Euler class groups and a theorem of Roitman

Mrinal Kanti Das*

Stat-Math Unit, Indian Statistical Institute

203 B. T. Road, Kolkata 700108 India

and

Raja Sridharan†

School of Mathematics

Tata Institute of Fundamental Research

Homi Bhabha Road, Mumbai 400005 India

July 3, 2010

1 Introduction

Let A be a commutative Noetherian ring of dimension d containing the field of rationals. In [D1] the notion of the dth Euler class group $E^d(A[T])$ of $A[T]$ has been defined. Apart from this group's own intrinsic properties, it is important to understand the relation between the Euler class groups $E^d(A)$ and $E^d(A[T])$ which we did to some extent in [D1, D2, D-RS], following the "Quillen-Suslin model" for $K_0(A)$ and $K_0(A[T])$. For example, as an analogue to the Quillen's localization theorem, we proved the local global principle for the Euler class groups which says that the following sequence of groups

$$0 \longrightarrow E^d(A) \longrightarrow E^d(A[T]) \longrightarrow \prod_{\mathfrak{m}} E^d(A_{\mathfrak{m}}[T])$$

is exact where the product runs over all maximal ideals $\mathfrak{m}$ of A.

*mrinal@isical.ac.in

†sraja@math.tifr.res.in

We now recall the following result of Roitman [Ro, Proposition 2], which is, in some sense, a converse of Quillen's result.

Theorem 1.1 *Let R be a commutative Noetherian ring and $S \subset R$ be multiplicatively closed. Suppose that all projective $R[X]$-modules are extended from R. Then all projective $R_S[X]$-modules are extended from R_S.*

In this paper we prove the following analogue of Roitman's result in the context of Euler class groups.

Theorem 1.2 *Let A be a commutative Noetherian ring of dimension $d \geq 2$ containing $\mathbb{Q}$. Let $S \subset A$ be multiplicatively closed. Suppose it is given that the canonical map $\phi : E^d(A) \longrightarrow E^d(A[T])$ is surjective. Then the canonical map $\phi_S : E^d(A_S) \longrightarrow E^d(A_S[T])$ is also surjective.*

The canonical maps ϕ and ϕ_S are always injective [D1, D2]. So the above theorem actually tells that ϕ_S is an isomorphism if ϕ is an isomorphism.

Now assume that A is a commutative Noetherian ring of dimension d and n be an integer such that $2n \geq d+3$. In section 3 we define the notion of the nth Euler class group $E^n(A[T])$ of $A[T]$, prove results analogous to [D1] and explore the relation between $E^n(A[T])$ and $E^n(A)$ (the group $E^n(A)$ has been introduced in [B-RS 3]). In this section we need to make the assumption that A is regular because we crucially use Theorem 2.11, which depends on Lemma 2.6, which in turn depends on a result of Lindel-Popescu (see [B-RS 1, 3.4]).

With A, d, n as in the above paragraph, we also prove Theorem 1.2 for the nth Euler class groups.

As mentioned above, section 3 depends very much on the following result of Bhatwadekar and Keshari (Theorem 2.11 in this paper). This result is a typical example of the *monic inversion principle* and can be thought of as an analogue of the Affine Horrocks Theorem in the context of complete intersections. Here $A(T)$ denotes the ring obtained from $A[T]$ by inverting all the monic polynomials.

Theorem 1.3 *[B-K, 4.9] Let A be a regular domain of dimension d and n be an integer such that $2n \geq d+3$. Let $I \subset A[T]$ be an ideal of height n for which it is given that $I = (f_1, \cdots, f_n) + (I^2T)$. Assume further that there exist $G_1, \cdots, G_n \in IA(T)$ such that $IA(T) = (G_1, \cdots, G_n)$ where $G_i - f_i \in I^2A(T)$. Then there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ where $g_i - f_i \in (I^2T)$.*

The case when $IA(T) = A(T)$ (i.e., when I contains a monic polynomial), has been proved by Mandal in [M 2] without any smoothness assumption. Therefore, an interesting open problem is to extend the above theorem to rings which are not necessarily regular. This has been done in the case when $n = d$ in [D1].

In section 2 we restructure the proof of Bhatwadekar and Keshari in the spirit of Quillen-Suslin solution of the Affine Horrocks Theorem. The core idea of Bhatwadekar and Keshari is presented in the form of a theorem (2.8 below). We name the ideal arising out of this theorem as the *Nori ideal*. This is our tribute to Madhav Nori, whose vision and insight initiated most of the recent research in the theory of projective modules and complete intersections.

2 The Nori ideal

The aim of this section is to revisit the Bhatwadekar-Keshari solution to a conjecture of Nori (Theorem 2.13 below). We rewrite their proof (in the free case) in the "Quillen-Suslin way". Analogous to the concept of the *Quillen ideal*, here we have the *Nori ideal*, which we highlight in Theorem 2.8. The reason for emphasizing Theorem 2.8 is that it provides a passage from local to global and thus proves a couple of major results in one stroke. For instance, Bhatwadekar-Keshari proved their result (Theorem 2.13) in this manner. Here we show that a monic inversion theorem [B-K, 4.9] (2.11 below) is a direct consequence of Theorem 2.8. In the next section we will obtain a local global principle for the Euler class groups as another application.

The results in this section are not at all new, neither the core ideas of the proofs. However, unlike [B-K] we do not use the monic inversion theorem [B-K, 4.9] to prove Theorem 2.8 or 2.13. We first prove a "local" version of the monic inversion theorem (2.1 below) and use it to prove Theorem 2.8. To prove the monic inversion theorem in general (2.11), we resort to the good old local global philosophy, as indicated in the above paragraph. This was precisely the way the Affine Horrocks Theorem was proved. We will also point out at the appropriate place how the same method gives a new proof of [D1, Theorem 3.10].

Towards the end of the proof of Theorem 2.8 we use some simple calculus of Euler classes. A reader familiar with basics of Euler class theory will probably find the proof a bit easier to understand.

In order to reorganise the ideas of [B-K], we need to modify some of their results. We start with the following "local" version of the monic inversion theorem which can be easily proved using [B-K, Lemmas 4.2, 4.6, 4.7]. We quickly sketch a proof.

Notation In what follows, $A(T)$ will denote the ring obtained from $A[T]$ by inverting all the monic polynomials.

Theorem 2.1 *Let A be a ring of dimension d and n be an integer such that $2n \geq d+3$. Assume that $\text{ht}\mathcal{J}(A) \geq n-1$. Let $I \subset A[T]$ be an ideal of height n such that $I = (f_1, \cdots, f_n) + (I^2T)$ and assume that $IA(T) = (G_1, \cdots, G_n)$ where $G_i - f_i \in I^2A(T)$. Then there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $g_i - f_i \in (I^2T)$.*

Proof Let $J = \mathcal{J}(A) \cap I \subset I \cap A$. Then $\text{ht}\, J \geq n-1$. It is easy to check that following the same proof as in [B-RS 1, Lemma 3.6] we can find $\alpha_1, \cdots, \alpha_n$ such that

1. $I = (\alpha_1, \cdots, \alpha_n) + (J^2T)$ such that $\alpha_i - f_i \in (I^2T)$
2. $(\alpha_1, \cdots, \alpha_n) = I \cap I_1$, where $\text{ht}\, I_1 \geq n$
3. $I_1 + (J^2T) = A[T]$.

The case when $I_1 = A[T]$ being trivial, let us assume that $\text{ht}\, I_1 = n$. We have $I_1 = (\alpha_1, \cdots, \alpha_n) + I_1^2$. Since $I_1(0) = A$, by [B-RS 1, Remark 3.9] we have $I_1 = (\beta_1, \cdots, \beta_n) + (I_1^2T)$ where $\beta_i - \alpha_i \in I_1^2$. Applying [B-RS 3, Proposition 3.2] on $A(T)$ we see that $I_1A(T) = (H_1, \cdots, H_n)$ such that $H_i - \alpha_i \in I_1^2A(T)$. Consequently, $H_i - \beta_i \in I_1^2A(T)$. Now as I_1 is comaximal with $\mathcal{J}(A)$, by [B-K, 4.6] we see that $I_1 = (\gamma_1, \cdots, \gamma_n)$ where $\gamma_i - \alpha_i \in (I_1^2T)$. Applying [B-K, 4.7] we are done. $\square$

Applying Theorem 2.1 we obtain the following subtraction principle.

Proposition 2.2 (Subtraction principle) *Let A be a ring of dimension d and n be an integer such that $2n \geq d+3$. Assume that $\text{ht}\mathcal{J}(A) \geq n-1$. Let $I, K \subset A[T]$ be two comaximal ideals, each of height n such that $I = (f_1, \cdots, f_n)$ and $I \cap K = (h_1, \cdots, h_n)$ where $h_i - f_i \in I^2$. Then there exist $g_1, \cdots, g_n$ such that $K = (g_1, \cdots, g_n)$ where $h_i - g_i \in K^2$.*

Proof The method has become standard now. One can follow the proof of [D1, Proposition 4.3] or [B-K, Corollary 4.11].

Now we quote three beautiful lemmas which are crucial for the proof of Theorem 2.8.

Lemma 2.3 (Square lemma) *[B-K, Lemma 3.4] Let R be a ring and $I \subset R$ be an ideal. Let $s \in R$ be such that $I + (s) = R$. Assume that $I = (f_1, \cdots, f_n)$. Let r be a positive integer. Then $I = (s^r f_1, \cdots, s^r f_n) + I^2$. If r is even, $s^r f_1, \cdots, s^r f_n$ can be lifted to a set of generators of I.*

Remark 2.4 Let R be a ring of dimension d and n be an integer such that $2n \geq d+3$. Then we can talk about the nth Euler class group $E^n(R)$ of R (see [B-RS 3]). Let $J \subset R$ be an ideal of height n and $\omega : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. Let $\Delta \in GL_n(R/J)$ be a diagonal matrix whose determinant is a square in $(R/J)^*$. If we write $\widetilde{\omega} = \omega\Delta$, then applying the square lemma it is easy to check that $(J, \omega) = (J, \widetilde{\omega})$ in $E^n(R)$.

Lemma 2.5 *[B-K, Lemma 3.5] Let R be a ring and $s, t \in R$ be such that $Rs + Rt = R$. Let I, L be ideals of R such that $L \subset I^2$. Let $I = (a_1, \cdots, a_n) + L$. Suppose that there exist $b_1, \cdots, b_n$ such that $I_t = (b_1, \cdots, b_n)$ where $b_i - a_i \in L_t$. Then there exist $c_1, \cdots, c_n$ such that $I = (c_1, \cdots, c_n) + (sL)$ where $c_i - a_i \in L$.*

Lemma 2.6 *[B-RS 1, Lemma 3.5] Let A be a regular domain containing a field and $I \subset A[T]$ be an ideal. Let $J = I \cap A$ and write $B = A_{1+J}$. Suppose it is given that $I = (f_1, \cdots, f_n) + (I^2T)$. Assume further that $I_{1+J} = (h_1, \cdots, h_n)$ where $h_i - f_i \in (I^2T)_{1+J}$. Then $I = (g_1, \cdots, g_n)$ with $g_i - f_i \in (I^2T)$.*

Remark 2.7 If dim$A = n \geq 3$, then the above lemma holds for an arbitrary ring A, as shown in [D1, 3.8].

Let A be a Noetherian ring of dimension d and n be an integer such that $2n \geq d+3$. Let $I \subset A[T]$ be an ideal of height n such that $\mu(I/(I^2T)) = n$. Suppose it is given that $I = (f_1, \cdots, f_n) + (I^2T)$. Now consider the set

$$N(I; \underline{f}) = \{s \in A \mid \exists g_1, \cdots, g_n \in I_s \text{ such that } I_s = (g_1, \cdots, g_n)\,;\, g_i - f_i \in (I^2T)_s\}.$$

The essential idea of Bhatwadekar-Keshari solution of Nori's conjecture is to prove that if A is a regular domain then $N(I; \underline{f})$ is an ideal. This is implicit in the proof of [B-K, Theorem 4.13]. We rewrite the proof below with some subtle modifications. We will call the ideal $N(I; \underline{f})$ as the *Nori ideal* associated to I and $f_1, \cdots, f_n$.

Notation Let R be a ring and $K \subset R$ be an ideal for which it is given that $K = (a_1, \cdots, a_n) + K^2$. This induces a surjection from $(R/K)^n$ to K/K^2. We will denote this surjection by $\omega_{(K,\underline{a})}$. This notation will be useful in the proof below when we will use Euler class computations.

Theorem 2.8 *Let A be a regular domain of dimension d containing a field and n be an integer such that $2n \geq d+3$. Let $I \subset A[T]$ be an ideal of height n such that $I = (f_1, \cdots, f_n) + (I^2T)$. Then $N(I; \underline{f})$ is an ideal of A.*

Proof Let $s \in N(I;\underline{f})$ and $a \in A$. Then clearly $sa \in N(I;\underline{f})$.

Let $s, t \in N(I;\underline{f})$. We need to prove that $s+t \in N(I;\underline{f})$. Inverting $s+t$ and replacing s, t by $\frac{s}{s+t}, \frac{t}{s+t}$ we assume that $s+t=1$. Thus we are reduced to prove that $1 \in N(I;\underline{f})$.

Let $J = I \cap A$ and write $B = A_{1+J}$. By Lemma 2.6, it is enough to show that $f_1, \cdots, f_n$ can be lifted to a set of generators of $IB[T]$. Since J is contained in the Jacobson radical $\mathcal{J}(B)$ of B, we note that ht$\mathcal{J}(B) \geq n-1$.

If s or t is a unit in B, we have nothing to prove. Therefore we assume that they are not units in B. As $JB \subset \mathcal{J}(B)$, it follows that $s \notin \sqrt{JB}$ and $t \notin \sqrt{JB}$. Now we write down the proof in steps.

Step 1. Since $t \in N(I;\underline{f})$ and t, s are comaximal, applying Lemma 2.5 with $L = (I^2T)$ and $R = B[T]$, it follows that there exist $\alpha_1, \cdots, \alpha_n$ such that

$$I = (\alpha_1, \cdots, \alpha_n) + (I^2Ts) \text{ where } \alpha_i - f_i \in (I^2T). \tag{1}$$

Applying [B-K, Lemma 4.2] we can find an ideal $I_1 \subset A[T]$ of height $\geq n$ which is comaximal with $I \cap (s)$ such that

$$I \cap I_1 = (\beta_1, \cdots, \beta_n) \text{ where } \beta_i - \alpha_i \in (I^2Ts) \tag{2}$$

If ht $I_1 > n$ then $I_1 = B[T]$ and we are done. Therefore we assume that ht $I_1 = n$. We have

$$I_1 = (\beta_1, \cdots, \beta_n) + I_1^2 \tag{3}$$

Since $s \in N(I;\underline{f})$, there exist $g_1, \cdots, g_n$ such that we eventually have

$$IB_s[T] = (g_1, \cdots, g_n) \text{ with } g_i - f_i \in (I^2T)B_s[T] \tag{4}$$

Now we note that the Jacobson radical of B_s contains JB_s and has height $\geq n-1$. In view of equations 2 and 4 above, applying Proposition 2.2 over $B_s[T]$ we obtain that

$$I_1B_s[T] = (\gamma_1, \cdots, \gamma_n) \text{ where } \gamma_i - \beta_i \in (I_1^2T)B_s[T] \tag{5}$$

Step 2. Now we will clear denominators and to do so effectively we will utilise the fact that $(I_1, s) = B[T]$. Let N be a positive even integer such that $s^N\gamma_i \in I_1$ for $1 \leq i \leq n$. Let us write $\delta_i = s^N\gamma_i$ for $1 \leq i \leq n$. Since $(I_1, s) = B[T]$, we have

$$I_1 = (\delta_1, \cdots, \delta_n) + I_1^2 \tag{6}$$

Clearly, $I_1B_s[T] = (\delta_1, \cdots, \delta_n)B_s[T]$. Adapting the proof of [B, Proposition 3.1] it is easy to see that there exists an elementary matrix $\Delta \in \mathcal{E}_n(B_s[T])$ such that if $(\delta_1, \cdots, \delta_n)\Delta = (\epsilon_1, \cdots, \epsilon_n)$ then $\epsilon_i \in I_1$ for $1 \le i \le n$ and $\epsilon_1, \cdots, \epsilon_n$ generate an ideal of height n in $B[T]$. It would imply that $(\epsilon_1, \cdots, \epsilon_n) = I_1 \cap I_2$ where I_2 is an ideal of height $\ge n$ such that $I_2B_s[T] = B_s[T]$. Since $(s) + I_1 = B[T]$ it follows that $I_1 + I_2 = B[T]$.

We note that $B[T]/I_1 = B_s[T]/I_1B_s[T]$. Therefore the element Δ gives rise to an element $\overline{\Delta} \in \mathcal{E}_n(B[T]/I_1)$. Let $(\delta_1, \cdots, \delta_n)\overline{\Delta} = (\phi_1, \cdots, \phi_n)$. Then

$$I_1 = (\phi_1, \cdots, \phi_n) + I_1^2 \tag{7}$$

Note that $\phi_i - \epsilon_i \in I_1^2$. We have

$$I_2 = (\epsilon_1, \cdots, \epsilon_n) + I_2^2 \tag{8}$$

and $I_2(0) = I_2(0) \cap B = I_2(0) \cap I_1(0) = (\epsilon_1(0), \cdots, \epsilon_n(0))$. Therefore we actually have $I_2 = (\epsilon_1, \cdots, \epsilon_n) + (I_2^2T)$.

We now show that $t \in N(I_2; \underline{\epsilon})$. Consider the ring $B_t(T)$ and its Euler class group $E(B_t(T))$. The following hold in $E(B_t(T))$:

- $(I_1B_t(T), \omega_{(I_1,\underline{\epsilon})}) + (I_2B_t(T), \omega_{(I_2,\underline{\epsilon})}) = 0$ (from $I_1 \cap I_2 = (\epsilon_1, \cdots, \epsilon_n)$)
- $(I_1B_t(T), \omega_{(I_1,\underline{\beta})}) = (I_1B_t(T), \omega_{(I_1,\underline{\gamma})}) = (I_1B_t(T), \omega_{(I_1,\underline{\delta})}) = (I_1B_t(T), \omega_{(I_1,\underline{\phi})}) = (I_1B_t(T), \omega_{(I_1,\underline{\epsilon})})$ (from equations 3 to 7 and 2.4)
- $(IB_t(T), \omega_{(I,\underline{f})}) + (I_1B_t(T), \omega_{(I_1,\underline{\beta})}) = 0$ (from equations 1, 2)

But since $t \in N(I; \underline{f})$, we have $(IB_t(T), \omega_{(I,\underline{f})}) = 0$ in $E(B_t(T))$. Consequently, we have $(I_2B_t(T), \omega_{(I_2,\underline{\epsilon})}) = 0$ in $E(B_t(T))$. But applying [B-RS 3, Theorem 4.2] and Theorem 2.1, this would imply that $t \in N(I_2; \underline{\epsilon})$. Since I_2 contains a power of s and $1 - s = t \in N(I_2; \underline{\epsilon})$, by Lemma 2.6 it follows that

$$I_2 = (\eta_1, \cdots, \eta_n) \text{ where } \eta_i - \epsilon_i \in (I_2^2T) \tag{9}$$

Using similar computations as above we show that the element $(IB(T), \omega_{(I,\underline{f})})$ is zero in the Euler class group $E(B(T))$. We have the following equations in $E(B(T))$

- $(IB(T), \omega_{(I,\underline{f})}) = (IB(T), \omega_{(I,\underline{\alpha})}) = (IB(T), \omega_{(I,\underline{\beta})})$ (from equations 1, 2)
- $(IB(T), \omega_{(I,\underline{\beta})}) + (I_1B(T), \omega_{(I_1,\underline{\beta})}) = 0$ (from equation 2).

- $(I_1B(T), \omega_{(I_1,\underline{\beta})}) = (I_1B(T), \omega_{(I_1,\underline{\gamma})}) = (I_1B(T), \omega_{(I_1,\underline{\delta})}) = (I_1B(T), \omega_{(I_1,\underline{\phi})}) = (I_1B(T), \omega_{(I_1,\underline{\epsilon})})$ (from equations 3 to 7 and 2.4)
- $(I_1B(T), \omega_{(I_1,\underline{\epsilon})}) + (I_2B(T), \omega_{(I_2,\underline{\epsilon})}) = 0$ (from equation 8)
- $(I_2B(T), \omega_{(I_2,\underline{\epsilon})}) = 0$ (from equation 9)

A simple checking shows that $(IB(T), \omega_{(I,\underline{f})}) = 0$.Therefore by [B-RS 3, Theorem 4.2] we have $IB(T) = (G_1, \cdots, G_n)$ with $G_i - f_i \in I^2B(T)$. Applying Theorem 2.1 we are done. □

Remark 2.9 In the above theorem regularity of A has only been used to apply Lemma 2.6. If $d = n$ and A is a commutative Noetherian ring containing the field of rationals, Lemma 2.6 holds without A being regular. Therefore, for such A, the same proof as above will show that $N(I; \underline{f})$ is an ideal of A.

Remark 2.10 The above theorem opens up the passage from local to global in the following way. Let $I \subset A[T]$ be an ideal as in the theorem and suppose it is given that $I = (f_1, \cdots, f_n) + (I^2T)$. Assume further that for each maximal ideal $\mathfrak{m}$ of A, the elements $f_1, \cdots, f_n$ can be lifted to a set of generators of $IA_{\mathfrak{m}}[T]$ (i.e., local solution exists for each $\mathfrak{m}$). Then Theorem 2.8 implies that $f_1, \cdots, f_n$ can be lifted to a set of generators of I (i.e., globally there is a solution). The proof is quite obvious. However we illustrate it in the next theorem (2.11).

As a first application of the above theorem we have the following result.

Theorem 2.11 *Let A be a regular domain of dimension d and n be an integer such that $2n \geq d+3$. Let $I \subset A[T]$ be an ideal of height n for which it is given that $I = (f_1, \cdots, f_n) + (I^2T)$. Assume further that there exist $G_1, \cdots, G_n \in IA(T)$ such that $IA(T) = (G_1, \cdots, G_n)$ where $G_i - f_i \in I^2A(T)$. Then there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ where $g_i - f_i \in (I^2T)$.*

Proof All we need to show that the ideal $N(I; \underline{f})$ of A is actually A. Suppose, if possible, that $N(I; \underline{f})$ is proper. Let $\mathfrak{m}$ be a maximal ideal of A containing $N(I; \underline{f})$. Now over $A_{\mathfrak{m}}[T]$ we have $IA_{\mathfrak{m}}[T] = (f_1, \cdots, f_n) + (I^2T)A_{\mathfrak{m}}[T]$. It follows from 2.1 that $f_1, \cdots, f_n$ can be lifted to a set of generators of $IA_{\mathfrak{m}}[T]$. Then it follows easily that there is $s \in A - \mathfrak{m}$ and $h_1, \cdots, h_n \in IA_s[T]$ such that $IA_s[T] = (h_1, \cdots, h_n)$ with $h_i - f_i \in (I^2T)A_s[T]$. By definition of $N(I; \underline{f})$, it means that $s \in N(I; \underline{f})$. As $s \notin \mathfrak{m}$, this is a contradiction and the result follows. □

Remark 2.12 As we noticed earlier that if A is a commutative Noetherian ring containing $\mathbb{Q}$ and $d = n$, then $N(I; \underline{f})$ is an ideal of A. The proof of 2.11 then gives an alternative proof of the main theorem [D1, 3.10].

Another application is the following proof of Nori's conjecture as done in [B-K, Theorem 4.13].

Theorem 2.13 *Let A be a smooth affine domain of dimension d over an infinite perfect field k and n be an integer such that $2n \geq d + 3$. Let $I \subset A[T]$ be an ideal of height n for which it is given that $I = (f_1, \cdots, f_n) + (I^2T)$. Then there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ where $g_i - f_i \in (I^2T)$.*

Proof Method of proof is the same as the above theorem. We only need to note that for a maximal ideal $\mathfrak{m}$ of A, by a result of Mandal-Varma [M-V] one has $IA_{\mathfrak{m}} = (h_1, \cdots, h_n)$ with $h_i - f_i \in (I^2T)A_{\mathfrak{m}}[T]$ $\square$

We will see another application of 2.8 in proving a "local global principle" for the Euler class groups in the next section.

3 Euler class group of $A[T]$

Let A be a ring of dimension d. In [D1] the notion of the dth Euler class group of $A[T]$ was first introduced. In this section we define and study the nth Euler class group of $A[T]$ when A is a regular ring and $2n \geq d + 3$.

So let A be a regular domain of dimension d and n be an integer such that $2n \geq d+3$. Let $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Two surjections $\alpha, \beta : (A[T]/I)^n \twoheadrightarrow I/I^2$ are said to be *related* if there exists an elementary matrix $\sigma \in \mathcal{E}_n(A[T]/I)$ such that $\alpha\sigma = \beta$. It easily follows that this defines an equivalence relation on the set of surjections from $(A[T]/I)^n$ to I/I^2. Further, since $\mathcal{E}_n(A[T]) \longrightarrow \mathcal{E}_n(A[T]/I)$ is surjective, it is easy to see that if a surjection α can be lifted to a surjective map $\theta : A[T]^n \twoheadrightarrow I$ then the same can be done for any β related to α.

Let G be the free abelian group on the set of pairs (I, ω_I) where $I \subset A[T]$ is an ideal of height n with the property that Spec $(A[T]/I)$ is connected and I/I^2 is generated by n elements, and $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ represents an equivalence class of surjections.

Let I be any ideal of $A[T]$ of height n such that I/I^2 is generated by n elements. Consider the unique decomposition, $I = I_1 \cap \cdots \cap I_k$, where each of Spec$(A[T]/I_i)$ is connected (for a proof see [B-RS 3] or [D1]), pairwise comaximal and $\operatorname{ht} I_i = n$.

Now if ω_I is a representative of a class of surjections then it naturally gives rise to $\omega_{I_i} : (A[T]/I_i)^n \twoheadrightarrow I_i/I_i^2$ for $1 \leq i \leq k$. By (I, ω_I) we mean the element $\sum(I_i, \omega_{I_i}) \in G$.

Let H be the subgroup of G generated by the set of pairs (I, ω_I) in G such that ω_I is induced by a surjection $\theta : (A[T]/I)^n \twoheadrightarrow I$. We define the nth Euler class group of $A[T]$ as $E^n(A[T]) = G/H$.

Now we prove some results on $E^n(A[T])$. These results were proved in detail in [D1] and [D2] for the case $d = n$ (and commutative Noetherian A containing $\mathbb{Q}$). Here instead of repeating those proofs we will either skip or sketch a proof or only indicate the subtle differences.

The first result we would like to have for $E^n(A[T])$ is the following.

Theorem 3.1 *Let A be a regular domain of dimension d and n be an integer such that $2n \geq d+3$. Let $I \subset A[T]$ be an ideal of height n and let $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ represent an equivalence class of surjections. Suppose that the image of (I, ω_I) is zero in $E^n(A[T])$. Then ω_I can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$.*

To prove this theorem we need some preparatory results. For instance, we need a group homomorphism from $E^n(A[T])$ to $E^n(A)$. To define such a group homomorphism, we need to improve the addition and subtraction principles of [B-RS 3] in the following forms.

Proposition 3.2 (Addition principle) *Let A be a Noetherian ring of dimension d and n be an integer such that $2n \geq d+3$. Let I, J be two comaximal ideals of A, each of height $\geq n-1$. Suppose it is given that $I = (a_1, \cdots, a_n)$ and $J = (b_1, \cdots, b_n)$. Then $I \cap J = (c_1, \cdots, c_n)$ where $c_i - a_i \in I^2$ and $c_i - b_i \in J^2$.*

Proof We follow the proof of [B-RS 3] and give a quick sketch. First of all, it is clear that elementary transformations on $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ are permissible. Let $B = A/(b_1, \cdots, b_n)$ and bar denote reduction modulo $(b_1, \cdots, b_n)$. Then $\dim B \leq d-n+1$. Note that $(\overline{a}_1, \cdots, \overline{a}_n)$ is a unimodular row on B. Since we have $n \geq \dim B + 2$, it follows that $(\overline{a}_1, \cdots, \overline{a}_n)$ can be elementarily transformed to $(\overline{1}, \overline{0}, \cdots, \overline{0})$. Applying [RS, Lemma 2] we can apply an elementary transformation and assume that $\operatorname{ht}(a_1, \cdots, a_{n-1}) = n-1$. Note that this transformation preserves the fact that $a_1 \equiv 1$ modulo J.

Now let $C = A/(a_1, \cdots, a_{n-1})$ and let bar denote reduction modulo $(a_1, \cdots, a_{n-1})$. Clearly the row $(\overline{b}_1, \cdots, \overline{b}_n)$ is unimodular over C. Similar arguments as above finally yields that (1) $(a_1, \cdots, a_{n-1}) + (b_1, \cdots, b_{n-1}) = A$ and (2) $\operatorname{ht}(a_1, \cdots, a_{n-1}) = \operatorname{ht}(b_1, \cdots, b_{n-1}) = n-1$.

In $A[T]$ consider the two ideals $I_1 = (a_1, \cdots, a_{n-1}, T+a_n)$ and $I_2 = (b_1, \cdots, b_{n-1}, T+b_n)$. Let $K = I_1 \cap I_2$. Then $\dim A[T]/K \le d - n + 1$. Using the Chinese remainder theorem we can choose $g_1(T), \cdots, g_n(T) \in K$ such that $K = (g_1(T), \cdots, g_n(T)) + K^2$ satisfying $g_i(T) \equiv a_i \bmod I_1^2$, $g_i(T) \equiv b_i \bmod I_2^2$, $1 \le i \le n-1$; $g_n(T) \equiv T + a_n \bmod I_1^2$, $g_n(T) \equiv T + b_n \bmod I_2^2$.

Applying a theorem of Mandal [M], it is now easy to conclude that there exist $h_1(T), \cdots, h_n(T)$ such that $K = (h_1(T), \cdots, h_n(T))$ where $h_i(T) \equiv g_i(T) \bmod K^2$. Take $c_i = h_i(0)$ Then $I \cap J = (c_1, \cdots, c_n)$ with the required properties. □

Proposition 3.3 (Subtraction principle) *Let A be as above and $I, J \subset A[T]$ be two co-maximal ideals, each of height $\ge n-1$. Suppose it is given that $I = (a_1, \cdots, a_n)$ and $I \cap J = (c_1, \cdots, c_n)$ where $c_i \equiv a_i$ mod I^2. Then there exist $b_1, \cdots, b_n \in J$ such that $J = (b_1, \cdots, b_n)$ with $b_i \equiv c_i$ mod J^2.*

Proof The proof follows the method of [B-RS 3] with necessary modifications as illustrated in the proof of the above proposition. □

Using the above two propositions together with [B-RS 3, 2.4] we have the following result, whose proof is same as [D2, 3.3] and hence omitted.

Theorem 3.4 *Let A be as above. There is a group homomorphism $\Psi : E^n(A[T]) \longrightarrow E^n(A)$ such that if $(I, \omega_I) \in E^n(A[T])$ has the property that $I(0)$ is an ideal of A of height n, then $\Psi((I, \omega_I)) = (I(0), \omega_{I(0)})$ in $E^n(A)$, where $\omega_{I(0)}$ is the surjection induced by ω_I. If $I(0) = A$, then $\Psi((I, \omega_I)) = 0$.*

Remark 3.5 It is easy to see that Ψ is surjective.

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: We have $(I, \omega_I) = 0$ in $E^n(A[T])$. Suppose that ω_I is given by $I = (f_1, \cdots, f_n) + I^2$. We first assume that $I(0)$ is a proper ideal of A. We have $I(0) = (f_1(0), \cdots, f_n(0)) + I(0)^2$. Suppose that $\Psi((I, \omega_I)) = -(K, \omega_K)$, where K is an ideal of A of height $\ge n$ such that $K \cap I(0) = (c_1, \cdots, c_n)$ where $c_i \equiv f_i(0) \bmod I(0)^2$ and ω_K is induced by $c_1, \cdots, c_n$. Since $(I, \omega_I) = 0$ in $E^n(A[T])$, we have $\Psi((I, \omega_I)) = 0$ in $E(A)$ and therefore, $(K, \omega_K) = 0$ in $E^n(A)$. his implies, by [B-RS 2], that $K = (a_1, \cdots, a_n)$ such that $a_i \equiv c_i \bmod K^2$. Now applying the subtraction principle given above we see that $I(0) = (b_1, \cdots, b_n)$ such that $b_i \equiv c_i \bmod I(0)^2$. Therefore, $b_i \equiv f_i(0) \bmod I(0)^2$. Now applying [B-RS 1] we can lift ω_I to a surjection $\alpha : A[T]^n \twoheadrightarrow I/(I^2T)$.

The element $(IA(T), \omega_{IA(T)}) \in E^n(A(T)$ is zero. It follows from [B-RS 3] that $\omega_{IA(T)}$ and hence $\alpha \otimes A(T)$ can be lifted to a set of generators of $IA(T)$. Now we can apply 2.11 to conclude that α can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$. Clearly θ lifts ω_I.

The case $I(0) = A$ is easier and along the same line as above. $\square$

Now we derive the following theorem whose assertion is quite natural.

Theorem 3.6 *There is a canonical group homomorphism $\Phi : E^n(A) \longrightarrow E^n(A[T])$ such that Φ is injective. Moreover, the composition $\Psi\Phi$ is identity on $E^n(A)$.*

Proof Let $(J, \omega_J) \in E^n(A)$. Define $\Phi((J, \omega_J) = (J[T], \omega_{J[T]})$, where $J[T]$ is the ideal of $A[T]$ which is extended from J and $\omega_{J[T]} : (A[T]/J[T])^n \twoheadrightarrow J[T]/J[T]^2$ is the surjection naturally induced by ω_J.

Assume that $(J[T], \omega_{J[T]}) = 0$ in $E^n(A[T])$. By 3.1 $\omega_{J[T]}$ has a lift $\theta : A[T]^n \twoheadrightarrow J[T]$. Clearly $\theta(0) : A^n \twoheadrightarrow J$ lifts ω_J. Therefore Φ is injective.

From the way Φ and Ψ are defined, it is clear that $\Psi\Phi = \mathrm{id}_{E^n(A)}$. $\square$

It is natural to ask the following question.

Question 3.7 Is $\Phi : E^n(A) \hookrightarrow E^n(A[T])$ an isomorphism?

We do not know any example of a regular domain A for which Φ fails to be an isomorphism. The following is the best result we have in this regard.

Theorem 3.8 *Let A be a smooth affine domain of dimension d over an infinite perfect field and n be an integer such that $2n \geq d + 3$. Then $\Phi : E^n(A) \hookrightarrow E^n(A[T])$ is an isomorphism.*

We will derive the proof from the following "local global principle" for the Euler class groups, which in turn is derived from Theorem 2.8.

Theorem 3.9 *Let A be a regular domain of dimension d and n be an integer such that $2n \geq d + 3$. Then we have the following exact sequence of groups*

$$0 \longrightarrow E^n(A) \longrightarrow E^n(A[T]) \longrightarrow \prod_{\mathfrak{m}} E^n(A_{\mathfrak{m}}[T]),$$

where the product runs over all maximal ideals $\mathfrak{m}$ of A.

Proof All we need to prove is that if $(I, \omega_I) \in E^n(A[T])$ is such that its image in $E^n(A_{\mathfrak{m}}[T])$ is zero for each maximal ideal $\mathfrak{m}$ of A, then there is an element $(J, \omega_J) \in E^n(A)$ such that $(J[T], \omega_{J[T]}) = (I, \omega_I)$ in $E^n(A[T])$.

Suppose ω_I is given by $I = (f_1, \cdots, f_n) + I^2$. Then $I(0) = (f_1(0), \cdots, f_n(0)) + I(0)^2$. Using some standard general position arguments we can find an ideal K of A of height

n and $a_1, \cdots, a_n \in I(0)$ such that $K + I \cap A = A$ and $I(0) \cap K = (a_1, \cdots, a_n)$, where $a_i \equiv f_i(0) \bmod I(0)^2$.

We have $K = (a_1, \cdots, a_n) + K^2$ and let $(K, \omega_K) \in E^n(A)$ be the corresponding element. Let $I_1 = I \cap K[T]$. Since $I + K[T] = A[T]$, applying the Chinese remainder theorem we see that ω_I and ω_K will induce $\omega_{I_1} : (A[T]/I_1)^n \twoheadrightarrow I_1/I_1^2$. Suppose ω_{I_1} is given by $I_1 = (g_1, \cdots, g_n) + I_1^2$. Note that $I_1(0) = I(0) \cap K = (a_1, \cdots, a_n)$ and we have $g_i(0) \equiv a_i \bmod I_1(0)^2$. Therefore we can lift $g_1, \cdots, g_n$ to a set of generators of $I_1/(I_1^2 T)$, which also corresponds to ω_{I_1}. In $E^n(A[T])$ we have the following equation

$$(I_1, \omega_{I_1}) = (I, \omega_I) + (K[T], \omega_{K[T]}).$$

Let $\mathfrak{m}$ be a maximal ideal of A. If we go to $E^n(A_{\mathfrak{m}}[T])$ the above equation and the assumption on (I, ω_I) would imply that $(I_1, \omega_{I_1}) = 0$ in $E^n(A_{\mathfrak{m}}[T])$. Since ω_{I_1} is actually induced by a set of generators of $I_1/(I_1^2 T)$, by 2.10 it follows that $(I_1, \omega_{I_1}) = 0$ in $E^n(A[T])$. Therefore, $(I, \omega_I) = -(K[T], \omega_{K[T]})$ in $E^n(A[T])$, implying that (I, ω_I) comes from $E^n(A)$. $\square$

Remark 3.10 As remarked earlier, if $n = d$, one does not need the regularity assumption in Theorem 2.8. Therefore, if A is a commutative Noetherian ring of dimension d containing $\mathbb{Q}$, the local global principle proved in [D1] can also be proved adapting the proof given above.

Proof of Theorem 3.8 : Let A be a smooth affine domain of dimension d and n be an integer such that $2n \geq d+3$. Let $\mathfrak{m}$ be a maximal ideal of A. It is an obvious consequence of the main result of Mandal-Varma [M-V], that $E^n(A_{\mathfrak{m}}[T]) = 0$. Now it follows from Theorem 3.9 that $\Phi : E^n(A) \hookrightarrow E^n(A[T])$ is surjective and hence an isomorphism. $\square$

4 Analogue of Roitman's theorem

Roitman [Ro, Proposition 2] proved the following result which is, in some sense, a converse of the Quillen localization theorem.

Proposition 4.1 *Let R be a commutative Noetherian ring and $S \subset R$ be multiplicatively closed. Suppose that all projective $R[X]$-modules are extended from R. Then all projective $R_S[X]$-modules are extended from R_S.*

In [D1] and Theorem 3.9 above we proved analogue of Quillen's result for the Euler class groups. Here we prove an analogue of Roitman's result. We first prove it in the case when R is a commutative Noetherian ring containing $\mathbb{Q}$ and $n = d = \dim R$ (the set up of [D1]). More precisely, we prove the following theorem.

Theorem 4.2 *Let R be a commutative Noetherian ring of dimension $n \geq 2$ containing $\mathbb{Q}$. Let $S \subset R$ be multiplicatively closed. Suppose that the canonical map $\phi : E^n(R) \longrightarrow E^n(R[X])$ is surjective. Then the canonical map $\phi_S : E^n(R_S) \longrightarrow E^n(R_S[X])$ is also surjective.*

Remark 4.3 The canonical maps ϕ and ϕ_S are always injective [D1, D2]. So the above theorem actually tells that ϕ_S is an isomorphism if ϕ is an isomorphism.

Remark 4.4 The proof given below is motivated by the proof of [Ra, Proposition 1].

Proof We have the following exact sequence of abelian groups

$$0 \longrightarrow E^n(R_S) \longrightarrow E^n(R_S[X]) \longrightarrow \prod_m E^n((R_S)_m[X]),$$

where m is a maximal ideal of R_S of height n. To prove the theorem, it is enough to show that $E^n((R_S)_m[X]) = 0$ for each such m. Since m is a maximal ideal of R which avoids S, we are reduced to showing that under the hypothesis of the theorem, $E^n(R_m) \longrightarrow E^n(R_m[X])$ is surjective. Since $E^n(R_m) = 0$, we need only prove that $E^n(R_m[X]) = 0$.

Let $(I, \omega_I) \in E^n(R_m[X])$ be an arbitrary element. We may apply a moving lemma [D1, Lemma 6.2] and assume that $I(0) = R_m$. Suppose ω_I is induced by the following set of generators of I/I^2

$$I = (f_1(X), \cdots, f_n(X)) + I^2.$$

Since $I(0) = R_m$, by [B-RS 1, Remark 3.9] we can lift the above set of generators to a set of n generators of $I/(I^2X)$. Let us still call them as f_i. So we have $R_m = I(0) = (f_1(0), \cdots, f_n(0))$. Since R_m is local, the unimodular row $(f_1(0), \cdots, f_n(0))$ can be transformed elementarily to $(1, 0, \cdots, 0)$. Applying this elementary transformation on $(f_1(X), \cdots, f_n(X))$ we may further assume that $f_1(X) = 1$ modulo X and $f_i(X) = 0$ modulo X for $i = 2, \cdots, n$. Therefore, ω_I is induced by the following (now we rename the generators):

$$I = (1 + Xv_1(X), Xv_2(X), \cdots, Xv_n(X)) + I^2.$$

We may adjust so that there is $s \in R \smallsetminus m$ and $v_i(X) \in R_s[X]$ for all i. Write $v_i(X) = u_i(X)/s^t$, for some $u_i(X) \in R[X]$ and integer $t \geq 0$. Since I is finitely generated, we may actually assume that $I \subset R_s[X]$. Let $Y = X/s^{t+1}$.

We have

$$I = (1 + sX\frac{u_1(X)}{s^{t+1}}, sX\frac{u_2(X)}{s^{t+1}}, \cdots, sX\frac{u_n(X)}{s^{t+1}}) + I^2$$

and rewriting with Y one gets

$$I = (1 + sYu_1(s^{t+1}Y), sYu_2(s^{t+1}Y), \cdots, sYu_n(s^{t+1}Y)) + I^2.$$

Now write $J = I \cap R[Y]$. Then $J_s = I$ and since J contains $1 + sYu_1(s^{t+1}Y)$, we observe that $J + (s) = R[Y]$. Further, note that the ring $R[Y]/J$ is isomorphic to $R_s[X]/I$ and consequently the modules J/J^2 and I/I^2 are isomorphic. Therefore,

$$J = (1 + sYu_1(s^{t+1}Y), sYu_2(s^{t+1}Y), \cdots, sYu_n(s^{t+1}Y)) + J^2.$$

Let ω_J denote the corresponding local orientation of J. Clearly the image of (J, ω_J) under the canonical map $E^n(R[Y]) \longrightarrow E^n(R_s[Y]) = E^n(R_s[X])$ is (I, ω_I). Now by the hypothesis of the theorem, (J, ω_J) is in the image of the canonical map $E^n(R) \longrightarrow E^n(R[Y])$. Therefore the element $(I, \omega_I) \in E^n(R_m[X])$ is actually induced by an element of $E^n(R)$. But this implies that $(I, \omega_I) = 0$ in $E^n(R_m[X])$. This proves the theorem. $\square$

It is easy to see that adapting the same proof one can obtain the analogous result in the case of the nth Euler class groups with the base ring being a d dimensional regular domain. We state the result.

Theorem 4.5 *Let R be a regular domain of dimension d containing a field and n be an integer such that $2n \geq d+3$. Let $S \subset R$ be multiplicatively closed. Suppose that the canonical map $\phi : E^n(R) \longrightarrow E^n(R[X])$ is surjective. Then the canonical map $\phi_S : E^n(R_S) \longrightarrow E^n(R_S[X])$ is also surjective.*

Now let K be the cokernel of the map ϕ and L be the cokernel of ϕ_S. In view of the above theorems it is natural to ask whether K and L are isomorphic. The answer is in the negative. To see this let R be a non-smooth affine domain over an infinite perfect field and let $s \in R$ be an element which belongs to the singular locus of R. Then R_s is smooth and in that case it follows that $E^n(R_s) \simeq E^n(R_s[X])$. Consequently, L is trivial. But K can be non-trivial. For such an example see [B-RS 1, Example 6.4].

References

[B] S. M. Bhatwadekar, Some results on a question of Quillen, In : Proc. Internat. Bombay Colloquium on Vector Bundles on Algebraic Varieties, Oxford Univ. Press, 1987, 107-125.

[B-K] S. M. Bhatwadekar and M. K. Keshari, A question of Nori : Projective generation of ideals, *K-Theory* **28** (2003), 329-351.

[B-RS 1] S. M. Bhatwadekar and Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain and a question of Nori, *Invent. Math.* **133** (1998), 161-192.

[B-RS 2] S. M. Bhatwadekar and Raja Sridharan, The Euler class group of a Noetherian ring, *Compositio Math.* **122** (2000), 183-222.

[B-RS 3] S. M. Bhatwadekar and Raja Sridharan, On Euler classes and stably free projective modules, in: Algebra, arithmetic and geometry, Part I, II(Mumbai, 2000), 139–158, Tata Inst. Fund. Res. Stud. Math., 16, Tata Inst. Fund. Res., Bombay, 2002.

[D1] M. K. Das, The Euler class group of a polynomial algebra, *J. Algebra* **264** (2003), 582-612.

[D2] M. K. Das, The Euler class group of a polynomial algebra II, *J. Algebra* **299** (2006), 94-114.

[D-RS] M. K. Das and Raja Sridharan, The Euler class groups of polynomial rings and unimodular elements in projective modules, *J. Pure and Appl. Algebra* **185** (2003), 73-86.

[M] S. Mandal, On efficient generation of ideals, *Invent. Math.* **75** (1984), 59-67.

[M 2] S. Mandal, Homotopy of sections of projective modules, *J. Algebraic Geometry* **1** (1992), 639-646.

[M-V] S. Mandal and P. L. N. Varma, On a question of Nori: the local case, *Communications in Algebra* **25** (1997), 451-457.

[Ra] R. A. Rao, A note on the Serre dimension of polynomial rings, *J. Pure and Appl. Algebra* **38** (1985), 87-90.

[RS] Raja Sridharan, Non-vanishing sections of algebraic vector bundles,*J. Algebra* **176** (1995), 947-958.

[Ro] M. Roitman, On projective modules over polynomial rings, J. Algebra 58 (1979), 51-63.