Good invariants for bad ideals

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1 Introduction

In [14, Section 5], Murthy defined the "Segre class" of a finitely generated module M over a smooth affine domain A over an algebraically closed field k. The Segre class of M, denoted $s_0(M)$, takes values in the Chow group $CH_0(A)$. Murthy proves that $s_0(M)$ is the precise obstruction for M to be efficiently generated. In other words, if this class is zero, then a certain Eisenbud-Evans estimate gives a bound for the number of generators of M. Further, Murthy studied the case when $M = I$ is an ideal of A and proved the following interesting results. The treatment of Segre class of an ideal is slightly different from the module case.

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Theorem 1.1 *Let A be a smooth affine domain of dimension $n \geq 2$ over an algebraically closed field k. Let $I \subset A$ be an ideal such that I/I^2 is generated by n elements. Then, we have,*

1. *I is generated by n elements if and only if $s_0(I) = 0$ in $CH_0(A)$.*

2. *Suppose J is another ideal as I such that $I + J = A$. Then, we have the following relation : $s_0(I \cap J) = s_0(I) + s_0(J)$.*

3. *Let I, J be as above. Suppose any two of the ideals I, J, $I \cap J$ are generated by n elements; then so is the third (follows from (1) and (2)).*

Later, the notion of Segre class reappeared in a paper of Mandal-Murthy [11]. Motivated by the above results of Murthy and taking a cue from Mandal-Murthy [11, Lemma 2.5, Theorem 2.6], we study in this paper the Segre classes of ideals in a more general set up. Let A be a commutative Noetherian ring of dimension $n \geq 2$. For an ideal J of A with $\text{ht}\,(J) \geq 2$ and a surjection $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$, we define the Segre class $s(J, \omega_J)$ which takes values in the Euler class group $E(A)$ of A. We prove that $s(J, \omega_J)$ is the precise obstruction for ω_J to be lifted to a surjection $\theta : A^n \twoheadrightarrow J$. We also prove the "additivity" of Segre classes as in Theorem 1.1(2).

A few words about the title of this paper are in order. Let A be a commutative Noetherian ring of dimension n. For us an ideal $J \subset A$ is "good" if $\text{ht}\,J = n = \mu(J/J^2)$ (for a module M, $\mu(M)$ denotes the minimal number of generators of M). In this paper we consider two types of "bad" ideals : (1) ideals J for which $\text{ht}\,J < \mu(J/J^2) = n$; (2) ideals J for which $\text{ht}\,J = n$ but $\mu(J/J^2)$ is not necessarily n. The notion of Segre class is defined to handle bad ideals of first type. This is the content of Section 3. In Section 5 we introduce the notion of *Northcott-Rees class* of a bad ideal J of type two where A is a smooth affine domain over a field k and prove that this class, $NR(J) = 0$ implies that J is a set-theoretic complete intersection in some interesting cases. The *Northcott-Rees class* of

J is actually the Chern class of any of its minimal reductions and takes values in the Chow group of zero cycles of A.

In Section 2 we prove the so called "addition" and "subtraction" principles in a more general set up than the available ones and collect other results which are crucial to later sections. Section 4 is about the Segre classes of ideals in a polynomial algebra. The paper ends with a separate section on historical motivation for the introduction of the notion of Segre classes.

2 Preliminaries

In this section first we prove the "addition" and "subtraction" principles in a slight more generality to fit our needs. Next we prove a "moving lemma" (Lemma 2.7) which plays an important role in defining Segre classes in the next section.

Proposition 2.1 (Addition Principle) *Let A be a Noetherian ring of dimension $n \geq 2$ and I, J be two comaximal ideals of A, each of height ≥ 2. Assume further that $I = (a_1, \cdots, a_n)$ and $J = (b_1, \cdots, b_n)$. Then, $I \cap J = (c_1, \cdots, c_n)$ such that $c_i = a_i$ mod I^2 and $c_i = b_i$ mod J^2.*

Proof The case when $n = 2$ has been proved in [4, Theorem 3.2]. Therefore we assume $n \geq 3$.

Note that we can always perform elementary transformations on the row $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ and no generality is lost doing so. To see this, let us assume that $(a_1, \cdots, a_n)$ is elementarily transformed to $(\widetilde{a_1}, \cdots, \widetilde{a_n})$ and $(b_1, \cdots, b_n)$ is elementarily transformed to $(\widetilde{b_1}, \cdots, \widetilde{b_n})$. Suppose we can find a set of generators $\widetilde{c_1}, \cdots, \widetilde{c_n}$ of $I \cap J$ satisfying $\widetilde{c_i} = \widetilde{a_i}$ mod I^2 and $\widetilde{c_i} = \widetilde{b_i}$ mod J^2. Then we can use the surjectivity of the canonical map $E_n(A/I \cap J) \longrightarrow E_n(A/I) \times E_n(A/J)$ to transform $(\widetilde{c_1}, \cdots, \widetilde{c_n})$ to $(c_1, \cdots, c_n)$, so that $I \cap J = (c_1, \cdots, c_n)$ with $c_i = a_i$ mod I^2 and $c_i = b_i$ mod J^2.

Let $B = A/(b_1, \cdots, b_n)$ and bar denote reduction modulo the ideal $(b_1, \cdots, b_n)$. Since $I + J = A$, $(\overline{a_1}, \cdots, \overline{a_n}) \in Um_n(B)$. Since $\dim B \leq n-2$, we can elementarily transform $(\overline{a_1}, \cdots, \overline{a_n})$ to $(\overline{1}, \cdots, \overline{0})$. Applying [16, Lemma 2] we can apply an elementary transformation and assume that $\text{ht}\,(a_1, \cdots, a_{n-1}) \geq 2$. Note that this transformation preserves the fact that $a_1 = 1$ modulo J. Therefore, $(a_1, \cdots, a_{n-1}) + J = A$.

Now let $C = A/(a_1, \cdots, a_{n-1})$ and bar denote reduction modulo the ideal $(a_1, \cdots, a_{n-1})$. Consider the unimodular row $(\overline{b_1}, \cdots, \overline{b_n}) \in Um_n(C)$. Using similar arguments as in the above paragraph we finally obtain :

1. $(a_1, \cdots, a_{n-1}) + (b_1, \cdots, b_{n-1}) = A$.

2. $\text{ht}\,(a_1, \cdots, a_{n-1}) \geq 2$ and $\text{ht}\,(b_1, \cdots, b_{n-1}) \geq 2$.

In $A[T]$ we consider the ideals

$$I_1 = (a_1, \cdots, a_{n-1}, T + a_n),\ I_2 = (b_1, \cdots, b_{n-1}, T + b_n)$$

and let $K = I_1 \cap I_2$. Note that $I_1 + I_2 = A[T]$. Therefore, using the Chinese remainder theorem we can choose $g_1(T), \cdots, g_n(T) \in K$ such that

$$K = (g_1(T), \cdots, g_n(T)) + K^2$$

satisfying $g_i(T) = a_i \bmod I_1^2$, $g_i(T) = b_i \bmod I_2^2$, $1 \leq i \leq n-1$; $g_n(T) = T + a_n \bmod I_1^2$, $g_n(T) = T + b_n \bmod I_2^2$.

Now $\text{ht}\,(a_1, \cdots, a_{n-1}) \geq 2$, $\text{ht}\,(b_1, \cdots, b_{n-1}) \geq 2$. Also note that

- $\dim A[T]/I_1 = \dim A/(a_1, \cdots, a_{n-1}) \leq n-2$, and
- $\dim A[T]/I_2 = \dim A/(b_1, \cdots, b_{n-1}) \leq n-2$.

It follows that $\dim A[T]/K \leq n-2$. Therefore, the conditions of [10, Theorem 1.2] are satisfied for K. Applying [10, Theorem 1.2], we obtain $K = (h_1(T), \cdots, h_n(T))$ such that $h_i(T) = g_i(T) \bmod K^2$. Let $h_i(0) = c_i$. Then $I \cap J = (c_1, \cdots, c_n)$ with $c_i = a_i \bmod I^2$ and $c_i = b_i \bmod J^2$. $\square$

Proposition 2.2 (Subtraction Principle) *Let A be a Noetherian ring of dimension $n \geq 2$ and I, J be two comaximal ideals of A, each of height ≥ 2. Assume further that $I = (a_1, \cdots, a_n)$ and $I \cap J = (c_1, \cdots, c_n)$ such that $c_i = a_i$ mod I^2. Then $J = (b_1, \cdots, b_n)$ such that $c_i = b_i$ mod J^2.*

Proof The case when $n = 2$ has been proved in [4, Theorem 3.3]. Therefore we assume $n \geq 3$.

First note that we can perform elementary transformations on the row $(a_1, \cdots, a_n)$ because we can apply the same elementary transformations on $(c_1, \cdots, c_n)$ to retain the relation that $c_i = a_i$ mod I^2. Let $B = A/J^2$ and bar denote reduction modulo J^2. Since ht $(J) \geq 2$, dim $B \leq n - 2$. Therefore, performing elementary transformations as in the proof of the above proposition we may assume that: (1) ht $(a_1, \cdots, a_{n-1}) \geq 2$, (2) $a_n = 1$ mod J^2.

Consider the following ideals in $A[T]$:

$$I_1 = (a_1, \cdots, a_{n-1}, T + a_n),\ I_2 = JA[T],\ K = I_1 \cap I_2.$$

Applying [12, Theorem 2.3] we obtain that $K = (h_1(T), \cdots, h_n(T))$ such that $h_i(0) = c_i$. Let $b_i = h_i(1 - a_n)$. Then $J = (b_1, \cdots, b_n)$. Since $a_n = 1$ mod J^2, $b_i - c_i = h_i(1 - a_n) - h_i(0) = 0$ mod J^2. This proves the proposition. $\square$

Next we proceed to prove Lemma 2.7.

The following is a consequence of Prime Avoidance Lemma. The proof is standard (see [8]) and hence omitted.

Lemma 2.3 *Let R be a ring and $\mathcal{P}_1, \cdots, \mathcal{P}_r$ be a set of prime ideals of R. Let $I = (a_1, \cdots, a_n) \subset R$ be an ideal such that $I \not\subseteq \mathcal{P}_i$, $i = 1, \cdots, r$. Then there exist $\lambda_2, \cdots, \lambda_n \in R$ such that $c = a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n \notin \cup_{i=1}^r \mathcal{P}_i$.*

The following lemma can be easily deduced from the above using general position arguments. For a proof see [8, Lemma 7.1.4]

Lemma 2.4 *Let R be a ring and $a_1, \cdots, a_n, s \in R$. Then there are elements $\lambda_i \in R$, $1 \leq i \leq n$ such that ht $(a_1 + s\lambda_1, \cdots, a_i + s\lambda_i)R_s \geq i$ for $1 \leq i \leq n$ in the ring R_s.*

Remark 2.5 If R is a geometrically reduced affine algebra over an infinite field then Swan's version of Bertini theorem, as given in [3, Theorem 2.11], states that $\lambda_1, \cdots, \lambda_n$ can be so chosen that the ideal $I_i = (a_1+s\lambda_1, \cdots, a_i+s\lambda_i)$ has the additional property that $(R/I_i)_s$ is a geometrically reduced ring for $1 \leq i \leq n$.

Lemma 2.6 *Let R be a Noetherian ring and $J \subset R$ be an ideal of R. Let $K \subset J$ and $L \subset J^2$ be two ideals of R such that $K + L = J$. Then $J = K + (e)$ for some $e \in L$ and $K = J \cap J'$ where $J' + L = R$.*

Proof Consider the Noetherian ring $\overline{R} = R/K$. Then in $\overline{R}$, we have

$$\overline{J}^2 = (J/K)^2 = (J^2 + K)/K \supseteq (L + K)/K = \overline{L} = J/K = \overline{J}.$$

Therefore, $\overline{J}$ is an idempotent ideal. Applying Nakayama lemma and the fact that $\overline{L} = \overline{J}$, it follows that there is an element $e \in L$ such that $J = K + (e)$ and $e(1-e) \in K$. Now we take $J' = K + (1-e)$. Then clearly $J' + L = R$. It is easy to check that $J' \cap J = K$. $\square$

We will refer to the following lemma as the "moving lemma". The proof of this lemma is implicit in [4, Corollary 2.14]. But the version we need in this paper is much simpler and we give a proof for the convenience of the reader.

Lemma 2.7 *Let A be a Noetherian ring of dimension $n \geq 2$. Let J be an ideal of A of height ≥ 1 such that $J = (a_1, \cdots, a_n) + J^2$. Let K be any ideal of A of height ≥ 1. Then there exists an ideal $J' \subset A$ such that :*

1. *J' is comaximal with $J \cap K$ and ht $J' \geq n$;*
2. *$J \cap J' = (c_1, \cdots, c_n)$ where $c_i \equiv a_i$ mod J^2.*

Proof Let $a \in K \cap J^2$ such that $\text{ht}\,(a) \geq 1$. Let bar denote reduction modulo a. Since $a \in J^2$, we have

$$\overline{J} = (\overline{a}_1, \cdots, \overline{a}_n) + \overline{J}^2.$$

We first show that $\overline{a}_1, \cdots, \overline{a}_n$ can be lifted to a set of n generators of $\overline{J}$. By Lemma 2.6, there is an element $e \in J^2$ such that $\overline{J} = (\overline{a}_1, \cdots, \overline{a}_n, \overline{e})$. Applying Lemma 2.4 we can choose elements $\lambda_1, \cdots, \lambda_n \in A$ such that the ideal $N = (\overline{a}_1 + \overline{\lambda}_1\overline{e}, \cdots, \overline{a}_n + \overline{\lambda}_n\overline{e})$ of $\overline{A}$ has the property that $\text{ht}\,(N_{\overline{e}}) \geq n$. Since $\dim\,(\overline{A}) \leq n-1$, it follows that N contains some positive power of $\overline{e}$. Combining this fact with the fact that $N + (\overline{e}) = \overline{J}$ implies that $N = \overline{J}$, as they are same locally. Note that $\overline{\lambda}_i\overline{e} \in \overline{J}^2$, $1 \leq i \leq n$.

Coming back to the ring A, we have $J = (b_1, \cdots, b_n, a)$ where $b_i = a_i + \lambda_i e$. Again applying Lemma 2.4 we see that there are elements $\gamma_1, \cdots, \gamma_n \in A$ such that the ideal $I = (b_1 + \gamma_1 a, \cdots, b_n + \gamma_n a)$ has the property that $\text{ht}\,(I_a) \geq n$. Note that $I + (a) = J$ and $(a) \subset J^2$. Applying Lemma 2.6 we see that there is an ideal J' such that

$$(b_1 + \gamma_1 a, \cdots, b_n + \gamma_n a) = J \cap J'$$

where $J' + (a) = A$. Now it is easy to deduce that $\text{ht}\,(J') \geq n$. $\square$

Remark 2.8 If A is a geometrically reduced affine algebra over an infinite field then using Swan's Bertini theorem (see Remark 2.5 above), one can choose J' to have the additional property that either $J' = A$ or J' is finite intersection of maximal ideals.

Before going to the next section we quickly sketch the definition of the Euler class group $E(A)$ where A is a Noetherian ring of dimension $n \geq 2$. For a detailed account we refer to [4].

Let G be the free abelian group on all pairs $(\mathcal{N}, \omega_{\mathcal{N}})$ where $\mathcal{N}$ is an $\mathcal{M}$-primary ideal of height n and $\omega_{\mathcal{N}} : (A/\mathcal{N})^n \twoheadrightarrow \mathcal{N}/\mathcal{N}^2$ is a surjection. Let I be any ideal of A of height n and $\omega_I : (A/I)^n \twoheadrightarrow I/I^2$ be a surjection (we call ω_I a *local orientation* of I). We take its irredundant primary decomposition $I = \mathcal{N}_1 \cap \cdots \cap \mathcal{N}_r$ and observe that ω_I induces local orientations $\omega_{\mathcal{N}_i}$ of $\mathcal{N}_i$, $1 \leq i \leq r$. We denote the element $\sum_{i=1}^{r} (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ of G by (I, ω_I).

Let H be the subgroup of G generated by all those (I, ω_I) for which ω_I can be lifted to a surjection $\theta : A^n \twoheadrightarrow I$ (such an ω_I is called a *global orientation* of I). The Euler class group $E(A)$ is defined as $E(A) = G/H$.

We will use the following result in the next section.

Theorem 2.9 *[4, Theorem 4.2] Let A be a Noetherian ring of dimension $n \geq 2$. Let I be an ideal of height n and $\omega_I : (A/I)^n \twoheadrightarrow I/I^2$ be a local orientation of I. If $(I, \omega_I) = 0$ in $E(A)$ then ω_I is a global orientation, i.e., ω_I can be lifted to a surjection $\theta : A^n \twoheadrightarrow I$.*

3 Segre classes

As a starting point, we first recall the definition of the Segre class from Mandal-Murthy [11].

3.1 Mandal-Murthy definition

Let A be a regular affine domain of dimension n over an algebraically closed field and $J \subset A$ be an ideal such that J/J^2 is generated by n elements. The *Segre class* of J is defined as follows (see [11, Lemma 2.5]).

Choose elements $a_1, \cdots, a_n \in J$ such that $J = (a_1, \cdots, a_n) + J^2$ and $(a_1, \cdots, a_n) = J \cap J_1$ where J_1 is an ideal of A which is the intersection of finitely many maximal ideals of A. Define the Segre class of J by

$$s(J) = -(J_1) \in CH_0(\text{Spec}\, A).$$

3.2 Our definition

Now let A be a commutative Noetherian ring of dimension $n \geq 2$. Let $J \subset A$ be an ideal of height ≥ 2 such that J/J^2 is generated by n elements. We proceed to define the Segre class of (J, ω_J), where $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ is a surjection.

The surjection ω_J induces $J = (a_1, \cdots, a_n) + J^2$. Applying the moving lemma (Lemma 2.7) we can find $c_1, \cdots, c_n \in J$ such that $(c_1, \cdots, c_n) =$

$J \cap J_1$ where $\operatorname{ht} J_1 \geq n$, $J_1 + J = A$ and $c_i = a_i$ modulo J^2. If J_1 is a proper ideal then $J_1 = (c_1, \cdots, c_n) + J_1^2$ and it induces a local orientation $\omega_{J_1} : (A/J_1)^n \twoheadrightarrow J_1/J_1^2$. We define the Segre class of (J, ω_J) by

$$s(J, \omega_J) = -(J_1, \omega_{J_1}) \in E(A),$$

where $E(A)$ is the Euler class group of A. If $J_1 = A$ then $J = (c_1, \cdots, c_n)$ and we define the Segre class $s(J, \omega_J) = 0 \in E(A)$.

Remark 3.1 If A is a geometrically reduced affine algebra of dimension n over an infinite field k of characteristic zero (not necessarily algebraically closed), then in the above definition we can choose J_1 with the additional property that J_1 is a finite intersection of maximal ideals. See Remark 2.8 for a discussion. Such a choice is indeed advantageous, for instance, as demonstrated in Proposition 6.2.

We need to show that our definition of Segre class of (J, ω_J) does not depend on the choice of J_1. We do this in the following proposition.

Proposition 3.2 *The Segre class of (J, ω_J), as described above, is well defined.*

Proof To show that our definition of Segre class of (J, ω_J) does not depend on the choice of J_1, let J_2 be an ideal of A of height $\geq n$ such that

1. $J + J_2 = A$;
2. $(d_1, \cdots, d_n) = J \cap J_2$, where $d_i = a_i$ modulo J^2.

If $J_2 = A$ then it is easy to check using addition and subtraction principles that $(J_1, \omega_{J_1}) = 0$ in $E(A)$. Therefore assume that J_2 is a proper ideal. In fact, in course of the proof we will assume all the ideals to be proper.

Let $\omega_{J_2} : (A/J_2)^n \twoheadrightarrow J_2/J_2^2$ be the local orientation induced by $d_1, \cdots, d_n$. We have to show that $(J_1, \omega_{J_1}) = (J_2, \omega_{J_2})$ in $E(A)$. In what follows, we prove this.

Using Lemma 2.7 we can find an ideal J_3 of A of height n and a local orientation ω_{J_3} such that : (i) J_3 is comaximal with each of J, J_1 and J_2, (ii) $(J_1, \omega_{J_1}) + (J_3, \omega_{J_3}) = 0$ in $E(A)$.

Now it is enough to prove that $(J_2, \omega_{J_2}) + (J_3, \omega_{J_3}) = 0$ in $E(A)$.

Again applying Lemma 2.7 we can find an ideal J_4 of A of height n such that $J \cap J_4$ is generated by n elements and J_4 is comaximal with each of J, J_1, J_2 and J_3.

Now the ideals $J_1 \cap J_3$ and $J \cap J_4$ are both generated by n elements and they are comaximal. Applying the addition principle (Proposition 2.1), the ideal $J_1 \cap J_3 \cap J \cap J_4$ is generated by n elements. Since $J_1 \cap J$ is generated by n elements, by the subtraction principle (Proposition 2.2) it follows that $J_3 \cap J_4$ is generated by n elements with appropriate set of generators.

Now we look at $J_2 \cap J_3 \cap J \cap J_4$. Since $J \cap J_2$ and $J_3 \cap J_4$ are both generated by n elements and they are comaximal, by the addition principle $J_2 \cap J_3 \cap J \cap J_4$ is generated by n elements with appropriate set of generators. Again since $J \cap J_4$ is n-generated, it follows using the subtraction principle that $J_2 \cap J_3$ is n-generated by the appropriate set of generators. Keeping track of the generators, it is easy to see that this implies $(J_2, \omega_{J_2}) + (J_3, \omega_{J_3}) = 0$ in $E(A)$.

Therefore, $s(J, \omega_J)$ is well defined. $\square$

The following theorem shows that the Segre class $s(J, \omega_J)$ is the precise obstruction for ω_J to be lifted to a surjection $\theta : A^n \twoheadrightarrow J$. Obviously if ω_J is induced by a set of generators of J then $s(J, \omega_J) = 0$ in $E(A)$. This can be seen from the definition of $s(J, \omega_J)$ and the addition principle. What we prove below is the converse.

Theorem 3.3 *Let A be a commutative Noetherian ring of dimension $n \geq 2$. Let $J \subset A$ be an ideal of height ≥ 2 and $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ be a surjection. Suppose that $s(J, \omega_J) = 0$ in $E(A)$. Then ω_J can be lifted to a surjection $\theta : A^n \twoheadrightarrow J$.*

Proof Suppose ω_J is given by $J = (a_1, \cdots, a_n) + J^2$.

As before applying Lemma 2.7 we can find $c_1, \cdots, c_n \in J$ such that $(c_1, \cdots, c_n) = J \cap J_1$ where $\operatorname{ht} J_1 = n$ and $c_i = a_i$ modulo J^2. Then $J_1 = (c_1, \cdots, c_n) + J_1^2$ and we obtain an induced local orientation $\omega_{J_1} : (A/J_1)^n \twoheadrightarrow J_1/J_1^2$. We defined the Segre class of (J, ω_J) by

$$s(J, \omega_J) = -(J_1, \omega_{J_1}) \in E(A),$$

where $E(A)$ is the Euler class group of A.

Now $s(J, \omega_J) = 0$ implies $(J_1, \omega_{J_1}) = 0$ in $E(A)$. Therefore, by [4, Theorem 4.2], ω_{J_1} is a global orientation of J_1. This means that there exist $d_1, \cdots, d_n \in J_1$ such that $J_1 = (d_1, \cdots, d_n)$ where $d_i = c_i$ modulo J_1^2. Now we can apply the subtraction principle to see that $J = (b_1, \cdots, b_n)$ with $b_i = a_i$ modulo J^2. This means ω_J has the desired lift. This proves the theorem. $\square$

The following theorem is on additivity of the Segre classes.

Theorem 3.4 *Let A be as above and J_1, J_2 be two comaximal ideals of A, each of height ≥ 2. Suppose that we have surjections $\omega_{J_1} : (A/J_1)^n \twoheadrightarrow J_1/J_1^2$ and $\omega_{J_2} : (A/J_2)^n \twoheadrightarrow J_2/J_2^2$. Then,*

$$s(J_1 \cap J_2, \omega_{J_1 \cap J_2}) = s(J_1, \omega_{J_1}) + s(J_2, \omega_{J_2})$$

in $E(A)$, where $\omega_{J_1 \cap J_2} : (A/J_1 \cap J_2)^n \twoheadrightarrow J_1 \cap J_2/(J_1 \cap J_2)^2$ is the surjection induced by ω_{J_1} and ω_{J_2}.

Proof Suppose ω_{J_1} is given by $J_1 = (a_1, \cdots, a_n) + J_1^2$ and ω_{J_2} is given by $J_2 = (b_1, \cdots, b_n) + J_2^2$. Proceeding as in the definition of the Segre class we can find an ideal I_1 of A of height n and a local orientation ω_{I_1} so that $s(J_1, \omega_{J_1}) = -(I_1, \omega_{I_1})$. Similarly, we can choose an ideal I_2 of A of height n and a local orientation ω_{I_2} so that $s(J_2, \omega_{J_2}) = -(I_2, \omega_{I_2})$. Note that we can choose I_1 to be comaximal with J_1 and J_2 and once I_1 is chosen, we can take I_2 to be comaximal with J_1, J_2 and I_1.

Now since I_1 and I_2 are comaximal, we have

$$(I_1, \omega_{I_1}) + (I_2, \omega_{I_2}) = (I_1 \cap I_2, \omega_{I_1 \cap I_2})$$

in $E(A)$, where $\omega_{I_1 \cap I_2}$ is the local orientation of $I_1 \cap I_2$ induced by ω_{I_1} and ω_{I_2}. Now $I_1 \cap I_2$ is comaximal with $J_1 \cap J_2$. Keeping track of the generators it is easy to see that $s(J_1 \cap J_2, \omega_{J_1 \cap J_2}) = -(I_1 \cap I_2, \omega_{I_1 \cap I_2})$ in $E(A)$. Therefore, $s(J_1 \cap J_2, \omega_{J_1 \cap J_2}) = s(J_1, \omega_{J_1}) + s(J_2, \omega_{J_2})$, as desired. $\square$

The following theorem shows the equivalence of our definition of Segre class with the definition of Mandal-Murthy when the ring in question is a regular affine domain over an algebraically closed field.

Theorem 3.5 *Let A be a regular affine domain over an algebraically closed field with dim $A = n$. Let J be an ideal of A of height ≥ 2 such that J/J^2 is generated by n elements and $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ be a surjection. Then, $s(J) = 0$ if and only if $s(J, \omega_J) = 0$.*

Proof First assume that $s(J, \omega_J) = 0$. Then, by Theorem 3.3 it follows that J is generated by n elements and therefore by [14, Corollary 5.5], $s(J) = 0$.

Now suppose that $s(J) = 0$. We want to prove that $s(J, \omega_J) = 0$. Suppose, as in the definition of the Segre class, $s(J, \omega_J) = -(J_1, \omega_{J_1})$, where J_1 is an ideal of height n with $J_1 + J = A$. Further, using Lemma 2.7 we can again find some ideal J_2 of height n and a local orientation ω_{J_2} such that $J_1 + J_2 = A$ and $(J_1, \omega_{J_1}) + (J_2, \omega_{J_2}) = 0$ in $E(A)$. Therefore, $s(J, \omega_J) = (J_2, \omega_{J_2})$. On the other hand, since J_1 and J_2 are local complete intersection ideals of height n, $J_1 + J_2 = A$ and their intersection is generated by n elements, it follows that $(J_1) + (J_2) = 0$ in $CH_0(A)$. Consequently, $s(J) = (J_2)$. Now $s(J) = 0$ implies that $(J_2) = 0$ in $CH_0(A)$. Applying [14, Corollary 3.4] we see that J_2 is a complete intersection. Again since A is a regular affine domain over an algebraically closed field, this would imply that $(J_2, \omega_{J_2}) = 0$ in $E(A)$. Consequently, $s(J, \omega_J) = 0$. $\square$

4 The case of polynomial algebras

Let A be a Noetherian ring (containing $\mathbb{Q}$) of dimension $n \geq 3$. Let $I \subset A[T]$ be an ideal of height ≥ 3 such that $\mu(I/I^2) = n$. For such ideals we would like to define the Segre class and derive analogous results in this section. The Segre class of I takes values in the Euler class group $E(A[T])$. For definition and results on $E(A[T])$, we refer to [5]. We remark here that for the definition of $E(A[T])$ we need to make the assumption that $\mathbb{Q} \subset A$.

First we recall a result from [5] which played a crucial role in defining the Euler class group of $A[T]$. Here $A(T)$ denotes the ring obtained from $A[T]$ by inverting all the monic polynomials.

Theorem 4.1 *[5, Theorem 3.10] Let A be a Noetherian ring (containing $\mathbb{Q}$) of dimension $n \geq 3$ and $I \subset A[T]$ be an ideal of height n such that $I = (f_1, \cdots, f_n) + (I^2T)$. Suppose that there exists $G_1, \cdots, G_n \in IA(T)$ such that $IA(T) = (G_1, \cdots, G_n)$ with $G_i - f_i \in I^2A(T)$. Then there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $g_i - f_i \in (I^2T)$.*

The following is an improvement of the above theorem where we relax the condition on the height of the ideal. One can actually try to mimic the proof of 4.1 from [5], modify appropriately, and give a straightforward proof. Here we will rather obtain it as an application of the above theorem.

Theorem 4.2 *Let A be a Noetherian ring (containing $\mathbb{Q}$) of dimension $n \geq 3$ and $I \subset A[T]$ be an ideal of height ≥ 3 such that $I = (f_1, \cdots, f_n) + (I^2T)$. Suppose that there exists $G_1, \cdots, G_n \in IA(T)$ such that $IA(T) = (G_1, \cdots, G_n)$ with $G_i - f_i \in I^2A(T)$. Then there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $g_i - f_i \in (I^2T)$.*

Proof Let $J = I \cap A$. Then $\text{ht}(J) \geq 2$. We can apply [5, Lemma 3.9] and obtain $h_1, \cdots, h_n \in I$ such that

1. $I = (h_1, \cdots, h_n) + (J^2T)$ with $h_i - f_i \in (I^2T)$.
2. $(h_1, \cdots, h_n) = I \cap I'$, where $\text{ht}(I') \geq n$.
3. $I' + (J^2T) = A[T]$.

If $I' = A[T]$, there is nothing to prove. So we assume that $\text{ht}(I') = n$. Observe that $I' = (h_1, \cdots, h_n) + I'^2$ and $I'(0) = A$. It is easy to see that we can lift $h_1, \cdots, h_n$ to a set of generators of $I'/(I'^2T)$. Let us do so and retain the same notation for the generators. Now consider the ideals $IA(T)$ and $I'A(T)$ in $A(T)$. We have

1. $IA(T) = (G_1, \cdots, G_n)$.
2. $IA(T) \cap I'A(T) = (h_1, \cdots, h_n)$ with $h_i - G_i \in I^2A(T)$.
3. $IA(T)$ and $I'A(T)$ are comaximal.

Applying Proposition 2.2 we conclude that there exist $H_1, \cdots, H_n \in I'A(T)$ such that $I'A(T) = (H_1, \cdots, H_n)$, where $H_i - h_i \in I'^2A(T)$. Recall that we have $I' = (h_1, \cdots, h_n) + (I'^2T)$. Now we can apply Theorem 4.1 and see that there exist $k_1, \cdots, k_n \in I'$ such that $I' = (k_1, \cdots, k_n)$ with the relations $k_i - h_i \in (I'^2T)$.

By [5, Lemma 3.8] (take the "free" case and note that nowhere in that lemma height of J is the issue), it is enough to prove that $IA_{1+J} = (u_1, \cdots, u_n)$ with $u_i - f_i \in (I^2T)_{1+J}$. The rest of the proof is devoted to proving this. This is a particular type of subtraction and the method is same as Steps 3 and 4 of the proof of [5, Theorem 3.10]. We will keep in mind that elementary transformations on $(k_1, \cdots, k_n)$ are always admissible.

Write $B = A_{1+J}$. Note that the row $(k_1, \cdots, k_n)$ is unimodular in $B[T]/(J^2T)B[T]$. Write $D = B[T]/(J^2T)B[T]$. Consider $(k_1, \cdots, k_n)$ as a unimodular row over D/JD. Note that $D/JD \simeq (A/J)[T]$. As $\dim(A/J) \leq n - 2$ and the length of the row is n, the row $(k_1, \cdots, k_n)$ is elementarily completable over D/JD. Since JD is contained in the Jacobson radical of D, we can lift the elementary transformation to conclude that $(k_1, \cdots, k_n)$ can be transformed elementarily to $(1, 0, \cdots, 0)$ over D. Since elementary transformations can be lifted via a surjection of rings, we can lift this elementary transformation to $E_n(B[T])$ and applying it on $(k_1, \cdots, k_n)$ we may assume that $(k_1, \cdots, k_n) = (1, 0, \cdots, 0)$ modulo $(J^2T)B[T]$. We may further apply elementary transformation to ensure that $\text{ht}(k_1, \cdots, k_{n-1}) =$

$n-1$ and $k_1 = 1$ modulo $(J^2T)B[T]$. Further, since $(J^2T)B[T]$ is contained in the Jacobson radical of B, we have $\dim(B[T]/(k_1, \cdots, k_{n-1}) \leq 1$.

Let $C = B[T]$. Consider the following ideals in $C[Y]$:

$$K_1 = (Y + k_1, k_2, \cdots, k_n), \quad K_2 = IC[Y], \quad K_3 = K_1 \cap K_2.$$

Now applying [12, Theorem 2.3] it follows that

$$K_3 = (U_1(T, Y), \cdots, U_n(T, Y))$$

such that $U_i(T, 0) = h_i$. Putting $Y = 1 - k_1$, we obtain

$$IB[T] = (U_1(T, 1 - k_1), \cdots, U_n(T, 1 - k_1)).$$

Since $k_1 = 1$ modulo $(J^2T)B[T]$ it follows that $U_i(T, 1 - k_1) = U_i(T, 0)$ modulo $(I^2T)B[T]$. If we write $u_i = U_i(T, 0)$ then $IB[T] = (u_1, \cdots, u_n)$ such that $u_i = f_i$ modulo $(I^2T)B[T]$. As mentioned earlier this is enough to prove the theorem. □

Now we can apply the above theorem to derive addition and subtraction principles for ideals in $A[T]$.

Corollary 4.3 (Addition principle) *Let A be a Noetherian ring (containing $\mathbb{Q}$) of dimension $n \geq 3$ and $I_1, I_2 \subset A[T]$ be two comaximal ideals, each of height ≥ 3 such that $I_1 = (f_1, \cdots, f_n)$ and $I_2 = (g_1, \cdots, g_n)$. Then $I_1 \cap I_2 = (h_1, \cdots, h_n)$ where $f_i - h_i \in I_1^2$ and $g_i - h_i \in I_2^2$.*

Proof Let us denote $I_1 \cap I_2$ by I_3. Then $\mathrm{ht}(I_3) \geq 3$. Further we note that the ideals $I_1(0)$ and $I_2(0)$ are comaximal and each has height ≥ 2. We have $I_1(0) = (f_1(0), \cdots, f_n(0))$ and $I_2(0) = (g_1(0), \cdots, g_n(0))$. Therefore by Proposition 2.1 we have, $I_3(0) = I_1(0) \cap I_2(0) = (c_1, \cdots, c_n)$ with $f_i(0) - c_i \in I_1(0)^2$ and $g_i(0) - c_i \in I_2(0)^2$.

By the Chinese Remainder Theorem, $I_3/{I_3}^2 \simeq I_1/{I_1}^2 \oplus I_2/{I_2}^2$. Therefore, the given generators of I_1 and I_2 together will induce a set of generators of $I_3/{I_3}^2$. This means we will have, $I_3 = (H_1, \cdots, H_n) + {I_3}^2$

where $f_i - H_i \in I_1^2$ and $g_i - H_i \in I_2^2$. Therefore, $H_i(0) - c_i \in I_1(0)^2$ and $H_i(0) - c_i \in I_2(0)^2$.

Combining the conclusions of the last two paragraphs we have,

1. $I_3 = (H_1, \cdots, H_n) + {I_3}^2$
2. $I_3(0) = (c_1, \cdots, c_n)$
3. $H_i(0) - c_i \in I_3(0)^2$

Applying [4, Remark 3.9], we can find $L_1, \cdots, L_n \in I_3$ such that

$$I_3 = (L_1, \cdots, L_n) + ({I_3}^2 T)$$

with the property that $L_i - H_i \in I_3^2$.

Consider the ring $A(T)$. Applying Proposition 2.1 to the two comaximal ideals $I_1 A(T)$ and $I_2 A(T)$, we have $I_3 A(T) = (U_1, \cdots, U_n)$, where $f_i - U_i \in I_1^2 A(T)$ and $g_i - U_i \in I_2^2 A(T)$. Therefore, $L_i - U_i \in I_3^2 A(T)$.

Now we can apply Theorem 4.2 to obtain the desired set of generators for I_3. □

Corollary 4.4 (Subtraction principle) *Let A be a Noetherian ring (containing $\mathbb{Q}$) of dimension $n \geq 3$ and $I_1, I_2 \subset A[T]$ be two comaximal ideals, each of height ≥ 3 such that $I_1 = (f_1, \cdots, f_n)$ and $I_1 \cap I_2 = (h_1, \cdots, h_n)$ where $f_i - h_i \in I_1^2$. Then there exist $g_1, \cdots, g_n$ such that $I_2 = (g_1, \cdots, g_n)$ where $g_i - h_i \in I_2^2$.*

Proof The method is similar to the proof of the above corollary and omitted.

Now we are ready to extend the definition of Segre class to $A[T]$ where A is a Noetherian ring (containing $\mathbb{Q}$) of dimension $n \geq 3$ and $I \subset A[T]$ is an ideal of height ≥ 3 such that $\mu(I/I^2) = n$.

Let $I \subset A[T]$ be an ideal of height ≥ 3 such that I/I^2 is generated by n elements. Let $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ be a surjection induced by $I = (f_1, \cdots, f_n) + I^2$. Applying [5, Lemma 2.12] we can find $g_1, \cdots, g_n$ and an ideal $I_1 \subset A[T]$ such that

1. $I \cap I_1 = (g_1, \cdots, g_n)$.

2. $I + I_1 = A[T]$ and $\text{ht}(I_1) \geq n$.

3. $g_i - f_i \in I^2$.

Clearly $g_1, \cdots, g_n$ will induce a local orientation on I_1, say, ω_{I_1}. We define the Segre class of (I, ω_I) by

$$s(I, \omega_I) = -(I_1, \omega_{I_1}) \in E(A[T]),$$

where $E(A[T])$ is the Euler class group of $A[T]$.

Remark 4.5 Proceeding as in Proposition 3.2 one can check that the Segre class of (I, ω_I) does not depend on the choice of I_1. Further, the proofs of the following results are similar to those in Section 4 and hence omitted. We mainly need the appropriate addition and subtraction principles which we have proved in this section.

Theorem 4.6 *Let A be a ring (containing $\mathbb{Q}$) of dimension $n \geq 3$ and $I \subset A[T]$ be an ideal of height ≥ 3 such that I/I^2 is generated by n elements. Let $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ be a surjection. Suppose that $s(I, \omega_I) = 0$ in $E(A[T])$. Then ω_I can be lifted to a surjection $\alpha : A[T]^n \twoheadrightarrow I$.*

Theorem 4.7 *Let A be as above and I_1, I_2 be two comaximal ideals of $A[T]$, each of height ≥ 3. Suppose that we have surjections $\omega_{I_1} : (A[T]/I_1)^n \twoheadrightarrow I_1/I_1^2$ and $\omega_{I_2} : (A[T]/I_2)^n \twoheadrightarrow I_2/I_2^2$. Then,*

$$s(I_1 \cap I_2, \omega_{I_1 \cap I_2}) = s(I_1, \omega_{I_1}) + s(I_2, \omega_{I_2})$$

in $E(A[T])$, where $\omega_{I_1 \cap I_2} : (A[T]/I_1 \cap I_2)^n \twoheadrightarrow I_1 \cap I_2/(I_1 \cap I_2)^2$ is the surjection induced by ω_{I_1} and ω_{I_2}.

5 Minimal reduction, Chern class and Northcott-Rees class

In this section we will deal with another class of the so called "bad" ideals in a ring A of dimension $n \geq 2$. We will consider ideals $I \subset A$ of height n which are not necessarily local complete intersections (equivalently, $\mu(I/I^2)$ is not necessarily equal to $\operatorname{ht} I (= n)$. An obvious idea to study the behaviour of such a bad ideal is to look for a way to associate it to a good ideal in a meaningful manner. In this context we will consider *reductions* of I. Recall that a subideal J of I is said to be a reduction of I if there exists a non-negative integer t such that $I^{t+1} = JI^t$. In a sense J can be regarded as a simplified version of I while it retains many properties of I. The study of reductions was initiated by Northcott and Rees [15] in 1950's and it has been the subject of much work since then. An excellent source to read the theory of reductions and its connection with other important topics like integral closure and multiplicity is [7].

Here we recall some definitions and collect some results on minimal reductions. For details, proofs and unexplained notations, see [7].

Definition 5.1 *Let R be a ring . Let $J \subseteq I$ be ideals. J is said to be a reduction of I if there exists a non-negative integer t such that $I^{t+1} = JI^t$.*

It is easy to see that if J is a reduction of I then $\sqrt{J} = \sqrt{I}$ and $\text{ht}J = \text{ht}I$.

The proof of the following proposition is essentially contained in the proof of [7, Theorem 8.73 (2)]; the theorem is due to Katz [9].

Proposition 5.2 *Let A be a Noetherian ring of dimension n containing $\mathbb{Q}$ and $I \subset A$ be an ideal of height n. Then I has a reduction K such that K/K^2 is generated by n elements.*

The above proposition actually implies that I has a minimal reduction. To see this note that for any maximal ideal m containing I, the ideal KA_m is a reduction of IA_m. Since KA_m is an ideal of height n which is also

n-generated, it is *basic* and therefore minimal. If we had another reduction K' of I contained in K, we would have $K'A_m \subset KA_m$ and by minimality of KA_m, $K'A_m = KA_m$. Since m was arbitrarily chosen, we conclude that $K' = K$.

Further, take any reduction $J \subset I$. We can apply the above proposition to J and see that J contains an ideal $\tilde{J}$ such that $\tilde{J}$ is a minimal reduction of I. This phenomenon is observed in local rings ([7, Theorem 8.3.6]), but may not be true for non-local rings in general ([7, Exercise 8.10]).

The upshot of the above proposition is that the reduction K of I (as in the proposition) has the property that it is minimal and $\mu(K/K^2) = n$. Conversely, take any minimal reduction J of I. Let m be any maximal ideal containing I. Since every minimal reduction of IA_m is n-generated, we have JA_m (which is a minimal reduction of IA_m) is n-generated. Since m is arbitrary, we conclude that $\mu(J/J^2) = n$.

Now suppose that there exists a reduction K of I such that K is generated by n elements. We may wonder whether this will imply that all minimal reductions will be generated by n elements. We prove below that such a result holds if we assume A to be a smooth affine domain over an algebraically closed field.

First we have the following proposition.

Proposition 5.3 *Let A be a smooth affine domain of dimension $n \geq 2$ over an infinite field k (not necessarily algebraically closed). Let $I \subset A$ be an ideal of height n. Then all minimal reductions of I have the same Chern class in $CH_0(A)$.*

Proof Let K be a minimal reduction of I. Let $m_1, \cdots, m_r$ be the set of maximal ideals containing I. Since $\sqrt{I} = \sqrt{K}$, these are precisely all the maximal ideals containing K. By definition of Chern class, the element in $CH_0(A)$ associated to K is the following

$$[K] = \sum_1^r \lambda(A/K)_{m_i}[m_i],$$

where λ stands for length.

Let m be any one of the maximal ideals $m_1, \cdots, m_r$. Since IA_m is an mA_m-primary ideal and KA_m is a minimal reduction of IA_m, by [7, Proposition 11.2.2], $e(IA_m; A_m) = \lambda(A_m/KA_m)$, where e denotes multiplicity. Therefore, we have

$$[K] = \sum_{1}^{r} e(IA_{m_i}; A_{m_i})[m_i].$$

Since the right hand side of the above equation depends only on I, the result follows. □

Corollary 5.4 *Let A be a smooth affine domain of dimension $n \geq 2$ over an algebraically closed field k and $I \subset A$ be an ideal of height n. Suppose there is a minimal reduction K of I such that K is a complete intersection. Then every minimal reduction of I is a complete intersection.*

Proof Let L be any minimal reduction of I. Then L is locally a complete intersection. Now since K is complete intersection, we have $[K] = 0$ in $CH_0(A)$. By the above proposition $[L] = 0$ in $CH_0(A)$. Now it follows from [14] that L is a complete intersection. □

Let A be a smooth affine domain of dimension $n \geq 2$ over an infinite field k (not necessarily algebraically closed) and $I \subset A$ be an ideal of height n. We may associate to I an element, say $NR(I)$, in $CH_0(A)$ in the following way

$$NR(I) = \sum_{1}^{r} e(IA_{m_i}; A_{m_i})[m_i]$$

and call it the *Northcott-Rees class* of I. Note that if I is a local complete intersection then the Northcott-Rees class and the Chern class of I will coincide.

Corollary 5.5 *Let A be a smooth affine domain of dimension $n \geq 2$ over an algebraically closed field k and $I \subset A$ be an ideal of height n. Suppose $NR(I) = 0$ in $CH_0(A)$. Then I is set-theoretic complete intersection.*

Proof Let K be any minimal reduction of I. Then K is a local complete intersection. By the condition of the corollary $[K] = NR(I) = 0$ in $CH_0(A)$. By Murthy's result [14], we see that K is a complete intersection. Since $\sqrt{I} = \sqrt{K}$, we have I set theoretically generated by n elements. $\square$

Remark 5.6 Bhatwadekar pointed out (personal communication) that in the situation of the above corollary, I will be set theoretic complete intersection if we assume $[I] = 0$ (the Chern class) in $CH_0(A)$ instead of $NR(I) = 0$. To see this, let us fix some notation. For an ideal J of height n with $J = (a_1, \cdots, a_n) + J^2$, we will denote the ideal $(a_1, \cdots, a_{n-1}) + J^r$ by $J^{(r)}$. Then $\sqrt{J} = \sqrt{J^{(r)}}$ and $J^{(r)}$ is a local complete intersection. Further, $[J^{(r)}] = r[J]$ in $CH_0(A)$. With these notations in mind consider the ideal $I' = \cap_1^r m_i^{(\lambda_i)}$, where $\lambda_i = \lambda(A_{m_i}/IA_{m_i})$. Then I' is a local complete intersection and $\sqrt{I} = \sqrt{I'}$. It is not hard to see that

$$\lambda(A_{m_i}/m_i^{(\lambda_i)}) = \lambda_i = \lambda(A_{m_i}/IA_{m_i}).$$

Therefore, in $CH_0(A)$, we have

$$[I] = \sum_1^r \lambda(A_{m_i}/IA_{m_i})[m_i] = \lambda(A_{m_i}/m_i^{(\lambda_i)})[m_i] = [I'].$$

Now if we assume $[I] = 0$ then we have $[I'] = 0$ in $CH_0(A)$ for the local complete intersection ideal I', and if k is algebraically closed, by [14], I' is a complete intersection. As a consequence, I is set theoretic complete intersection.

Remark 5.7 In Corollary 5.5 we took A to be a smooth affine domain of dimension $n \geq 2$ over an algebraically closed field to apply Murthy's result which says that *for a local complete intersection ideal $J \subset A$ of height n, $[J] = 0$ in $CH_0(A)$ implies that J is a complete intersection*. One can deduce similar corollaries in a set up where results similar to Murthy's holds. For instance, we have the following results.

Corollary 5.8 *Let $X = SpecA$ be a smooth affine variety of dimension $n \geq 2$ over $\mathbb{R}$ such that the set of real points of X is not empty. Assume that the smooth manifold consisting of real points of X has no compact connected component. Let $I \subset A$ be an ideal of height n. Suppose $NR(I) = 0$ in $CH_0(A)$. Then I is set-theoretic complete intersection.*

Proof Let K be any minimal reduction of I. Then K is a local complete intersection. By the condition of the corollary $[K] = NR(I) = 0$ in $CH_0(A)$. Now it follows from [3, Theorem 5.10] that K is a complete intersection. Consequently I is set-theoretic complete intersection. □

For a smooth affine variety $X = \text{Spec}A$ over $\mathbb{R}$ let $\mathbb{R}(X)$ denote the ring obtained from A by inverting all the elements which do not belong to any real maximal ideal.

Corollary 5.9 *Let $X = SpecA$ be a smooth affine surface over $\mathbb{R}$ such that the set of real points of X is not empty. Assume that the smooth manifold consisting of real points of X is compact and connected and the canonical module $K_{\mathbb{R}(X)}$ is not trivial. Let $I \subset A$ be an ideal of height n. Suppose $NR(I) = 0$ in $CH_0(A)$. Then I is set-theoretic complete intersection.*

Proof Let K be any minimal reduction of I. Then K is a local complete intersection. By the condition of the corollary $[K] = NR(I) = 0$ in $CH_0(A)$. Now it follows from [1, Theorem 4.30, Remark 4.31] that K is a complete intersection. Consequently I is set-theoretic complete intersection. □

6 Some historical motivation

In this section we see how concepts similar to those of the Segre classes have been used in the earlier literature in the proof of a conjecture of Förster. We begin by discussing the conjecture of Förster.

Let $A = k[X_1, \cdots, X_n]$ be the polynomial ring in n variables over a field k and $\mathcal{P} \subset A$ be a prime ideal. Then as A is Noetherian, $\mathcal{P}$ is finitely generated. Now by Krull's dimension theorem, $\mu(\mathcal{P}) \geq \mathrm{ht}\,(\mathcal{P})$. One can ask if the function $\mu(\mathcal{P})$ is bounded as $\mathcal{P}$ varies over all prime ideals of the polynomial ring. This is however false. There are classical examples due to Macaulay of prime ideals $\mathcal{P}$ in $\mathbb{C}[X_1, X_2, X_3]$ of height 2 such that the function $\mu(\mathcal{P})$ is unbounded. In Macaulay's examples the ring $\mathbb{C}[X_1, X_2, X_3]/\mathcal{P}$ has a singularity at the origin.

By contrast if $A = k[X_1, \cdots, X_n]$ and $\mathcal{P} \subset A$ is a prime ideal such that $A/\mathcal{P}$ is regular, Förster [6] proved that $\mathcal{P}$ is generated by $n + 1$ elements and conjectured that in this case n elements suffice to generate $\mathcal{P}$. The conjecture of Förster was proved by Sathaye [17] in the case where k is infinite and shortly afterwards by Mohan Kumar [13] in general.

The method of Sathaye's proof is the following. Let $\mathcal{P}$ be a prime ideal of $A = k[X_1, \cdots, X_n]$. If $\mathrm{ht}\,(\mathcal{P}) = 1$ then $\mathcal{P}$ is principal, so we may assume that $\mathrm{ht}\,(\mathcal{P}) \geq 2$. It follows from a theorem of Förster [6, 18] that if $A/\mathcal{P}$ is regular then $\mathcal{P}/\mathcal{P}^2$ is generated by n elements. We may assume, by Swan's Bertini theorem, that $(f_1, \cdots, f_n) = \mathcal{P} \cap m_1 \cap \cdots \cap m_r$ where $m_i \subset A$ are maximal and $m_i + \mathcal{P} = A$. Now, since m_i are maximal, each of the m_i is generated by n elements. The method of Sathaye (which is inspired by an argument of Abhyankar), is to successively eliminate each of the maximal ideals, and to show that the ideal $\mathcal{P} \cap m_1 \cap \cdots \cap m_i$ is generated by n elements for every $i < r$. In particular we conclude at the last stage that $\mathcal{P}$ is generated by n elements. We refer the reader to [17] for the details of the proof. We will indicate however a proof of the argument of Abhyankar which inspired Sathaye's proof. Before doing this, we point out the relevance of Segre classes in the above discussion. In the language of Segre classes $\mathcal{P}$ is generated by n elements because $s(\mathcal{P}, \omega_{\mathcal{P}}) = 0$, where $\omega_{\mathcal{P}}$ is the set of generators $f_1, \cdots, f_n$ of $\mathcal{P}/\mathcal{P}^2$.

In a similar vein we show how the notion of Segre classes can be applied to prove the following theorem which was a conjecture of Eisenbud-Evans.

Theorem 6.1 *Let R be a Noetherian ring of dimension d and $I \subset R[T]$ be an ideal of height ≥ 2. Suppose I/I^2 is generated by $d+1$ elements. Then I is generated by $d+1$ elements.*

That the above theorem implies Förster's conjecture was proved by Sathaye in [17] for affine domains A over infinite fields en route to his proof of the Förster's conjecture and by Mohan Kumar [13] in general. we give a proof of the theorem using the notion of Segre classes. Let $f_1, \cdots, f_{d+1}$ generate I/I^2. We may assume that $(f_1, \cdots, f_{d+1}) = I \cap I'$, where ht$(I') \geq d+1$. Since ht$(I') \geq d = 1$, I' contains a monic polynomial. Using a theorem of Mandal [10], it follows that $s(I, \omega_I) = 0$ where ω_I is the set of generators $f_1, \cdots, f_{d+1}$ of I/I^2. Hence I is generated by $d+1$ elements.

We conclude by sketching the argument of Abhyankar that inspired Sathaye's proof. The argument of Abhyankar is based on a method that Seshadri used to prove that projective modules over $K[X, Y]$ are free (K a field).

We prove the following proposition which gives the flavour of Abhyankar's argument. We remark that Abhyankar's argument applies to more general situations.

Proposition 6.2 *Let k be an algebraically closed field and $I \subset K[X, Y] = A$ be an ideal such that dim$(A/I) = 0$. Suppose that I/I^2 is generated by two elements. Then I s generated by two elements.*

Proof Let $f_1, f_2 \in I$ generate I/I^2. We may assume that $(f_1, f_2) = I \cap I'$, where $I + I' = A$ and $I' = m_1 \cap \cdots \cap m_r$ is the intersection of finitely many maximal ideals. We have $m_i = (X - a_i, Y - b_i)$ and by a change of variables we may assume that $m_1 = (X, Y)$ The ideal $(f_1(X, 0), f_2(X, 0))$ of $k[X]$ is principal and therefore, using the Euclidean algorithm, can be transformed to $(\lambda(X), 0)$, where $\lambda(X)$ is the g. c. d. of $(f_1(X, 0), f_2(X, 0))$. Considering these transformations as elements of $GL_2(k[X]) \subset GL_2(k[X, Y])$, we see that we can transform the row $(f_1(X,Y), f_2(X,Y))$ to another row $(h_1(X,Y), h'_2(X,Y))$ such that

1. the ideals $(f_1(X,Y), f_2(X,Y))$ and $(h_1(X,Y), h'_2(X,Y))$ are equal;

2. $h_1(X,0) = \lambda(X)$, $h'_2(X,0) = 0$, that is, $h'_2(X,Y) = Yh_2(X,Y)$.

This implies that

$$(h_1(X,Y), Yh_2(X,Y)) = I \cap m_1 \cap \cdots \cap m_r.$$

Since $m_1 = (X,Y)$, by a linear change of variables, for example replacing Y by $Y + cX$, we may assume that the line $Y = 0$ does not pass through the finitely many points belonging to $V(I)$ and the points (a_i, b_i), $2 \le i \le r$. This implies that the element $Y \in A$ is a unit modulo I, and modulo m_i, $2 \le i \le r$. Now since the elements $h_1(X,Y), Yh_2(X,Y)$ generate I/I^2 and $m_i/{m_i}^2$, $2 \le i \le r$, we have $h_1(X,Y), h_2(X,Y)$ generate I/I^2 and $m_i/{m_i}^2$, $2 \le i \le r$. Since $(h_1(X,Y), Yh_2(X,Y)) + {m_1}^2 = m_1$ and $m_1 = (X,Y)$, it follows that $Yh_2(X,Y) \notin {m_1}^2$ and hence $h_2(0,0) \ne 0$. Hence $h_2(X,Y) \notin m_1$. It follows, from the previous discussion that

$$(h_1(X,Y), h_2(X,Y)) = I \cap m_2 \cap \cdots \cap m_r.$$

Continuing the above process, we see that I is generated by two elements. □

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