A Riemann-Roch Theorem

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1 Introduction

We begin with the statement of the Riemann-Roch theorem ([F]), without denominators, for Chow groups of zero cycles.

Theorem 1.1 (**[F, page 297]**) *Suppose X is a non-singular variety of dimension n. Let $F^nK_0(X)$ denote the subgroup of the Grothendieck group $K_0(X)$ generated by points in X and $CH^n(X)$ be the Chow group of zero cycles. Let $\varphi : F^nK_0(X) \to CH^n(X)$ be defined by $\varphi(x) = c^n(x)$, where $c^n(x)$ is the $n-th$ Chern class of x and let $\psi : CH^n(X) \to F^nK_0(X)$ be the natural map.*

Then $\psi\varphi = (-1)^{n-1}(n-1)!Id$ and $\varphi\psi = (-1)^{n-1}(n-1)!Id$.

In this paper, we prove an analogue of this theorem for weak Euler classes. Weak Euler classes take values in weak Euler class Groups.

The definitions of the weak Euler class group and weak Euler classes were achieved by relaxing the definitions of the Euler class group and Euler classes. The original definitions of the Euler class group and the Euler classes, for

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smooth affine algebras over fields, were given by M. V. Nori. For background literature we refer to [Ma1], [MS], [MV],[BRS1], [BRS2], [Mu2].

For noetherian commutative rings A, with $\dim A = n \geq 2$ and a line bundle L on $Spec(A)$, the weak Euler class group $E_0(A, L)$ has been defined in ([BRS2]) as the quotient group $E_0(A, L) = G_0/H_0(L)$, where G_0 is the free abelian group generated by the set of all primary ideals N of height n such that N/N^2 is generated by n elements and $H_0(L)$ is the subgroup of G_0 generated by the set of all global $L-$cycles in G_0 (see the section on preliminaries for details).

Now assume that the field of rationals $\mathbb{Q}$ is contained in A. For projective $A-$modules P with $rank(P) = n$ and $det(P) = L$ the weak Euler class $e_0(P) \in E_0(A, L)$ of P has been defined in [BRS2] (see the section on preliminaries for details).

For a noetherian commutative ring A with $\dim A = n$, we define

$$F^n K_0(A) =$$

$$\{[A/I] \in K_0(A) : I \textit{ is local complete intersection ideal of height } n\}$$

In fact, $F^n K_0(A)$ is a subgroup of $K_0(A)$ ([Ma3]). It was also established in [Ma3] that, for a reduced affine algebra A over an algebraically closed field k, $F^n K_0(A)$ is the subgroup of $K_0(A)$ generated by the smooth maximal ideals of height n. This subgroup was investigated by Levine ([L]) and Srinivas ([Sr]).

When $\mathbb{Q} \subseteq A$, the weak Euler class induces a group homomorphism

$$\varphi_L : F^n K_0(A) \to E_0(A, L).$$

Conversely, there is a natural homomorphism

$$\psi_L : E_0(A, L) \to F^n K_0(A).$$

One of our main theorems is the following analogue of the Riemann-Roch theorem.

Theorem 1.2 *Let A be a Cohen-Macaulay ring of dimension $n \geq 2$. Assume that A contains the field of rationals $\mathbb{Q}$. Then φ_L and ψ_L are group homomorphisms. Further,*

$$\varphi_L \psi_L = -(n-1)! Id_{E_0(A,L)}$$

and

$$\psi_L \varphi_L = -(n-1)! Id_{F^n K_0(A)}.$$

It is known ([BRS2]) that, for any noetherian commutative ring A of dimension $n \geq 2$ with $\mathbb{Q} \subseteq A$ and line bundles L on $Spec(A)$, there is a natural isomorphism

$$\eta_L : E_0(A, A) \xrightarrow{\sim} E_0(A, L).$$

We also prove that φ_L and φ_A behave naturally with respect to η_L. That means $\varphi_L = \eta_L \varphi_A$. We give examples that such a natural property fails when we extend φ_L to a larger subgroup $F^2K_0(A)$ of $K_0(A)$. Similar examples also show that such an extension of φ_L fails to be a group homomorphism.

It was observed by S. M. Bhatwadekar that $kernel(\psi_A)$ is a torsion subgroup of $E_0(A, A)$. This result is a direct consequence of the theorem stated above. Our investigation on Riemann-Roch type of theorems on weak Euler classes originated out of a discussion on this result.

Assume that A is smooth over a field k with $char(k) = 0$ and $\dim(A) = n \geq 2$. Recall ([BRS3, 2.5]) the natural map $\pi_A : E_0(A, A) \to CH^n(A)$ that sends that cycle $(J) \in E_0(A, A)$ of a local complete intersection ideal J of height n to the Chow cycle $[J] \in CH^n(A)$. The map is well defined because the Chow cycle $[J] = 0$ when J is a complete intersection ideal. Following is a well known open question ([BRS3], [Mu2]).

Question 1.1 ([BRS3, Remark 3.13], [Mu2, Question 5.3]) *Let A be a smooth affine domain over an infinite field k and* $\dim A = n \geq 2$. *Let $CH^n(A)$ be the Chow group of zero cycles of $X = Spec(A)$ and $\pi_A : E_0(A, A) \to CH^n(A)$ be the natural homomorphism. Is π_A an isomorphism?*

Note that π_A is surjective. This question has affirmative answers when the field k is algebraically closed ([Mu2, page 163]) and when $k = \mathbb{R}$ ([BRS3, Theorem 5.5]).

It follows from our main theorem that, for a non-singular affine algebra A over a field k with $char(k) = 0$ and $\dim A = n \geq 2$, the kernel of the natural map $\pi_A : E_0(A, A) \to CH^n(A)$ is $(n-1)!-$ torsion. In particular, $\mathbb{Q} \otimes E_0(A, A) \approx \mathbb{Q} \otimes CH^n(A)$.

In section 2, we discuss some of the preliminaries and set up some definitions and notations. In section 3, we discuss the natural behavior of φ_A and φ_L with respect to η_L. In section 4, we prove our main theorem on Riemann-Roch and the consequences.

All rings we consider in this paper are noetherian and commutative with dimension at least 2. All modules we consider are finitely generated.

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2 Preliminaries

In this section we give definition of the weak Euler class group and the weak Euler class of projective modules. We also accumulate some of the results on $K-$theory and complete intersections that we use.

2.1 Definition of the weak Euler class groups

Let A be a noetherian commutative ring with $\dim A = n \geq 2$. Let L be a projective A-module of rank one. Bhatwadekar and Raja Sridharan ([BRS2]) defined the weak Euler class group $E_0(A, L)$ and weak Euler classes $e_0(P) \in E_0(A, L)$ of projective $A-$modules P, with $rank(P) = n$ and $det(P) = L$, as follows.

Definition 2.1 *Let A be a noetherian commutative ring with* $\dim A = n \geq 2$ *and let L be a rank one projective $A-$module. Write $F = L \oplus A^{n-1}$*

1. *Let G_0 be the free abelian group generated by the set*

$$S = \{N : N \textit{ is a primary ideal of height } n, \textit{ and } \mu(N/N^2) = n.\}$$

   (The minimal number of generators of a module will be denoted by μ.)

2. *Let J be an ideal of height n and $\mu(J/J^2) = n$. Let $J = N_1 \cap N_2 \cap \cdots \cap N_k$ be an irredundant primary decomposition of J. Then $\mu(N_i/N_i^2) = n$ and $N_i \in S$ for $i = 1, \ldots, k$. Let (J) denote the element $\sum_{i=1}^{k} N_i \in G_0$. We say (J) is the (weak Euler) $L-$cycle determined by J.*

3. *A cycle $(J) \in G_0$ is said to be a global (weak Euler) $L-$cycle if F maps onto J.*

4. *Let $H_0(L)$ be the subgroup of G_0 generated by all the global (weak Euler) $L-$cycles.*

5. *Define $E_0(A, L) = G_0/H_0(L)$. This group is called the weak Euler class group (relative to L).* Elements in $E_0(A, L)$ will also be called **weak Euler $L-$cycles.** We use the word 'cycle' in analogy to cycles in Chow groups.

6. **Notation:** The image of a cycle $(J) \in G_0$, determined by an ideal J, in $E_0(A, L)$ will be denoted by the same notation (J). It will be clear from the context, whether we mean (J) in G_0 or in $E_0(A, L)$.

7. *Now assume that the field of rationals $\mathbb{Q} \subseteq A$. Let P be a projective $A-$module with $rank(P) = n$ and $det(P) = L$. Let $f : P \to J$ be a surjective homomorphism, where J is an ideal of height n. Define the weak Euler class $e_0(P)$ of P as $e_0(P) = (J) \in E_0(A, L)$. In fact* ([BRS2, page 207]), *this association $e_0(P) = (J)$ is well defined.*

We quote the following theorem from [BRS2].

Theorem 2.1 ([**BRS2, Theorem 6.8**]) *Let A be a noetherian commutative ring of dimension $n \geq 2$ such that the field of rationals $\mathbb{Q}$ is contained in A. Then the natural map*

$$\eta_L : E_0(A, A) \xrightarrow{\sim} E_0(A, L)$$

is well defined and is an isomorphism of groups.

We now quote the following lemma from ([BRS2] or see [RS]).

Lemma 2.1 ([**BRS2, Proposition 6.7**]) *Let A be a noetherian commutative ring and P, Q be two projective modules of rank n such that $P \oplus A \simeq Q \oplus A$. Then there exists an ideal J of A, with height$(J) \geq n$, such that J is surjective image of both P and Q.*

Proof. Since $P \oplus A \simeq Q \oplus A$, we have an exact sequence

$$0 \longrightarrow Q \xrightarrow{i} P \oplus A \xrightarrow{(f,a)} A \longrightarrow 0.$$

Let $J = f(P)$. By a theorem of Eisenbud-Evans ([EE] or see [Ma2]), we may assume that J has height $\geq n$. Let $g : P \oplus A \to A$ be defined as $g(p, x) = x$. One can easily check that $g\,i(Q) = f(P)$. In other words, Q maps onto J.

The following follows immediately from the above lemma.

Lemma 2.2 (**[BRS2]**) *Let A be a noetherian commutative ring of dimension $n \geq 2$ with $\mathbb{Q} \subseteq A$ and L be a line bundle on $Spec(A)$. $K_0(A)$ will denote the Grothendieck group of finitely generated projective $A-$modules. Suppose P and Q are two projective $A-$module of rank n with $det(P) = L$. If $[P] = [Q] \in K_0(A)$ then $e_0(P) = e_0(Q) \in E_0(A, L)$.*

Proof. We have $rank(P) = rank(Q) = \dim A$. We have P and Q are stably isomorphic. Since rank $n+1$ projective modules are cancellative, by Bass cancellation theorem, it follows that $P \oplus A \approx Q \oplus A$. It also follows that $det(P) = det(Q) = L$. Now, by the above lemma, there is an ideal J of A of height n such that J is surjective image of both P and Q. Therefore, it is clear from the definition of the weak Euler class of a projective module that $e_0(P) = e_0(Q)$ in $E_0(A, L)$.

2.2 On the Grothendieck Group and the Chow Group

In this subsection we set up some notations and definitions regarding the Grothendieck group and the Chow group.

Notation 2.1 Let A be a noetherian commutative ring of dimension n and $X = Spec(A)$.

1. As usual, $K_0(A)$ (resp, $K_0(X)$) will denote the Grothendieck group of finitely generated projective $A-$modules.

2. $F^1K_0(A)$ will denote the kernel of the rank map $\epsilon : K_0(A) \to \mathbb{Z}$.

3. Define
$$F^2K_0(A) = \{x \in F^1K_0(A) : det(x) = A\}.$$

4. Define
$$F^nK_0(A) =$$
$$\{[A/I] \in K_0(A) : I\ is\ a\ local\ complete\ intersection\ ideal\ of\ height\ n\}.$$

   It was established in [Ma3, Theorem 1.1] that $F^nK_0(A)$ is a subgroup of $K_0(A)$.

5. When $n = 2$ the two notations for $F^2K_0(A)$ agree with each other. This follows from the fact that any unimodular row (a, b) of length 2 is first row of a matrix $\alpha \in SL_2(A)$.

6. We also write $K_0(X) = K_0(A), F^1K_0(X) = F^1K_0(A), F^2K_0(X) = F^2K_0(A)$.

Lemma 2.2 can be used to see that weak Euler classes define maps on $F^2K_0(A)$.

Definition 2.2 *Let A be a commutative noetherian ring of dimension $n \geq 2$ with $\mathbb{Q} \subseteq A$ and L be a line bundle on Spec(A). Write $F = L \oplus A^{n-1}$. Given any $x \in F^2K_0(A)$ we can write $x = [P] - [F]$ where rank$(P) = n$ and det$(P) = L$.*

Define $\Phi_L(x) = e_0(P)$.

It follows from the lemma 2.2 above that

$$\Phi_L : F^2K_0(A) \to E_0(A, L)$$

is a well defined map.

We will be more concerned with the restriction map

$$\varphi_L : F^nK_0(A) \to E_0(A, L)$$

of Φ_L to $F^nK_0(A)$.

Both the maps Φ_L and φ_L will be called the **weak Euler class map.**

Remark 2.1 In section 3, we will see that φ_A and φ_L behave naturally, with respect to the natural isomorphism $\eta_L : E_0(A, A) \xrightarrow{\sim} E_0(A, L)$, in the sense that $\varphi_L = \eta_L\varphi_A$. We will give examples to show that Φ_A and Φ_L fail to have the same natural property with respect to η_L. In section 4, we will also see that φ_L is a group homomorphism, while Φ_L is not.

Following are some standard notations regarding Chow Groups and Chern classes that will be useful for our later discussions.

Notation 2.2 Let X be a non-singular algebraic scheme of dimension n over a field k.

1. The Chow group of codimension r cycles will be denoted by $CH^r(X)$.
2. $CH(X) = \oplus_{r=0}^{n} CH^r(X)$ will denote the total Chow ring.
3. For $x \in K_0(X)$ the rth Chern class will be denoted by $c^r(x)$. Note that $c^r(x) \in CH^r(X)$.
4. $c(x) = 1 + c^1(x) + \cdots + c^n(x)$ will be called the total Chern class of x.

For general reference on Chow groups and Chern classes we refer to [F].

2.3 On Complete Intersections and $K-$theory

In this subsection we recall some of the key ingredients from complete intersections and $K-$theory. The first one among these results is the following theorem of Suslin ([S] or see [Ma2]).

Theorem 2.2 ([**S**]) *Let A be any commutative ring and $(a_1, \ldots, a_{n-1}, a_n)$ be a unimodular element. Then there is an invertible matrix $\alpha \in GL_n(A)$ such that first row of α is $(a_1, \ldots, a_{n-1}, a_n^{(n-1)!})$.*

Boratynski ([B] or see [Ma2]) used this theorem of Suslin to prove the following theorem.

Theorem 2.3 ([**B**]) *Let R be any commutative ring. Let I be an ideal in R and $I = (f_1, \ldots, f_{n-1}, f_n) + I^2$. Write $J = (f_1, \ldots, f_{n-1}) + I^{(n-1)!}$. Then J is image of a projective $R-$module P with rank$(P) = n$.*

This theorem of Boratynski served as a central motivation for some of the developments in this theory and of some techniques. We introduce the following notation.

Notation 2.3 *Let I be an ideal of a ring A such that I/I^2 is generated by n elements. Suppose $I = (f_1, \cdots, f_n) + I^2$. Let*

$$B(I) = B(I, \underline{f}) = B(I, f_1, \ldots, f_n) = (f_1, \ldots, f_{n-1}) + I^{(n-1)!}.$$

We quote the following from [Ma3].

Theorem 2.4 ([**Ma3, page 445**]) *Let A and I be as above. Further, assume that A is Cohen-Macaulay.*

1. *Then I is a local complete intersection ideal of height n if and only if so is $B(I, \underline{f})$.*

2. *If I is local complete intersection of height n then*
$$[A/B(I, \underline{f})] = (n-1)![A/I]$$
in $K_0(A)$.

The following version of Boratynski's theorem, due to Murthy ([Mu1]), is crucial for our later discussions.

Theorem 2.5 (**[Mu1, Theorem 2.2]**) *Let A be a noetherian commutative ring and $I \subset A$ be a local complete intersection ideal of height r. Suppose $I = (f_1, \cdots, f_r) + I^2$ and $J = (f_1, \cdots, f_{r-1}) + I^{(r-1)!}$. Assume $f_1, \ldots, f_r$ is a regular sequence. Then there is a projective A-module P of rank r and a surjective homomorphism $P \to J$, such that $[P] - [A^r] = -[A/I] \in K_0(A)$.*

We give the proof of Murthy's theorem to capture some of the technical details that will be useful in later sections.
Proof. It follows that $(1+s)I \subseteq (f_1, \ldots, f_r)$ for some $s \in I$. So, $\sum_{i=1}^r f_i g_i = s(1+s)$. Let
$$A_r = \mathbb{Z}[X_1, \ldots, X_r, Y_1, \ldots, Y_r, Z]/(\sum X_i Y_i - Z(1+Z))$$
Consider the map
$$A_r \longrightarrow A$$
that sends X_i, Y_i, respectively, to f_i, g_i and Z to s.

By Theorem 2.2, there is an invertible matrix $\beta \in GL_r((A_r)_{z(1+z)})$ whose first row is $(x_1, \ldots, x_{r-1}, x_r^{(r-1)!})$. Let α be the image of β in $GL_r(A_{s(1+s)})$.

Now, P be the projective $A-$module defined by the following fiber product diagram:

$$\begin{tikzcd}
P \arrow[rr] \arrow[d] & & A_s^r \arrow[d] \\
A_t^r \arrow[r] & A_{st}^r \arrow[r, "\alpha"] & A_{st}^r
\end{tikzcd}$$

Here $t = 1+s$. Then P maps onto J and $[P] - [A^r] = -[A/I]$.

The main technical point is, since α comes from $(A_r)_{z(1+z)}$, we have $[P] - [A^r] = -[A/I]$.

Also note that, if $r \geq 2$ then the determinant of the projective module P above is trivial. This follows from the fact (see [Sw]) that $Pic(A_r) = 0$.

3 The Natural Property

In this section we prove that the maps $\varphi_A : F^n K_0(A) \to E_0(A, A)$ and $\varphi_L : F^n K_0(A) \to E_0(A, L)$ behave naturally with respect to the natural isomorphism $\eta_L : E_0(A, A) \xrightarrow{\sim} E(A, L)$.

Theorem 3.1 *Let A be a commutative noetherian ring of dimension $n \geq 2$. Suppose J is a local complete intersection ideal of height n and $J = (f_1, f_2, \ldots, f_n) + J^2$ where $f_1, f_2, \ldots, f_n$ is a regular sequence. Let $I = (f_1, f_2, \ldots, f_{n-1}) + J^{(n-1)!}$. Assume that L is a projective $A-$module of rank one. We will construct a projective $A-$module Q of rank n such that*

1. *Q maps onto I*
2. *$[Q] - [L \oplus A^{n-1}] = -[A/J]$ in $K_0(A)$.*
3. *$\Lambda^n Q = L$.*

Proof. Since $\dim A = n = height(J)$, we can assume that L is an invertible ideal and $J + L = A$. Find $t_1 = 1 + s_1 \in 1 + J$ such that $t_1 J \subseteq (f_1, \ldots, f_n)$. Also let $t_2 = 1 + s_2 \in (1 + J) \cap L$. Let $t = t_1 t_2 = (1 + s_1 + s_2 + s_1 s_2) = 1 + s$ where $t \in L, s \in J$. We have $tJ \subseteq (f_1, \ldots, f_n)$. So, $I_t = (f_1, \ldots, f_{n-1}, f_n^{(n-1)!})$.

Let

$$A_n = \mathbb{Z}[X_1, \ldots, x_n, Y_1, \ldots, Y_n, Z]/(\sum X_i Y_i - Z(1 + Z))$$

There is a natural map $A_n \to A$, as described in the proof of Theorem 2.5. By the theorem of Suslin (theorem 2.2) there is a matrix $\alpha \in M_n(A) \cap GL_n(A_{st})$ such that

1. α is image of a matrix in $M_n(A_n) \cap GL_n((A_n)_{z(1+z)})$,
2. the first row of α is $(f_1, \ldots, f_{n-1}, f_n^{(n-1)!})$ and
3. $det(\alpha) = (st)^k$ for some integer $k \geq 0$.

Now consider the following fiber product diagram.

$$
\begin{tikzcd}
P \arrow[rrr] \arrow[dr, "g"] \arrow[dd] & & & A_s^n \arrow[dr, "g_2"] \arrow[dd] & \\
& I \arrow[rrr] \arrow[dd] & & & I_s \arrow[dd] \\
A_t^n \arrow[rr] \arrow[dr, "g_1"] & & A_{st}^n \arrow[r, "\alpha"] \arrow[dr, "g_1"] & A_{st}^n \arrow[dr, "g_2"] & \\
& I_t \arrow[rr] & & I_{st} \arrow[r] & I_{st}
\end{tikzcd}
$$

Here P is the projective $A-$module obtained by patching A_t^n and A_s^n via α. The homomorphism g_1 is given by $(f_1, \dots, f_{n-1}, f_n^{(n-1)!})$ and g_2 is given by $(1, 0, \dots, 0)$.

Because of the arguments given in the proof of theorem 2.5, we have $[P] - [A^n] = -[A/J]$ and $\Lambda^n P = A$.

Let $h : L \oplus P \to A = L + I$ be the surjective map defined by $h(l,p) = l - g(p)$. Let $Q = kernel(h)$. Then the following sequence

$$0 \to Q \to L \oplus P \xrightarrow{h} A \to 0$$

is exact. So, $Q \oplus A \approx L \oplus P$ and also

$$Q = \{(l,p) \in L \oplus P : l = g(p)\}.$$

Let $\varphi : Q \to L$ be defined by $\varphi(l,p) = l$. Then $\varphi : Q \to LI$ is a surjective homomorphism.

Write $F = L \oplus A^{n-1}$ and let $\psi : F \to L$ be the projection to L.

Let $K = A^{n-1}$ and $K' = ker(\varphi)$. Then sequences

$$0 \to K \to F \xrightarrow{\psi} L \to 0$$

and

$$0 \to K' \to Q \xrightarrow{\varphi} IL \to 0$$

are exact.

We will see that there is a homomorphism $\eta : Q \to F$, such that

1. η_s is an isomorphism,
2. $\varphi = \psi\eta$.

Now, we will split the proof into two parts. In Part-I of the proof, we assume the existence of a homomorphism η as above and complete the proof of the theorem. In Part-II, will prove the existence of the homomorphism η with the above properties.

Part-I: Some of the arguments in this part of the proof are similar to that in [Mu1]. It is enough to prove that there is a surjective map $f : F \oplus Q \to I \oplus F$. To see this, suppose there is such a surjection f. Let $Q' = f^{-1}(I)$. Then Q' maps onto I and $Q' \oplus F = Q \oplus F$. So, $[Q'] - [F] = [Q] - [F] = [P] - [A^n] = -[A/J]$. Also note that $det(Q') = L$.

Therefore, we will construct such a surjection $f : F \oplus Q \to I \oplus F$.

Let $M = \psi^{-1}(LI)$. Then there is an exact sequence

$$0 \to M \to I \oplus F \xrightarrow{\gamma} A \to 0$$

where the last map γ is defined by $\gamma(x, p) = x - \psi(p)$ and the first map sends $p \in M$ to $(\psi(p), p)$.

So, we will prove that $L \oplus A^{n-2} \oplus Q$ maps onto M. Look at the following diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \longrightarrow & Q & \xrightarrow{\varphi} & IL & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & M & \xrightarrow{\psi} & IL & \longrightarrow & 0 \end{array}$$

Note $\varphi = \psi\eta$. It follows that $K/\eta(K') \approx M/\eta(Q)$.

Since η_s is isomorphism, it follows that $K_s = \eta(K')_s$. So, $s^r K \subseteq \eta(K')$ for some integer $r > 0$.

The map $K/(s^r K) \to K/\eta(K')$ is surjective. Since, $t = 1 + s \in L$, we have $L/s^r L \approx A/s^r A$. So, $K/s^r K \approx (L \oplus A^{n-2})/(s^r(L \oplus A^{n-2})$. Therefore there is a surjective map $L \oplus A^{n-2} \to K/\eta(K')$.

Now $K/\eta(K') \approx M/\eta(Q)$. It follows that M is surjective image of $Q \oplus L \oplus A^{n-2}$. This completes the proof of Part-I.

Part-II: In this part, we establish that there is a homomorphism $Q \to F$ as described above. We write $Q_1 = Q_t$ and $Q_2 = Q_s$.

First, note that

$$Q_1 = \{(l, x_1, \ldots, x_n) \in L_t \oplus A_t^n : x_1 f_1 + \ldots + x_{n-1} f_{n-1} + x_n f_n^{(n-1)!} = l\}$$

and

$$Q_2 = \{(l, y_1, \dots, y_n) \in L_s \oplus A_s^n : y_1 = l\}.$$

Define

$$\eta_1 : Q_1 \to F_t$$

as follows:
We have $F_t = A_t^n$. Since $\alpha \in M_n(A)$, it defines a map $\alpha : A_t^n \to A_t^n$. Let $(l, x_1, \dots, x_n) \in Q_1$ and write $\alpha(x_1, \dots, x_n)^T = (z_1, z_2, \dots, z_n)$. We define $\eta_1(l, x_1, \dots, x_n) = (l, z_2, \dots, z_n)$. In fact, $\eta_1(l, x_1, \dots, x_n) = (z_1, z_2, \dots, z_n)$.

Define

$$\eta_2 : Q_2 \to F_s$$

as $\eta_2(l, y_1, \dots, y_n) = (l, y_2, \dots, y_n)$. In fact, $\eta_2(l, y_1, \dots, y_n) = (y_1, y_2, \dots, y_n)$.

Consider the fiber product diagram:

$$\begin{tikzcd}
Q \arrow[rrr, "\pi_2"] \arrow[dr, "\eta"] \arrow[dd, "\pi_1"'] & & & Q_2 \arrow[dr, "\eta_2"] \arrow[dd] & \\
& F \arrow[rrr] \arrow[dd] & & & F_s \arrow[dd] \\
Q_1 \arrow[rr] \arrow[dr, "\eta_1"'] & & Q_{1s} \arrow[r, "\delta"] \arrow[dr, "\eta_1"] & Q_{2t} \arrow[dr, "\eta_2"] & \\
& F_t \arrow[rr] & & F_{st} \arrow[r, "Id"] & F_{st}
\end{tikzcd}$$

Here δ is the restriction of $Id_L \oplus \alpha$.

We want to see that $\eta_2\delta = \eta_1$.
Let $(l, x_1, \dots, x_n) \in (Q_1)_s$. Let $(y_1, \dots, y_n)^T = \alpha(x_1, \dots, x_n)^T$.
Then $\eta_1(l, x_1, \dots, x_n) = (y_1, y_2, \dots, y_n) = (l, y_2, \dots, y_n)$. So, $\eta_2\delta = \eta_1$.

The homomorphism $\eta : Q \to F$ is given by the properties of fiber product diagrams. Since η_2 is isomorphism, so is η_s.

Note that $\eta(l, q) = (l, z_2, \dots, z_n)$ for some $z_2, \dots, z_n$. So, $\varphi = \psi\eta$. This completes the proof of the theorem.

Remark 3.1 The assumption in Theorem 3.1 that $height(I) = n = dimA$ was used only to arrange that L is an invertible ideal with $L + J = A$. The proof of the theorem shows that the Theorem 3.1 is also valid for any local complete intersection ideal $J = (f_1, \dots, f_r) + J^2$ of height $r \leq \dim(A)$ and invertible ideals L with $J + L = A$.

As a consequence of the above theorem the natural property of φ_L follows.

Corollary 3.1 *Let A be a commutative noetherian ring of dimension $n \geq 2$ with $\mathbb{Q} \subseteq A$ and L be a line bundle on Spec(A). Then $\varphi_L = \eta_L \varphi_A$, where $\varphi_A : F^n K_0(A) \to E_0(A, A)$ and $\varphi_L : F^n K_0(A) \to E_0(A, L)$ are the weak Euler class maps, as defined in Definition 2.2 and $\eta_L : E_0(A, A) \to E_0(A, L)$ is the natural isomorphism.*

Proof. Let $x = -[A/J] \in F^n K_0(A)$ where J is a local complete intersection ideal of height n. We can write $J = (f_1, \dots, f_n) + J^2$ where $f_1, \dots, f_n$ is a regular sequence. Let $I = (f_1, \dots, f_{n-1}) + J^{(n-1)!}$. By theorem 3.1, there are projective $A-$modules P, Q of rank n such that

1. Both P and Q map onto I,
2. $[P] - [A^n] = -[A/J] = x$ and $[Q] - [L \oplus A^{n-1}] = -[A/J] = x$ in $K_0(A)$,
3. $det(P) = A$ and $det(Q) = L$.

By definition, $\varphi_A(x) = (I)$ in $E_0(A, A)$ and $\varphi_L(x) = (I)$ in $E_0(A, L)$. So, it follows that $\varphi_L(x) = \eta\varphi_A(x)$. This completes the proof of the corollary.

Contrary to the above corollary, Φ_L and Φ_A fail to behave naturally likewise. To give an example, we recall the construction of Mohan Kumar ([MK]) below.

Example 3.1 (The Examples of Mohan Kumar [MK]) Let $k = \mathbb{Q}(t)$, where t is a transcendental element. Let p be a prime number and $f(T) = T^p - t$, where T is a polynomial variable. Note that $f(T^{p^r})$ is irreducible polynomial for all integer $r \geq 1$. Mohan Kumar defined homogeneous polynomials $F_n \in k[T_0, \dots, T_n]$, inductively, such that $F_1(T_0, T_1) = F(T_0, T_1) = T_1^p f(T_0/T_1)$ and $F_{n+1} = F(F_n, t^{u_n} T_{n+1}^{p^n})$ where $u_n = \sum_{1=0)}^{n-1} p^i$.

It follows that F_n is an irreducible polynomial of degree p^n. So, $S_n = V(F_n)$ is an irreducible hypersurface of the projective space $\mathbb{P}^n(k)$. Let $X_n = \mathbb{P}^n_k \setminus S_n$ be the affine open subset $F^n \neq 0$.

We recall some generalities regarding Chow groups ([F]) and summarize some facts from [MK]:

1. X_n is an affine smooth variety over k, with $\dim X_n = n$.

2. Let $\zeta \in CH^1(\mathbb{P}^n_k)$ denote the codimension one cycle defined by a linear equation. Then $CH(\mathbb{P}^n_k)$ is generated, as a $\mathbb{Z}-$algebra, by ζ.

3. The restriction map $j : CH(\mathbb{P}^n_k) \to CH(X_n)$ is a surjective ring homomorphism.

4. Let $a = j(\zeta) \in CH^1(X_n)$. Then $CH(X_n)$ is generated by a as an algebra over $\mathbb{Z}$.

5. Therefore, $CH^r(X_n) = (a^r)$ is generated by a^r.

6. $CH^n(X_n) = \mathbb{Z}/p\mathbb{Z}$ is nonzero.

7. Therefore $a^r \neq 0$ for $r = 1, \ldots, n$.

Following example shows that Φ_A and Φ_L do not behave naturally with respect to the natural isomorphism η_L.

Example 3.2 Let $X = X_3 = Spec(A)$ be the smooth affine 3-fold over a field $k = \mathbb{Q}(t)$, as in the above example of Mohan Kumar. We will show that $\eta_L \Phi_A \neq \Phi_L$ for some line bundle L.

Let $CH(X) = \oplus_{r=0}^3 CH^r(X)$ be the total Chow ring of X.

There is a line bundle L on X such that the first Chern class $c^1(L) = a$.

Let $P = L \oplus L^{-1} \oplus A$ and $Q = L \oplus L^{-1} \oplus L$. The total Chern classes of P, Q are given by $C(P) = 1 - a^2$ and $C(Q) = C(L)^2 C(L^{-1})) = (1+a)^2(1-a) = 1 + a - a^2 - a^3$. So, $c^3(P) = 0$ and $c^3(Q) = -a^3$. Since $a^3 \neq 0$, we have $c^3(P) \neq c^3(Q)$.

There is a natural homomorphism $\pi_L : E_0(A, L) \to CH^3(X)$ (see [Mu2]). Then $c^3(P) = (-1)^3 \pi_A(\Phi_A(x))$ and $c^3(Q) = (-1)^3 \pi_L(\Phi_L(x))$. Also note that $\pi_A = \pi_L \eta_L$.

Let $x = [P] - [A^3]$. Then $x \in F^2K_0(A)$ and also $x = [Q] - [L \oplus A^2]$. We claim that $\eta_L \Phi_A(x) \neq \Phi_L(x)$.

There is a surjective homomorphism $P \to J$ where J is a local complete intersection ideal of height 3. Since $det(P) = A$, by definition $\Phi_A(x) = (J)$ in $E_0(A, A)$, and $\eta_L \Phi_A(x) = (J)$ in $E_0(A, L)$.

We claim that $\Phi_L(x) \neq (J)$. For, if $\Phi_L(x) = (J)$, then
$C^3(Q) = (-1)^3 cycle(J) = C(P)$, which is a contradiction. Therefore, $\Phi_L \neq \eta_L \Phi_A$.

4 Results on Riemann-Roch

In this section, we discuss our results on Riemann-Roch. First, we have the following lemma on classes of ideals in $E_0(A, L)$.

Lemma 4.1 *Let A be a noetherian commutative ring of dimension $n \geq 2$ and J be a local complete intersection ideal of height n. Let $J = (f_1, \ldots, f_n) + J^2$ and $J_r = (f_1, \ldots, f_{n-1}) + J^r$. If $f_1, \ldots, f_n$ is a regular sequence, then the class $(J_r) = r(J)$ in $E_0(A, L)$.*

Proof. We will write $f_n = g_1$. There exists a local complete intersection ideal K_1 of height n such that $J \cap K_1 = (f_1, \ldots, f_{n-1}, g_1)$ and $J + K_1 = A$.

By induction, we can find, for $i = 1, \ldots, r$, elements $g_i \in J$ and local complete intersection ideals K_i of height n such that

1. $J = (f_1, f_2, \ldots, f_{n-1}, g_i) + J^2$.
2. $J \cap K_i = (f_1, f_2, \ldots, f_{n-1}, g_i)$.
3. $J + K_i = A$ and $K_i + K_j = A$, for $i \neq j$.

We will indicate the proof of the inductive step. Suppose we have picked $g_1, \ldots, g_k$. We will pick g_{k+1}. Let $\mathcal{P}_1 = \{\wp \in Spec(A) : K_1 \cap \cdots \cap K_k \subseteq \wp\}$ and let $\mathcal{P}_2$ be the set of all associated primes of $(f_1, \ldots, f_{n-1})$. Write $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Let $\wp_1, \ldots, \wp_l, \wp_{l+1}, \ldots, \wp_m$ be the maximal elements in $\mathcal{P}$. Assume $f_n \notin \wp_i$ for $i = 1$ to l and $f_n \in \wp_i$ for $i = l+1$ to m. Pick $\lambda \in I^2 \cap (\cap_{i=1}^{l} \wp_i) \setminus (\cup_{i=l+1}^{m} \wp_i)$. Write $g_{k+1} = f_n + \lambda$. This completes the proof of the inductive step.

Let 'overline' denote mod $(f_1, \ldots, f_{r-1})$.
Then $\overline{J_r K_1} \cdots \overline{K_r} = \overline{J}^r \overline{K_1} \cdots \overline{K_r} = \prod (JK_i) = \overline{g_1 \cdots g_r}$.
So, $J_r \cap K_1 \ldots \cap K_r = (f_1, f_2, \ldots, f_{r-1}, g)$ where $g = g_1 g_2 \cdots g_r$.

Therefore, it follows that $(J_r) = -\sum (K_i) = r(J)$ in $E_0(A, A)$.

Also, since the natural map $\eta_L : E_0(A, A) \to E_0(A, L)$ is an isomorphism (Theorem 2.1), it follows that $(J_r) = r(J)$ in $E_0(A, L)$. This completes the proof of the lemma.

Before we state our next theorem, we define a map in the opposite direction to that of the weak Euler class map $\varphi_L : F^n K_0(A) \to E_0(A, L)$.

Definition 4.1 *Let A be a Cohen-Macaulay ring of dimension $n \geq 2$. Define $\psi_L : E_0(A, L) \to F^n K_0(A)$ as the natural map that sends the class (J) of an ideal J to the class $[A/J]$. Since J is locally n generated and A is Cohen-Macaulay, it follows that that J is a local complete intersection ideal and $[A/J] \in F^n K_0(A)$.*

It follows immediately that $\psi_L = \eta_L \psi_A$.

We will see that ψ_L is a well defined group homomorphism in the following theorem. Following is the statement of our main theorem.

Theorem 4.1 *Let A be a Cohen-Macaulay ring of dimension $n \geq 2$ that contains the field of rationals $\mathbb{Q}$. Then,*

1. *the maps $\varphi_L : F^n K_0(A) \to E_0(A, L)$ and $\psi_L : E_0(A, L) \to F^n K_0(A)$ are well defined group homomorphisms,*

2. $$\varphi_L \psi_L = -(n-1)! Id_{E_0(A,L)}$$
   and

3. $$\psi_L \varphi_L = -(n-1)! Id_{F^n K_0(A)}.$$

Proof. Write $F = L \oplus A^{n-1}$ and $E_0(A, L) = G_0/H_0(L)$ as in the definition 2.1. First, we want to establish that ψ_L is well defined. Clearly, since A is Cohen-Macaulay, the assignment that sends the class $(J) \in G_0$ of an ideal J to $[A/J] \in F^n K_0(A)$ defines a group homomorphism $G_0 \to F^n K_0(A)$. Now suppose that there is a surjective map $F \to J$ where J is an ideal of height n. Since A is a Cohen-Macaulay ring, J is a local complete intersection ideal. It follows that $[A/J] = \sum_{i=0}^{n} (-1)^r [\Lambda^r F] = 0$ in $K_0(A)$. Therefore, $\psi_L : E_0(A, L) \to F^n K_0(A)$ is a well defined group homomorphism. (*Note that we did not use the hypothesis that $\mathbb{Q} \subseteq A$ to prove that ψ_L is well defined.*)

We have already seen that the weak Euler class map φ_L is well defined. We will prove that φ_L is a group homomorphism. First, we will prove that $\varphi_A : F^n K_0(A) \to E_0(A, A)$ is a group homomorphism.

Since the weak Euler class $e_0(A^n) = 0$, it follows that $\varphi_A(0) = 0$. Now let $x, y \in F^n K_0(A)$. We can write $x = -[A/I]$ and $y = -[A/J]$, where I, J are local complete intersection ideals of height n. We can assume that $I+J=A$. Let $I=(f_1,\dots,f_n)+I^2$ and $J=(g_1,\dots,g_n)+J^2$. Write $r=(n-1)!$ and let $I_r=(f_1,\dots,f_{n-1})+I^r$ and $J_r=(g_1,\dots,g_{n-1})+J^r$. By Theorem 2.5, there are projective $A-$modules P,P' of rank n such that

1. There is a surjective map $P\to I_r$ and $[P]-[A^n]=-[A/I]=x$.
2. There is a surjective map $P'\to J_r$ and $[P']-[A^n]=-[A/J]=y$.

We have,

$$\varphi_A(x)+\varphi_A(y)=e_0(P)+e_0(P')=(I_r)+(J_r)=r(I)+r(J)$$

in $E_0(A,A)$.

Let $K=I\cap J$ and $K=(h_1,\dots,h_n)+K^2$. Write $K_r=(h_1,\dots,h_{n-1})+K^r$. Again, by Theorem 2.5, there is a projective $A-$module Q of rank n such that $[Q]-[A^n]=-[A/K]$ and there is a surjective map $Q\to K_r$.

So, $[Q]-[A^n]=-[A/K]=-[A/I]-[A/J]=x+y$. Therefore,

$$\varphi_A(x+y)=e_0(Q)=(K_r)=r(K)=r(I)+r(J)=\varphi_A(x)+\varphi_A(y).$$

Therefore, φ_A is a group homomorphism. Since $\varphi_L=\eta_L\varphi_A$, it follows that φ_L is a group homomorphism for any line bundle L on $Spec(A)$.

Now we prove that $\varphi_A\psi_A=-(n-1)!Id_{E_0(A,A)}$. Let $x=(I)\in E_0(A,A)$, where I is an ideal of height n with $\mu(I/I^2)=n$.

Let $I=(f_1,\dots,f_n)+I^2$. Since A is a Cohen-Macaulay ring, we can assume that $f_1,\dots,f_n$ is a regular sequence. We can write $(f_1,\dots,f_n)=I\cap J$, where J is a local complete intersection ideal of height n and $I+J=A$. So, $\psi_A(x)=[A/I]=-[A/J]$. Then $J=(f_1,\dots,f_n)+J^2$. Write $B(J)=(f_1,\dots,f_{n-1})+J^{(n-1)!)}$. By Theorem 2.5, there is a projective $A-$module P of rank n such that $[P]-[A^n]=-[A/J]=\psi_A(x)$ and P maps onto $B(J)$.

By definition $\varphi_A(\psi_A(x))=\varphi_A(-[A/J])=e_0(P)=(B(J))$. Now it follows from Lemma 4.1 that the class $(B(J))=(n-1)!(J)=-(n-1)!(I)$ in $E_0(A,A)$.

So, $\varphi_A(\psi_A(x))=e_0(P)=(B(J))=-(n-1)!(I)=-(n-1)!x$.

We shall now prove that $\psi_A\varphi_A=-(n-1)!Id_{F^nK_0(A)}$. Let $x=-[A/J]\in F^nK_0(A)$. Again, let $B(J),P$ be as above. Then $\varphi_A(x)=e_0(P)=(B(J))$. By theorem 2.4, we have $\psi_A(B(J))=[A/B(J)]=(n-1)![A/J]=-(n-1)!x$.

This completes the proof of the theorem when $L = A$.

In the general case, $\varphi_L\psi_L = \eta_L(\varphi_A\psi_A)\eta_L^{-1} = -(n-1)!Id_{E_0(A,L)}$. Similarly, $\psi_L\varphi_L = \psi_L\eta_L\varphi_A = \psi_A\varphi_A = -(n-1)!Id_{F^nK_0(A)}$. This completes the proof of the theorem.

Following corollary is a partial answer to Question 1.1 stated in the introduction.

Corollary 4.1 *Let A be a regular ring containing the field of rationals $\mathbb{Q}$ and* $\dim A = n \geq 2$. *Let $\pi_A : E_0(A, A) \to CH^n(A)$ be the natural homomorphism. Then kernel(π_A) is $(n-1)!-$torsion. So, $\mathbb{Q} \otimes E_0(A, A) \approx \mathbb{Q} \otimes CH^n(A)$. Also,in particular, if kernel(π_A) has no $(n-1)!-$torsion, then π_A is an isomorphism.*

Proof. Let $\zeta : CH^n(A) \to F^nK_0(A)$ be the natural homomorphism. Let $\pi_A(x) = 0$. Since that $\zeta\pi_A = \psi_A$, it follows that $\psi_A(x) = 0$. So, $(n-1)!x = -\phi_A\psi_A(x) = 0$. This completes the proof of the corollary.

The following result was orally communicated to one of the authors by S. M. Bhatwadekar. The result is a consequence of theorem 4.1.

Corollary 4.2 (**Bhatwadekar**) *Let A be a Cohen-Macaulay ring of dimension $n \geq 2$ that contains the field of rationals $\mathbb{Q}$. Then, kernel(ψ_A) is $(n-1)!$ torsion.*

Following is also a corollary to Theorem 4.1.

Corollary 4.3 *Let A be a Cohen-Macaulay ring of dimension $n \geq 2$ containing the field of rationals $\mathbb{Q}$. Then, the image $\varphi_L(F^nK_0(A)) = (n-1)!E_0(A, L)$.*

Proof. It is enough to prove $\varphi_A(F^nK_0(A)) = (n-1)!E_0(A, A)$. Let $x = \varphi_A(y)$ be in the image $\varphi_A(F^nK_0(A))$, where $y \in F^nK_0(A)$. Note that the map $\psi_A : E_0(A, A) \to F^nK_0(A)$ is surjective. So, $\psi_A(z) = y$ for some $z \in E_0(A, A)$. Therefore, $x = \varphi_A(y) = \varphi_A(\psi_A(z)) = -(n-1)!z$ is in $(n-1)!E_0(A, A)$.

Conversely, let $x = (n-1)!(I) \in (n-1)!E_0(A, A)$, where I is an ideal of height n and $I = (f_1, \ldots, f_{n-1}, f_n) + I^2$. Since A is Cohen-Macaulay, we can assume that $f_1, \ldots, f_{n-1}, f_n$ is a regular sequence. Write $J = (f_1, \ldots, f_{n-1})+$

$I^{(n-1)!}$. Then $x = (J)$ and by Theorem 2.5, x is in $\varphi_A(F^n K_0(A))$. This completes the proof of this corollary.

The following example shows that Φ_L fails to be a group homomorphism.

Example 4.1 Let X_4 be the affine open subset of $\mathbb{P}_k^4$ in Mohan Kumar's example 3.1 and $X_4 = Spec(A)$. Let a be the generator of $CH^1(A)$, as given in the example. Then we know that $a^4 \neq 0$. Let L be a line bundle on X_4 such that the first Chern class $c^1(L) = a$.

We shall see that Φ_A is not a group homomorphism on $F^2K_0(A)$.

Let $P = P' = L \oplus L^{-1} \oplus A^2$. We can write $P \oplus P' = Q \oplus A^4$ for some projective $A-$module Q of rank 4.

So, the total Chern classes $C(P) = C(P') = 1 - a^2$ and $C(Q) = 1 - 2a^2 + a^4$. So, the top Chern classes $c^4(P) = c^4(P') = 0$ and $c^4(Q) = a^4 \neq 0$.

Let $x = [P] - [A^4], y = [P'] - [A^4]$. Then $x + y = [Q] - [A^4]$. We have $0 = c^4(P) + c^4(P') \neq c^4(Q)$.

Let $\pi_A : E_0(A, A) \to CH^4(A)$ be the natural map. We have $\pi_A\Phi_A = (-1)^4c^4$ on $F^2K_0(A)$. Therefore, it follows that $\Phi_A(x+y) \neq \Phi_A(x) + \Phi_A(y)$.

So, Φ_A is not a group homomorphism.

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