i=1}^n f_ig_i = st$ for some $g_1, \ldots, g_n$ in A. There is a homomorphism

$$A_n \longrightarrow A$$

that sends X_i to f_i, Y_i to g_i and Z to s.

By the theorem of Suslin ([S]), there is a matrix $\Gamma \in \mathbb{M}_n(A_n)$ with first row $(x_1, \ldots, x_{n-1}, x_n^{(n-1)!})$ and $det(\Gamma) = (z(1+z))^r$ for some integer $r \geq 0$. Let $\alpha \in M_n(A)$ be an the image of Γ. Then,

1. the first row of α is $(f_1, \ldots, f_{n-1}, f_n^{(n-1)!})$,

2. $det(\alpha) = s^rt^r$ for some integer $r \geq 1$.

Define $\varphi_1 : F_t \to I_t$ by

$$\varphi_1(x_1, x_2, \ldots, x_{n-1}, l_n) = f_1x_1 + \cdots + f_{n-1}x_{n-1} + f_n^{(n-1)!}l_n$$

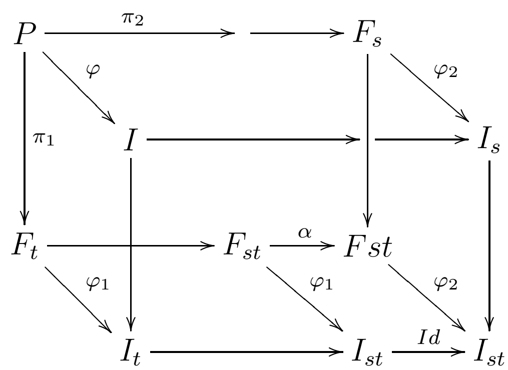
for $(x_1, x_2, \ldots, x_{n-1}, l_n) \in F_t = A_t^{n-1} \oplus L_t$.

Define $\varphi_2 : F_s \to I_s$ by

$$\varphi_2(x_1, x_2, \ldots, x_{n-1}, l_n) = x_1$$

for $(x_1, x_2, \ldots, x_{n-1}, l_n) \in F_s = A_s^{n-1} \oplus L_s$.

Note that both φ_1, φ_2 are surjective. Also note that α defines a homomorphism $\alpha : F_{st} \to F_{st}$. Consider the following



fiber product diagram. Here P is the fiber product of F_t and F_s via α. Note that $\varphi_2 \alpha = \varphi_1$. The map $\varphi : P \to I$ is obtained by the properties of fiber product diagrams. Since φ_1, φ_2 are surjective, so is φ.

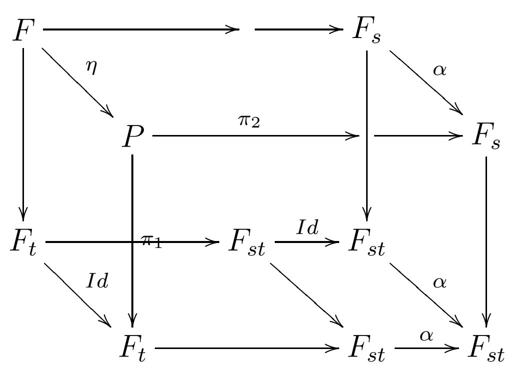
We will now construct an orientation $\chi : L \to \wedge^n P$.

Let $\chi_1 = (\wedge^n \pi_1)_t^{-1} (\chi_0)_t : L_t \to \wedge^n P_t$ and $\chi_2 = (\wedge^n \pi_2)_s^{-1} (\chi_0)_s : L_s \to \wedge^n P_s$.

Now $(\chi_2)_t^{-1} (\chi_1)_s = (\chi_0)_{st}^{-1} (\wedge^n \pi_2)_{st}^{-1} (\wedge^n \pi_1)_{st} (\chi_0)_{st} = det(\alpha) = s^r t^r$.

Let $\chi_1' = t^{-r} \chi_1 : L_t \to \wedge^n P_t$ and $\chi_2' = s^r \chi_2 : L_s \to \wedge^n P_s$. Now $(\chi_1')_s = (\chi_2')_t$. Let $\chi : L \to \wedge^n P$ be the isoporphism such that $\chi_t = \chi_1'$ and $\chi_s = \chi_2' = t^{-r} \wedge^n (\pi_t)^{-1} (\chi_0)_t$. So, $\chi : L \to \wedge^n P$ is an orientation of P.

Note that α also defines a map $\alpha : F_s \to F_s$ (this is possible because L is an ideal). We define a map $\eta : F \to P$ by the following



fiber product diagram. We also have the following

$$
\begin{array}{ccccc}
F_t & \xrightarrow{\eta_t} & P_t & \xrightarrow{\varphi_t} & I_t \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle Id} \\
F_t & \xrightarrow{Id} & F_t & \xrightarrow{\varphi_1} & I_t
\end{array}
$$

commutative diagram. It follows froms this that $\varphi\eta \equiv \omega_I$ (modulo I).

Also the diagram

$$
\begin{array}{ccccc}
L_t & \xrightarrow{(\chi_0)_t} & \wedge^n F_t & \xrightarrow{\wedge^n \eta_t} & \wedge^n P_t \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \wedge^n(\pi_1)_t} \\
L_t & \xrightarrow{(\chi_0)_t} & \wedge^n F_t & \xrightarrow{Id} & \wedge^n F_t
\end{array}
$$

is commutative. So, $\wedge^n \eta_t(\chi_0)_t = \wedge^n(\pi_1)_t^{-1}(\chi_0)_t = (\chi_1)_t$.

Let "overline" denote (*modulo* I). Therefore $\overline{\wedge^n \eta_t(\chi_0)_t} = \overline{\chi_1} = \overline{\chi_1'}$. So,$\overline{\wedge^n \eta \chi_0} = \overline{\chi}$.

This establishes that $e(P, \chi) = \omega_I$.

It remains to prove that $[P] - [F] = -[A/J]$. Let $F_0 = A^n$ be the free $A-$module of rank n. Let P' be the projective $A-$module obtained by patching $(F_0)_t$ and $(F_0)_s$ via $\alpha : (F_0)_{st} \to (F_0)_{st}$. Since α comes from A_n, it follows, from Murthy's version ([Mu]) of Boratynski's([B]) theorem 2.1, that $[P'] - [A^n] = -[A/J]$ and $det(P') = A$. Therefore, it is enough to prove that $P' \oplus L \approx P \oplus A$.

Recall that $F = A^{n-1} \oplus L$ and $F_0 = A^n$. Define $\epsilon : F \oplus A \to F_0 \oplus L$ by

$$\epsilon(x_1, \ldots, x_{n-1}, l_n, x_{n+1}) = (x_1, \ldots, x_{n-1}, -x_{n+1}, l_n)$$

where $x_i \in A$ and $l_n \in L$. So, ϵ switches the last two coordinates with a sign adjustment.

Similarly, define $\epsilon' : (F_0 \oplus L)_{st} \to (F_0 \oplus L)_{st}$ by

$$\epsilon'(x_1, \ldots, x_{n-1}, x_n, x_{n+1}) = (x_1, \ldots, x_{n-1}, -x_{n+1}, x_n)$$

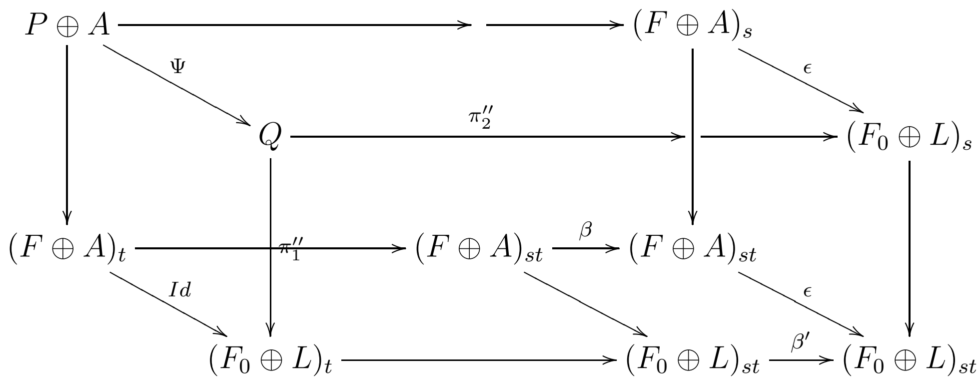
Note that $L_{st} = A_{st}$ and so $(F_0 \oplus L)_{st} = A_{st}^{n+1}$. Note also that ϵ' is in $SL_n(\mathbb{Z})$. So, ϵ' is an elementary matrix and hence isotopic to identity.

Let $\beta = (\alpha \oplus Id_A) : (F \oplus A)_{st} \to (F \oplus A)_{st}$.

Let $\beta' = \epsilon'(\alpha \oplus Id) : (F_0 \oplus L)_{st} \to (F_0 \oplus L)_{st}$. Then

$$\epsilon\beta = \beta'.$$

Let Q be the projective $A-$module obtained by patching $(F_0 \oplus L)_t$ and $(F_0 \oplus L)_s$ via β'. Consider the following



fiber product diagram. Here the homomorphism $\Psi : P \oplus A \to Q$ is obtained by the properties of fiber product diagrams. Since, the maps on upper right hand and lower left hand corners are isomprphisms, Ψ is also an isomorphism.

By lemma 2.3, we have $Q = \mathcal{P}((F_0 \oplus L)_t, (F_0 \oplus L)_s, \beta') \approx \mathcal{P}((F_0 \oplus L)_t, (F_0 \oplus L)_s, \alpha \oplus Id) = P' \oplus L$ So, $P \oplus A \overset{\Psi}{\approx} Q \approx P' \oplus L$. This completes the proof of the theorem.

Remark 3.1 Theorem 2.1 holds for any locally complete intersection ideal J with J/J^2 free and invertible ideals L with $L + J = A$. Although, the last item that $e(P, \chi) = (I, \omega)$, in the statement of the theorem, needs to be appropriately formulated.

Now we are ready to prove our main theorem.

Theorem 3.2 *Let A be a commutative noetherian ring of dimension n. Let L be a line bundle on Spec(A) and $F = A^{n-1} \oplus L$. Let J be a local complete intersection ideal of height n and $J = (f_1, \ldots, f_{n-1}, f_n) + J^2$. Let*

$$I = (f_1, \ldots, f_{n-1}) + J^{(n-1)!}.$$

Let $\omega : F/IF \to I/I^2$ be any surjective homomorphism.

Then there is an $L-$oriented projective $A-$module (P, χ) of rank n such that

1. $[P]-[F]=-[A/J]$ *in* $K_0(A)$,

2. P *maps onto* I,

3. the Euler class $e(P,\chi)=(I,\omega)\in E(A,L)$, *if* A *contains the field of rationals* $\mathbb{Q}$.

Proof. As in the proof of Theorem 3.1, we can assume that $f_1,\ldots,f_{n-1},f_n$ is a regular sequence. Since I/I^2 is free, we have ω is an isomorphism. Let $e_1,\ldots,e_{n-1}$ be the standard basis of $A^{n-1}\subseteq F$ and $e_n=(0,\ldots,0,l)\in F$ is such that $L/IL=(A/I)l$. Let "overline" denote modulo I. So, $\overline{e_1},\ldots,\overline{e_{n-1}},\overline{e_n}$ is a basis of F/IF.

Since $\dim A/I=0$, there is a $\sigma\in SL(F/IF)$ such that $\omega_1=\omega\sigma$ has the following form:

$\omega_1(\overline{e_1})=\overline{uf_1}$,where $u\in A$ and image of u is a unit in A/I, $\omega_1(\overline{e_i})=\overline{f_i}$ for $i=2,\ldots,n-1$ and $\omega_1(\overline{e_n})=\overline{f_n^{(n-1)!}}$.

In fact, we can assume that u is a non zero divisor. To see this, let $uv+x=1$ for some $x\in J$. Let $\wp_1,\ldots,\wp_k,\wp_{k+1},\ldots,\wp_m$ be maximal among the associated primes of A such that u is in $\wp_{k+1},\ldots,\wp_m$ and not in $\wp_1,\ldots,\wp_k$. Pick $\lambda\in J^{2(n-1)!}\cap\wp_1\cap\cdots\cap\wp_k\setminus\cup_{i=k+1}^m\wp_i$. Let $U=u+\lambda$. Then $Uv+(x-\lambda v)=1$ and U is a nonzero divisor on A. Replacing u by U, we will assume that u is a non zero divisor.

Write $F_1=uf_1$. Since u is also a unit in A/J we have

$J=(F_1,f_2,\ldots,f_n)+J^2$ and F_1 is a non zero divisor. We also have $I=(F_1,f_2,\ldots,f_{n-1})+J^{(n-1)!}$.

We will pick $F_2,\ldots,F_n$ such that

1. $J=(F_1,F_2,\ldots,F_n)+J^2$,
2. $F_1,F_2,\ldots,F_n$ is a regular sequence,
3. for $i=2,\ldots,n$ we have $F_i-f_i\in J^{2(n-1)!}\subseteq I^2$.

The proof, by induction, is similar to above. Suppose we have picked $F_1,\ldots,F_r$. Let $\wp_1,\ldots,\wp_k,\wp_{k+1},\ldots,\wp_m$ be the maximal among the associate primes of $A/(F_1,\ldots,F_r)$ such that f_{r+1} is in $\wp_{k+1},\ldots,\wp_m$ and not in $\wp_1,\ldots,\wp_k$. We pick $\lambda\in J^{2(n-1)!}\cap\wp_1\cap\cdots\cap\wp_k\setminus\cup_{i=k+1}^m\wp_i$. Write $F_{r+1}=f_{r+1}+\lambda$. Then

1. $J=(F_1,F_2,\ldots,F_{r+1},f_{r+2},\ldots,f_n)+J^2$,

2. $F_1, F_2, \ldots, F_{r+1}$ is a regular sequence,

3. for $i = 2, \ldots, r+1$ we have $F_i - f_i \in J^{2(n-1)!}$.

For $i = 2, \ldots, n-1$ we have $f_i - F_i \in J^{(n-1)!}$. So, $I = (F_1, f_2, \ldots, f_{n-1}) + J^{(n-1)!} = (F_1, F_2, \ldots, F_{n-1}) + J^{(n-1)!}$.

Also, since $f_n - F_n \in I^2$, we have $f_n^{(n-1)!} - F_n^{(n-1)!} \in I^2$.

So, $\omega_1(\overline{e_1}) = \overline{uf_1} = \overline{F_1}$, $\omega_1(\overline{e_i}) = \overline{f_i} = \overline{F_i}$ for $i = 2, \ldots, n-1$ and $\omega_1(\overline{e_n}) = \overline{f_n^{(n-1)!}} = \overline{F_n^{(n-1)!}}$.

So, ω_1 is a local $L-$orientation of the type in Theorem 3.1 above, with respect to the generators $F_1, \ldots, F_n$ of J/J^2. By Theorem 3.1, there is an $L-$oriented projective $A-$module (P, χ) that satisfies the assertions of the theorem. This completes the proof of the theorem.

Remark 3.2 For a noetherian commutative ring A of dimension n and a line bundle L over $Spec(A)$, not all cycles $x \in E(A, L)$ can be realized as Euler class of an oriented projective projecive $A-$module (P, χ). This follows from the fact that the top Chern class map $C^n : K_0(A) \to CH^n(A)$ is not always surjective.

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