The Euler class groups of polynomial rings and unimodular elements in projective modules

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1 Introduction

Let A be a commutative noetherian ring of dimension n. Let P be a projective $A[T]$-module. Plumstead ([P]) proved that if $\operatorname{rank} P > \dim A$ then P splits off a free summand of rank one. It is natural to ask what happens when $\operatorname{rank} P = \dim A$. In this paper we investigate this question when P has trivial determinant. Let $\alpha : P \twoheadrightarrow I$ be a generic surjection (i.e. $I \subset A[T]$ is an ideal of height n). It is proved in ([D]) that if P splits off a free summand of rank one then I is generated by n elements. It is natural to ask whether the converse holds, i.e., if I is generated by n elements then whether P has a free summand of rank one. This has been proved in ([B-RS 4]) if A is an affine domain over an algebraically closed field. But the following example shows that this converse is false in general. Let A be the coordinate ring of the even dimensional real sphere and $\widetilde{P}$ be the tangent bundle. It can be shown that $\widetilde{P}$ does not have a free summand of rank one whereas there is a generic surjection $\beta : \widetilde{P} \twoheadrightarrow J$ such that J is generated by n elements. Tensoring with $A[T]$ we obtain a generic surjection $\beta \otimes A[T] : \widetilde{P}[T] \twoheadrightarrow J[T]$ showing that the converse is false. This leads us to the following

Question : *Let A be a noetherian ring with* $\dim A = n$ *and P be a projective*

$A[T]$-module of rank n having trivial determinant. Suppose that there exists a surjection $\alpha : P \twoheadrightarrow I$ where $I \subset A[T]$ is an ideal of height n which is generated by n elements. Assume further that the A-module P/TP has a free summand of rank one. Does P split off a free summand of rank one ?

We prove in this paper (3.5) that the question has an affirmative answer in the case where n is even and A contains the field of rationals.

If A is a noetherian ring of dimension n containing $\mathbb{Q}$, in ([D]) a group called the Euler class group of $A[T]$, denoted by $E(A[T])$, is defined and it is shown that if P is a projective $A[T]$-module of rank n (with trivial determinant) then P splits off a free summand of rank one if and only if the Euler class of P in $E(A[T])$ vanishes. The method of proof of (3.5), in brief, is to show that under the hypothesis of the question, the Euler class of P vanishes.

If A is a smooth affine domain over reals, then the set X of real points of Spec A is a manifold of dimension n and the groups $E(A)$ and $E(A[T])$ are algebraic analogues of the n-th cohomology groups $H^n(X)$ and $H^n(X \times I)$. In view of the homotopy axiom of cohomology it is natural to ask whether the canonical map from $E(A)$ to $E(A[T])$ is an isomorphism. It is shown in ([D]) that this is the case if A is smooth. It is also remarked in ([D]) that in general $E(A)$ and $E(A[T])$ are not canonically isomorphic. It is natural to ask if one can give a natural description of the quotient $E(A[T])/E(A)$. We show that if $\dim A$ is even, $E(A[T])/E(A)$ is isomorphic to the group $E_0(A[T])/E_0(A)$ (see ([D]) or Section 2 for definition). This is of interest because the groups $E_0(A[T])$ and $E_0(A)$ are analogues of the groups $F^nK_0(A[T])$ and $F^nK_0(A)$.

We also develop a "Quillen-Suslin theory" for the weak Euler class groups in the case when $\dim A = n$ is even.

2 Some preliminaries

In this section we define some of the terms used in the paper and record some results which are used in later sections.

All rings considered in this paper are commutative and noetherian and all modules considered are assumed to be finitely generated. For a module M over a ring, $\mu(M)$ will denote the minimal number of generators of M.

Definition 2.1 *Let A be a ring. A row $(a_1, a_2, \cdots, a_n) \in A^n$ is said to be unimodular if there exist $b_1, b_2, \cdots, b_n$ in A such that $a_1 b_1 + \cdots + a_n b_n = 1$.*

Definition 2.2 *Let A be a noetherian ring. Let P be a projective A-module. An element $p \in P$ is said to be unimodular if there exists a linear map $\phi : P \to A$ such that $\phi(p) = 1$.*

Remark 2.3 Note that a projective A-module of rank n has a unimodular element if and only if $P = Q \oplus A$ for some projective A-module Q of rank $n - 1$.

Now we state a useful lemma. The proof of this lemma can be found in ([B-RS 1], 3.3).

Lemma 2.4 *Let A be a noetherian ring containing an infinite field k and let $I \subset A[T]$ be an ideal of height n. Then there exists $\lambda \in k$ such that either $I(\lambda) = A$ or $I(\lambda) \subset A$ is an ideal of height n, where $I(\lambda) = \{f(\lambda) : f(T) \in I\}$.*

The following lemma is a consequence of a theorem of Eisenbud-Evans. For a proof one can look at ([B-RS 3], 2.13)

Lemma 2.5 *Let A be a ring and P be a projective A-module of rank n. Let $(\alpha, a) \in (P^* \oplus A)$. Then there exists an element $\beta \in P^*$ such that* $\operatorname{ht}(I_a) \geq n$*, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq n$ then* $\operatorname{ht} I \geq n$*. Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and I is a proper ideal of A, then* $\operatorname{ht} I = n$*.*

Definition 2.6 *Let A be a commutative noetherian ring and P be a projective A-module of rank $n \leq \dim A$. By a generic surjection of P we mean a surjection $\alpha : P \twoheadrightarrow J$ where J is an ideal of A of height n.*

Lemma (2.5) actually ensures that generic surjections always exist.

The following proposition is a consequence of a result of Ravi Rao ([R], Corollary 2.5) and Quillen's local-global principle ([Q], Theorem 1).

Proposition 2.7 *Let A be a noetherian ring of dimension n. Suppose $n!$ is invertible in A. Then any projective module given by a unimodular row over $A[T]$ of* *length $n+1$ is extended from A. In other words, all stably free $A[T]$-modules of rank n are extended from A.*

Definition 2.8 *Let A be a commutative noetherian ring, P a projective $A[T]$-module. Let $J(A,P) \subset A$ consist of all those $a \in A$ such that P_a is extended from A_a. It follows from* ([Q], Theorem 1)*, that $J(A,P)$ is an ideal and $J(A,P) = \sqrt{J(A,P)}$. This is called the Quillen ideal of P in A.*

Remark 2.9 It is easy to deduce from Quillen-Suslin theorem ([Q], [S]) that ht $J(A,P) \geq 1$. If determinant of P is extended from A, then by ([B-R], 3.1), ht $J(A,P) \geq 2$.

Proof Let S be the (multiplicative) set of nonzero divisors of A. Then $S^{-1}A[T] \simeq k_1[T] \times \cdots k_r[T]$ for some fields $k_1, \cdots, k_r$. Then by Quillen-Suslin theorem ([Q], [S]), $S^{-1}P$ is free. So there exists $s \in S$ such that P_s is free (in particular, extended from A_s). So $s \in J(A,P)$ and hence height of $J(A,P)$ is at least one.

Now suppose that the determinant of P is extended from A. Let $L[T]$ be the determinant of P. Let $\mathcal{P}$ be any prime ideal of A of height one. Consider $A_{\mathcal{P}}[T]$. Since dim $A_{\mathcal{P}} = 1$, by ([B-R], 3.1) $P_{\mathcal{P}} = L_{\mathcal{P}}[T] \oplus A_{\mathcal{P}}[T]^m$ for some positive integer m. So $P_{\mathcal{P}}$ is extended and therefore there exists $t \in A - \mathcal{P}$ such that P_t is extended from A_t, i.e., $t \in J(A,P)$. Now it can be easily seen that ht $J(A,P) \geq 2$. $\square$

In the rest of this section we briefly sketch the definitions of the *Euler class group* $E(A[T])$ and the *weak Euler class group* $E_0(A[T])$ (where A is a commutative noetherian ring containing $\mathbb{Q}$ with dim $A = n \geq 2$) and quote some results relevant to this paper. These two notions have been defined and studied in ([D]) which we refer to for a detailed account.

Let $I \subset A[T]$ be an ideal of height n such that $\mu(I/I^2) = n$. Two surjections α and β from $(A[T]/I)^n \twoheadrightarrow I/I^2$ are said to be *related* if there exists $\sigma \in SL_n(A[T]/I)$ such that $\alpha\sigma = \beta$. This is an equivalence relation on the set of surjections from $(A[T]/I)^n$ to I/I^2. Let $[\alpha]$ denote the equivalence class of α. We call such an equivalence class $[\alpha]$ a *local orientation of* I.

It was shown in ([D]) that if $\alpha : (A[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$ then so can any β equivalent to α. We call a local orientation $[\alpha]$ of I a *global orientation of* I if the surjection $\alpha : (A[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$.

Let G be the free abelian group on the set of pairs (I, ω_I) where $I \subset A[T]$ is an ideal of height n such that Spec $(A[T]/I)$ is connected and $\mu(I/I^2) = n$, and $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ is a local orientation of I.

Let $I \subset A[T]$ be an ideal of height n and ω_I a local orientation of I. Now I can be decomposed uniquely as $I = I_1 \cap \cdots \cap I_r$, where the I_k's are ideals of $A[T]$ of height n, pairwise comaximal and Spec $(A[T]/I_k)$ is connected for each k. Clearly ω_I induces local orientations ω_{I_k} of I_k for $1 \leq k \leq r$. By (I, ω_I) we mean the element $\Sigma(I_k, \omega_{I_k})$ of G.

Let H be the subgroup of G generated by set of pairs (I, ω_I), where I is an ideal of $A[T]$ of height n and ω_I is a global orientation of I. We define the Euler class group of $A[T]$, denoted by $E(A[T])$, to be G/H.

The weak Euler class group $E_0(A[T])$ is defined in a similar way, just dropping the orientations, as follows:

Let F be the free abelian group on the set of ideals $\mathcal{I}$ where $\text{ht}\,\mathcal{I} = n$, $\mu(\mathcal{I}/\mathcal{I}^2) = n$ and Spec $(A[T]/\mathcal{I})$ is connected. For an ideal I of $A[T]$ of height n with $\mu(I/I^2) = n$ we take its decomposition into connected components (as above), say, $I = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_r$, and associate to I the element $(I) := \Sigma \mathcal{I}_k$ of F. Let K be the subgroup of F generated by elements of the type (I), where $I \subset A[T]$ is an ideal of height n and $\mu(I) = n$. We define $E_0(A[T])$ to be F/K.

Let P be a projective $A[T]$-module of rank n with trivial determinant. Fix a trivialization $\chi : A[T] \simeq \wedge^n(P)$. Let $\alpha : P \twoheadrightarrow I$ be a generic surjection. Note that P/IP is a free $A[T]/I$-module. Composing $\alpha \otimes A[T]/I$ with some isomorphism $\gamma : (A[T]/I)^n \simeq P/IP$ with the property $\wedge^n(\gamma) = \chi \otimes A[T]/I$ we get a local orientation, say ω_I, of I. To the pair (P, χ) we attach the element (I, ω_I) of $E(A[T])$, denote it by $e(P, \chi)$ and call it the *Euler class* of P. The Euler class of P is well defined.

The following results were proved in ([D]).

Theorem 2.10 *Let A be a noetherian ring containing $\mathbb{Q}$ with* $\dim A = n \geq 2$. *Let $I \subset A[T]$ be an ideal of $A[T]$ of height n such that $\mu(I/I^2) = n$ and ω_I be a local orientation of I. Let P be a rank n projective $A[T]$-module with trivial determinant with a trivialization $\chi : A[T] \simeq \wedge^n(P)$. Then,*

(a) Suppose that the image of (I, ω_I) is zero in $E(A[T])$. Then ω_I is a global orientation of I.
(b) Suppose that $e(P, \chi) = (I, \omega_I)$ in $E(A[T])$. Then there exists a surjection $\alpha : P \twoheadrightarrow I$ such that ω_I is induced by α and χ (as described above).
(c) P has a unimodular element if and only if $e(P, \chi) = 0$ in $E(A[T])$.

The following theorem about $E_0(A[T])$ is crucial for the next two sections. It has also been proved in ([D]).

Theorem 2.11 *Let A be a noetherian ring containing $\mathbb{Q}$ with $\dim A = n$ (n even). $I \subset A[T]$ be an ideal of height n such that $\mu(I/I^2) = n$. Then, $(I) = 0$ in $E_0(A[T])$ if and only if I is image of a stably free $A[T]$-module of rank n.*
More precisely, if ω_I is a local orientation of I and $(I) = 0$ in $E_0(A[T])$, then there exists a stably free $A[T]$-module Q of rank n and a trivialization χ_Q of $\wedge^n(Q)$ such that $e(Q, \chi_Q) = (I, \omega_I)$ in $E(A[T])$. (See also 3.4)

3 On unimodular elements in projective modules

We begin with a lemma. Proof of this lemma can be found in ([B], 2.2).

Lemma 3.1 *Let A be a ring and $J \subset A$ be an ideal of height r. Let $\overline{\alpha} : (A/J)^r \twoheadrightarrow J/J^2$ and $\overline{\beta} : (A/J)^r \twoheadrightarrow J/J^2$ be two surjections. Let $\overline{\psi} \in M_r(A/J)$ be such that $\overline{\beta\psi} = \overline{\alpha}$. Then $\overline{\psi} \in GL_r(A/J)$.*

Remark 3.2 Let A be a noetherian ring with dim $A = n \geq 2$. Let $I \subset A[T]$ be an ideal of height n and ω_I be a local orientation of I. Let $\overline{F} \in A[T]/I$ be a unit. Composing ω_I with an automorphism of $(A[T]/I)^n$ with determinant $\overline{F}$, we obtain another local orientation of I which we denote by $\overline{F}\omega_I$. On the other hand, let ω_I, $\widetilde{\omega_I}$ be two local orientations of I. Then, it is easy to see from (3.1), that $\widetilde{\omega_I} = \overline{F}\omega_I$ for some unit $\overline{F} \in A[T]/I$.

The method of proof of the following lemma follows ([RS], pp 956).

Lemma 3.3 *Let B be any commutative noetherian ring and $J \subset B$ be an ideal such that $J = (a_1, \cdots, a_n)$, n even. Let $u, v \in B$ be such that $uv = 1$ modulo*

J. Assume further that the unimodular row $(v, a_1, \cdots, a_n)$ is completable. Then there exists $\sigma \in M_n(B)$ with det $(\sigma) = u$ modulo J such that if $(a_1, \cdots, a_n)\sigma = (b_1, \cdots, b_n)$ then $b_1, \cdots, b_n$ generate J.

Proof Let $\{e_0, \cdots, e_n\}$ be the standard basis of B^{n+1}. Consider the surjection f from B^{n+1} to J which sends e_0 to 0, e_i to a_{i+1} if i is odd and e_i to $-a_{i-1}$ if i is even. Clearly the image of the row $(v, a_1, \cdots, a_n)$ under f is zero. Since the unimodular row $(v, a_1, \cdots, a_n)$ is completable, there is a matrix $\alpha \in SL_{n+1}(B)$ whose first row is $(v, a_1, \cdots, a_n)$. Since the rows of α form a basis of B^{n+1}, the images of the rows under f generate J and since first row is in the kernel of f, it follows that the images of the rest n rows of α generate J. Denoting image of $(i+1)^{\text{st}}$ row by b_i, we have, $J = (b_1, \cdots, b_n)$. Let σ be the matrix obtained from α by deleting the first row and the first column of α. It is easily seen that $(a_1, \cdots, a_n)\sigma = (b_1, \cdots, b_n)$ and since $\alpha \in SL_{n+1}(B)$, $\det \sigma = u$ modulo J. This proves the lemma. □

Lemma 3.4 *Let A be a commutative noetherian ring containing $\mathbb{Q}$ with* dim $A = n$ *(n even). Let $I = (f_1, \cdots, f_n)$ be an ideal of $A[T]$ of height n. Let $\widetilde{\omega}$ be any local orientation of I. Then, there exists a stably free $A[T]$-module Q of rank n and a generator χ of $\wedge^n(Q)$ such that $e(Q, \chi) = (I, \widetilde{\omega})$ in $E(A[T])$.*

Proof Let us denote the orientation of I given by the generators $f_1, \cdots, f_n$ by ω. Then, by (3.1), $\widetilde{\omega} = \overline{F}\omega$ where $F \in A[T]$ is a unit modulo I. Let $G \in A[T]$ be such that $FG = 1$ modulo I. Consider the unimodular row $(G, f_1, \cdots, f_n)$ over $A[T]$ and let $Q = A[T]^{n+1}/(G, f_1, \cdots, f_n)$. Then Q is a stably free $A[T]$-module of rank n. Let $e_0, e_1, \cdots, e_n$ be the standard basis of $A[T]^{n+1}$ and $q_0, q_1, \cdots, q_n$ be the images of $e_0, e_1, \cdots, e_n$ in Q. Then $q_0 G + \Sigma_1^n q_i f_i = 0$.

We define a surjection $\alpha : Q \twoheadrightarrow I$ as follows:
(i) $\alpha(q_0) = 0$,
(ii) $\alpha(q_i) = f_{i+1}$ if i is odd, and
(iii) $\alpha(q_i) = -f_{i-1}$ if i is even.

Since $FG - 1 \in I = (f_1, \cdots, f_n)$, there exist $g_1, \cdots, g_n \in A[T]$ such that $FG + \Sigma_1^n f_i g_i = 1$. It is easy to check that the element

$$\chi = Gq_1 \wedge q_2 \cdots \wedge q_n - g_1 q_0 \wedge q_2 \cdots \wedge q_n + g_2 q_0 \wedge q_1 \wedge q_3 \cdots \wedge q_n - \cdots$$

is a generator of $\wedge^n(Q)$. Now one can compute the Euler class of Q induced by $\alpha : Q \twoheadrightarrow I$ and χ to obtain $e(Q,\chi) = (I, \overline{F}\omega)$ in $E(A[T])$. (We point out that local orientations of I are equivalence classes of surjections from $(A[T]/I)^n$ to I/I^2 under obvious $SL_n(A[T]/I)$ action.) $\square$

Now we answer the question raised in the introduction. We give two proofs. We believe that both the proofs are of independent interest.

Theorem 3.5 *Let A be a commutative noetherian ring containing the field of rationals with* dim $A = n$ *(n even) and let P be a projective $A[T]$-module of rank n such that its determinant is free. Suppose there is a surjection $\alpha : P \twoheadrightarrow I$ where I is an ideal of $A[T]$ of height n which is generated by n elements. Assume further that P/TP has a unimodular element. Then P has a unimodular element.*

Proof 1 Fix a trivialization $\chi : A[T] \simeq \wedge^n P$. Then (α,χ) induces $e(P,\chi) = (I,\omega_I)$ in $E(A[T])$, where ω_I is a local orientation of I. Let $J = J(A,P)$ where $J(A,P)$ denotes the Quillen ideal of P in A. Since P has trivial determinant, it follows that ht $J \geq 2$ (see remark 2.9).

Suppose that ω_I is given by $I = (g_1,\cdots,g_n) + I^2$. Let $B = A_{1+J}$. Therefore, $IB[T] = (g_1,\cdots,g_n) + I^2B[T]$. We first want to show that P_{1+J} has a unimodular element. Since P_{1+J} has a unimodular element if and only if $P \otimes B(T)$ has a unimodular element ([B-RS 4], 3.4), it is enough to show that $\omega_I \otimes B(T)$ can be lifted to a set of generators of $IB(T)$ (Since then, $0 = (IB(T), \omega_I \otimes B(T)) = e(P \otimes B(T), \chi \otimes B(T))$ and applying ([B-RS 3], 4.4) we are done). In the following paragraph we do this.

Now I is generated by n elements, say, $I = (f_1,\cdots,f_n)$. By (3.1) there exists a matrix $\tau \in GL_n(B[T]/IB[T])$ such that $(\overline{f_1},\cdots,\overline{f_n}) = (\overline{g_1},\cdots,\overline{g_n})\tau$, where bar denotes reduction modulo $IB[T]$. Let $\det\tau = \overline{F}$ and let $G \in B[T]$ be such that $FG = 1$ modulo $IB[T]$. By (2.7) every unimodular row of length $(n+1)$ over $B[T]$ is extended from B and since height of the Jacobson radical of B is greater than one, it is easy to see that the unimodular row $(G, f_1,\cdots,f_n) \in Um_{n+1}(B[T])$ is completable. Therefore applying lemma (3.3) we can find a matrix $\sigma \in M_n(B[T])$ with $\det\sigma = G$ modulo $IB[T]$ such that if $(f_1,\cdots,f_n)\sigma = (h_1,\cdots,h_n)$ then $IB[T] = (h_1,\cdots,h_n)$. We have $(\overline{g_1},\cdots,\overline{g_n})\tau\overline{\sigma} = (\overline{h_1},\cdots,\overline{h_n})$. Note that $\tau\overline{\sigma} \in SL_n(B[T]/IB[T])$. Now we move to the ring $B(T)$. Since $\dim(B(T)/IB(T)) = 0$, we have $SL_n(B(T)/IB(T)) = E_n(B(T)/IB(T))$ and hence the canonical map from $SL_n(B(T))$ to $SL_n(B(T)/IB(T))$ is surjective. Therefore we can find a

$\Delta \in SL_n(B(T))$ which lifts $\tau\overline{\sigma}$ of $SL_n(B(T)/IB(T))$. Then $(h_1, \cdots, h_n)\Delta^{-1}$ is the desired set of generators of $IB(T)$.

Thus P_{1+J} has a unimodular element. Let us call it p_1. Let $p \in P/TP$ be a unimodular element. We claim that there is an elementary automorphism σ of P_{1+J} such that $\bar{\sigma}\bar{p_1} = \bar{p}$, where "bar" denotes reduction modulo T. To see this, let us consider the ring $C = B/J(B)$ where $J(B)$ denotes the Jacobson radical of B. Since $\dim C \leq n-2$ it follows that there is an elementary automorphism τ of $P_{1+J} \otimes C$ such that $\tau\overline{p_1} = p$ over C. Since elementary automorphisms can be lifted via a surjection of rings ([B-R], 4.1), we have, by repeated use of this argument, a $\sigma \in E(P_{1+J})$ such that $\bar{\sigma}\bar{p_1} = \bar{p}$. Let q denote the unimodular element σp_1 of P_{1+J}.

Since P_{1+J} has a unimodular element, we can find $s \in J$ such that P_{1+sA} has a unimodular element. We still call it q. Since P_s is extended from A_s, it has a unimodular element, namely p. Since p and q are equal modulo T, i.e. over $A_{s(1+sA)}$, it follows using a patching argument in ([P]) that P has a unimodular element. This completes the proof. □

Proof 2 Fix a trivialization $\chi : A[T] \simeq \wedge^n P$. Then (α, χ) induces $e(P, \chi) = (I, \omega_I)$ in $E(A[T])$, where ω_I is a local orientation of I (induced by α and χ). Now I is generated by n elements, say, $f_1 \cdots, f_n$. Therefore, applying (3.4) we see that there exists a stably free $A[T]$-module Q' of rank n, a generator χ_1 of $\wedge^n(Q')$ such that $e(Q', \chi_1) = (I, \omega_I)$ in $E(A[T])$. Since Q' is stably free of rank n and A contains $\mathbb{Q}$, by (2.7) $Q' = Q[T]$ for some stably free A-module Q. So we have $e(Q[T], \chi_1) = (I, \omega_I)$ in $E(A[T])$.

Therefore, in order to prove that P has a unimodular element it is enough to prove that $Q[T]$ has a unimodular element. In what follows we prove that the A-module Q has a unimodular element.

Let us take a generic surjection $\alpha : P/TP \twoheadrightarrow J$. Then, since P is projective, we can find an $A[T]$-linear map $\phi : P \to (J, T)$ which lifts α (here (J, T) is the ideal generated by J and T in $A[T]$). Let us write $\phi(P) = K$. Consider the element $(\phi, T) \in P^* \oplus A[T]$. Applying (2.5) we can find some $\psi \in P^*$ such that the ideal $(\phi + T\psi)(P) = L$ (say) in $A[T]$ satisfies the property that if $\mathcal{P} \in \text{Spec}\,(A[T])$, $\mathcal{P} \supset L$, $T \notin \mathcal{P}$ then height $\mathcal{P} \geq n$. On the other hand, if $\mathcal{P} \supset L$ and $T \in \mathcal{P}$ then $\mathcal{P} \supset L(0) = K(0)$ and hence has height $\geq n$. Consequently, height $L \geq n$. If $L = A[T]$ then it implies that P has a unimodular element and so we are done. If L is a proper ideal then, since L is locally generated by n elements, by Krull's theorem, height $L \leq n$.

Therefore, in this case, height $L = n$. Let us write $\Phi = (\phi + T\psi)$. Note that $L(0) = J$ and $\Phi(0) = \alpha$.

Now (Φ, χ) induces $e(P, \chi) = (L, \omega_L)$ in $E(A[T])$. Since Euler class of P is well defined, we have $(I, \omega_I) = (L, \omega_L)$ in $E(A[T])$. As a consequence, $e(Q[T], \chi_1) = (L, \omega_L)$.

On the other hand, $(\alpha, \chi \otimes A[T]/(T))$ induces $e(P/TP, \chi \otimes A[T]/(T)) = (J, \omega_J)$ in $E(A)$. It is clear that the equation $e(Q[T], \chi_1) = (L, \omega_L)$, when specialized at $T = 0$, becomes $e(Q, \chi_1 \otimes A[T]/(T)) = (J, \omega_J)$ in $E(A)$. Since P/TP has a unimodular element, we have $(J, \omega_J) = 0$ and hence it follows that Q has a unimodular element. This completes the proof.

Note that in the above proof we have to consider the ideal L as $I(0)$ may not necessarily have height n. $\square$

4 A "Quillen-Suslin theory" for the weak Euler class groups

In what follows A will denote a commutative noetherian ring containing the field of rationals with $\dim A \geq 2$.

The following lemma has been proved in ([B-K]).

Lemma 4.1 *Let A be a noetherian ring with $\dim A/J(A) = r$ where $J(A)$ denotes the Jacobson radical of A. Let $I \subset A[T]$ be an ideal containing a monic polynomial and $I = (f_1, \cdots, f_n) + L$ where $L \subset I^2$ is an ideal containing a monic polynomial and $n \geq r + 2$. Then $I = (g_1, \cdots, g_n)$ such that $f_i = g_i$ modulo L and g_1 is a monic polynomial.*

Using the above lemma and following the method of proof of ([B-Ra], 2.2) we get the following proposition. We give the proof for the sake of completeness.

Proposition 4.2 *Let A be a noetherian ring containing the field of rationals with $\dim A = n$. Suppose that the Jacobson radical of A contains an element which is not a zero divisor. Then any stably free $A(T)$-module of rank n is free (in other words, any unimodular row over $A(T)$ of length $n + 1$ is completable).*

Proof Let $(a_0, \cdots, a_n) \in Um_{n+1}(A(T))$ and let P be the associated stably free module.

Let $Y = T^{-1}$, $B = A[Y]_{1+YA[Y]}$. Then it is easy to see that $A(T) = B_Y$. Clearing denominators we can assume that $a_i \in B$ for $0 \leq i \leq n$. Let us denote the ideal in B generated by $a_0, \cdots, a_n$ by I.

Since $(a_0, \cdots, a_n) \in Um_{n+1}(B_Y)$, we have $Y^l \in I$ for some positive integer l. Let $J = I \cap A[Y]$. Then clearly $JB = I$. Note that $A[Y]/(Y^{2l})$ is isomorphic to $B/(Y^{2l})$ and hence the modules $J/(Y^{2l})$ and $I/(Y^{2l})$ are isomorphic. Therefore, clearing denominators we can assume that $J = (f_0, \cdots, f_n) + (Y^{2l})$. Now we can apply the above lemma and get $J = (g_0, \cdots, g_n)$ such that $g_i = f_i$ modulo (Y^{2l}). Note that we can choose g_0 to be monic.

Since Y belongs to the Jacobson radical of B, and the way the g_i's are obtained, it follows that there exists $\sigma \in GL_{n+1}(B)$ such that $(g_0, \cdots, g_n)\sigma = (a_0, \cdots, a_n)$.

Since $Y^l \in J = (g_0, \cdots, g_n)$, $(g_0, \cdots, g_n)$ is a unimodular row over $A[Y, Y^{-1}]$. Let Q' be the associated stably free $A[Y, Y^{-1}]$-module. Since $(g_0, \cdots, g_n)\sigma = (a_0, \cdots, a_n)$, we have $P \simeq Q' \otimes_{A[Y,Y^{-1}]} B_Y$. But Q'_{g_0} is free and g_0 is monic in Y. Hence by ([Sw], 1.3) there exists a projective $A[Y^{-1}]$-module Q such that $Q' \simeq Q \otimes_{A[Y^{-1}]} A[Y, Y^{-1}]$. Since $Q' \oplus A[Y, Y^{-1}] \simeq A[Y, Y^{-1}]^{n+1}$, we have

$$(Q \oplus A[Y^{-1}])_{Y^{-1}} \simeq A[Y, Y^{-1}]^{n+1}.$$

Hence, by ([Q], [S]), $Q \oplus A[Y^{-1}] \simeq A[Y^{-1}]^{n+1}$.

Since A contains $\mathbb{Q}$, $\dim A = n$ and rank of the stably free $A[Y^{-1}]$-module Q is n, it follows from (2.7) that Q is extended from A. Since the Jacobson radical of A has height at least one, it is easily deduced that Q is actually free. As a consequence, P is free. $\square$

It has been proved in ([D]) that if the Jacobson radical of A has height at least one, then, the canonical map from $E(A[T])$ to $E(A(T))$ is injective. It is natural to ask whether a similar result holds for the weak Euler class groups also. The answer is affirmative when $\dim A$ is even, as the following theorem shows.

Theorem 4.3 *Suppose* $\dim A = n$ *is even and that the Jacobson radical of* A

has height at least one. Then the canonical map from $E_0(A[T])$ to $E_0(A(T))$ is injective.

Proof Let I be an ideal of A of height n such that I/I^2 is generated by n elements and consider $(I) \in E_0(A[T])$. Suppose that $(IA(T)) = 0$ in $E_0(A(T))$. We want to prove that $(I) = 0$ in $E_0(A[T])$.

Since n is even, by ([B-RS 3], 6.2) there exists a stably free $A(T)$-module P of rank n such that P maps onto $IA(T)$. By the above proposition P is free.

So $IA(T)$ is generated by n elements. Now consider any local orientation ω_I of I. Since $IA(T)$ is generated by n elements, and n is even, it follows by ([B-RS 3], 5.1) that there is a stably free $A(T)$ module P_1 of rank n and a trivialization $\chi : A(T) \simeq \wedge^n(P_1)$ such that $e(P_1, \chi) = (IA(T), \omega_I \otimes A(T))$ in $E(A(T))$. By the above proposition it follows that P_1 is free. Therefore, $(IA(T), \omega_I \otimes A(T)) = 0$ in $E(A(T))$ and consequently by the injectivity of the canonical map from $E(A[T])$ to $E(A(T))$, we have $(I, \omega_I) = 0$ in $E(A[T])$. Hence $(I) = 0$ in $E_0(A[T])$. This completes the proof. □

It is easily seen that the canonical map from $E(A)$ to $E(A[T])$ is injective. In the the following proposition we consider the same question for the weak Euler class groups. In this proposition we do not assume that dim A is even.

Proposition 4.4 *The canonical map from $E_0(A)$ to $E_0(A[T])$ is injective.*

Proof Let J be an ideal of height n in A such that J/J^2 is generated by n elements and let $(J[T]) = 0$ in $E_0(A[T])$. Take a local orientation ω_J of J and consider $(J, \omega_J) \in E(A)$. Since $(J[T]) = 0$ in $E_0(A[T])$, the following equation holds in $E(A[T])$:

$$(J[T], \omega_{J[T]}) + \Sigma_{l=r+1}^{r+s}(I_l, \omega_l) = \Sigma_{t=1}^{r}(I_t, \omega_t)$$

where I_l, I_t are generated by n elements (see 3.3, [B-RS 2]. Note that the same proof works in this case also). Now we can assume by (2.4) that each of the ideals $I_i(0)$ in A is either of height n or is equal to A. Specializing the above equation at $T = 0$ we get an equation in $E(A)$ and it is easy to see that $(J) = 0$ in $E_0(A)$. □

Proposition 4.5 *Let H be the kernel of the canonical surjection $E(A) \twoheadrightarrow E_0(A)$ and K be the kernel of the canonical surjection $E(A[T]) \twoheadrightarrow E_0(A[T])$. If $\dim A = n$ is even, H is isomorphic to K.*

Proof Let $(I, \omega_I) \in K$. Since $(I) = 0$ in $E_0(A[T])$ and n is even, by (3.4) I is the image of a stably free $A[T]$-module P of rank n such that $e(P, \chi) = (I, \omega_I)$ in $E(A[T]$, where χ is a suitable generator of $\wedge^n P$. Since by (2.7) stably free $A[T]$-modules of rank n are extended from A, we have a stably free A-module Q such that $P = Q[T]$. We can assume that $I(0)$ is of height n or $I(0) = A$. Suppose that height $I(0) = n$. Clearly, $e(Q, \chi \otimes A[T]/(T)) = (I(0), \omega_{I(0)})$ in $E(A)$. Now since the Euler class of a projective module is well defined, we have $(I(0)[T], \omega_{I(0)[T]}) = (I, \omega_I)$ in $E(A[T])$. On the other hand, if $I(0) = A$, it follows that P has a unimodular element and hence $(I, \omega_I) = 0$. Therefore, the canonical map from H to K is surjective. Since it is always injective, result follows. $\square$

Remark 4.6 We do not know what happens when $\dim A$ is odd. However, if A is a smooth affine domain, we have the answer in affirmative as shown in the following proposition. We note that it has been proved in ([D]) that if A is a smooth affine domain, then $E(A)$ is canonically isomorphic to $E(A[T])$.

Proposition 4.7 *Let A be a smooth affine domain of dimension $n \geq 2$. Then $E_0(A)$ is isomorphic to $E_0(A[T])$.*

Proof It is enough to prove that H and K are isomorphic, where H and K are as in the above proposition. Since the canonical map from H to K is always injective, it remains to show that it is surjective.

Let us take $(I, \omega_I) \in K$. Since $(I, \omega_I) \in E(A[T])$ and since $E(A) \simeq E(A[T])$ (note that A is a smooth affine domain), there exists $(J, \omega_J) \in E(A)$ such that $(J[T], \omega_J \otimes A[T]) = (I, \omega_I)$ in $E(A[T])$. So it is enough to prove that $(J, \omega_J) \in H$. Now since $(I) = 0$ in $E_0(A[T])$, it follows that $(J[T]) = 0$ in $E_0(A[T])$. Since the map from $E_0(A)$ to $E_0(A[T])$ is injective, we have $(J) = 0$ in $E_0(A)$. In other words, $(J) \in H$. $\square$

We know that the canonical map from $E(A)$ to $E(A[T])$ is injective. We have seen that the same is true for the weak Euler class groups also.

Now suppose C be the cokernel of the map from $E(A)$ to $E(A[T])$ and C_0 be the cokernel of the map from $E_0(A)$ to $E_0(A[T])$. It is natural to ask whether C and C_0 are isomorphic i.e., if $E(A[T])/E(A)$ is isomorphic to $E_0(A[T])/E_0(A)$. As mentioned in the introduction, this is of interest because the groups $E_0(A[T])$ and $E_0(A)$ can be thought of as analogues of the groups $F^nK_0(A[T])$ and $F^nK_0(A)$. In the following proposition we give the answer in the affirmative when dim A is even.

Proposition 4.8 *Let* dim $A = n$ *be even. Then the cokernels* C *and* C_0*, as mentioned above, are isomorphic.*

Proof Since the canonical map from C to C_0 is always surjective we have only to show the injectivity. Let $(I, \omega_I) \in E(A[T])$ be such that the weak Euler class $(I) \in E_0(A[T])$ comes from $E_0(A)$. We prove that then, (I, ω_I) comes from $E(A)$.

Suppose $J \in A$ be an ideal of height n such that J/J^2 is generated by n elements and $(J[T]) = (I)$ in $E_0(A[T])$. Let $I \cap A = L$. Applying "moving lemma" ([B-RS 3], 2.14) we can find an ideal K such that K is comaximal with $J \cap L$ and $J \cap K$ is generated by n elements. It therefore follows that $(I) + (K[T]) = 0$ in $E_0(A[T])$. Let us fix a local orientation ω_K of K. Then by the Chinese remainder theorem we see that $\omega_K \otimes A[T]$ and ω_I together induce a local orientation of $I \cap K[T]$, say $\omega_{I\cap K[T]}$. Now since $(I \cap K[T]) = 0$ in $E_0(A[T])$ and n is even, by (3.4) we have a stably free $A[T]$ module P of rank n and a generator χ of $\wedge^n(P)$ such that $e(P, \chi) = (I \cap K[T], \omega_{I\cap K[T]})$ in $E(A[T])$. Since P is stably free of rank n by (2.7) it is extended, say $P = Q[T]$ for some A-module Q. Therefore we have the following equation in $E(A[T])$

$$e(Q[T], \chi) = (I, \omega_I) + (K[T], \omega_K \otimes A[T]).$$

Since both $e(Q[T], \chi)$ and $(K[T], \omega_K \otimes A[T])$ are extended from $E(A)$ (i.e. they are both zero in C), it follows that (I, ω_I) also comes from $E(A)$ (i.e. it is zero in C). This completes the proof.

□

In ([D]), a "local-global principle" for the Euler class groups was proved. Here we prove a similar result with the weak Euler class groups in the case where dimension of A is even.

Proposition 4.9 *Let* dim $A = n$ *be even. Then the following sequence of groups is exact*

$$0 \longrightarrow E_0(A) \longrightarrow E_0(A[T]) \longrightarrow \Pi_m E_0(A_m[T]),$$

where the direct product runs over all maximal ideals of A *of height* n.

Proof It is obvious that the sequence is a complex. Injectivity of the first map has been proved in (4.4) (even if the dimension of A is odd). Now suppose $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Suppose that $(I) \in E_0(A[T])$ is trivial in $E_0(A_m[T])$ for all maximal ideals m of A of height n. We prove that then, (I) comes from $E_0(A)$.

Let us choose any local orientation ω_I of I. Consider $(I, \omega_I) \in E(A[T])$. Let $m \subset A$ be a maximal ideal of height n. Since $(I) = 0$ in $E_0(A_m[T])$, it follows from (2.11) that there is a stably free $A_m[T]$-module P of rank n and a generator χ of $\wedge^n(P)$ such that $e(P, \chi) = (I, \omega_I)$ in $E(A_m[T])$. Since P is stably free of rank n, by (2.7) it is free and hence it follows that $(I, \omega_I) = 0$ in $E(A_m[T])$.

Therefore, we have, $(I, \omega_I) = 0$ in $E(A_m[T])$ for all maximal ideals m of A of height n. Now applying the local-global principle for the Euler class groups ([D], 5.4), we have some $(J, \omega_J) \in E(A)$ such that $(J[T], \omega_J \otimes A[T]) = (I, \omega_I)$ in $E(A[T])$. Consequently, $(J[T]) = (I)$ in $E_0(A[T])$. Hence the proof of the proposition is complete. □

Remark 4.10 Some of the proofs in this paper can be shortened using the local-global principle for the Euler class groups ([D]). However, we preferred the approach followed here as it is more elementary.

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