ON INVARIANCE OF THE EULER CLASS GROUPS UNDER A SUBINTEGRAL BASE CHANGE

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1. INTRODUCTION

Let R be a commutative Noetherian ring. An extension $R \hookrightarrow S$ is called subintegral if: (1) it is integral, (2) the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is bijective, and (3) the induced field extensions $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow S_{\mathfrak{P}}/\mathfrak{P}S_{\mathfrak{P}}$ are all trivial, where $\mathfrak{P} \in \mathrm{Spec}(S)$ and $\mathfrak{p} = \mathfrak{P} \cap R$. Subintegral ring extensions, apart from their intrinsic appeal, played an important role in studying projective modules. As evidence, we mention a few results below.

Let $R \hookrightarrow S$ be a subintegral extension and P be a finitely generated projective R-module. In [I], where Ischebeck studied the behaviour of some K-theoretic functors under the extension $R \hookrightarrow S$, there is a result [I, Proposition 8] which asserts that P is free if and only if $P \otimes_R S$ is a free S-module (P is assumed to have trivial determinant). Later, while answering a conjecture of Murthy, Swan proved in [Sw2, 14.1] that if the module $P \otimes_R S$ has a direct sum decomposition into projective S-modules, then there is a similar decomposition for P (see [Sw2] for the precise statement). We now mention another result which inspired us to investigate the questions we are going to describe in a while. In [B 1], to address some question on existence of unimodular elements, Bhatwadekar implicitly proves that if $P \otimes_R S \simeq Q' \oplus S$ for some S-module Q', then there is an R-module Q such that $P \simeq Q \oplus R$ (in other words, P has a unimodular element if and only if so does $P \otimes_R S$). This result is not mentioned anywhere but it can be derived using the techniques of [B 1].

We now digress a bit. Let A be a commutative Noetherian ring of dimension $n \geq 2$. Let P be a projective A-module of rank n and for simplicity, assume that the determinant of P is trivial. Fix $\chi : A \xrightarrow{\sim} \wedge^n(P)$. By a theorem of Eisenbud-Evans [E-E], there exists a surjective map $\alpha : P \twoheadrightarrow J$, where $J \subset A$ is an ideal of height n. The map α will induce a surjection $\overline{\alpha} : P/JP \twoheadrightarrow J/J^2$. As $\dim(A/J) = 0$, the A/J-module P/JP is free. Choose an isomorphism $\sigma : (A/J)^n \xrightarrow{\sim} P/JP$ such that $\wedge^n(\sigma) = \chi \otimes A/J$ and take the composite surjection $\omega_J := \overline{\alpha}\sigma : (A/J)^n \xrightarrow{\sim} P/JP \twoheadrightarrow J/J^2$. A remarkable result of

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Bhatwadekar and Raja Sridharan [B-RS 3, 4.4] asserts that if $\mathbb{Q} \subset A$, then $P \simeq Q \oplus A$ for some A-module Q if and only if ω_J can be lifted to a surjection $\theta : A^n \twoheadrightarrow J$. This phenomenon is formalized in the theory of the Euler class groups in [B-RS 1, B-RS 3].

We now assume that $R \hookrightarrow S$ is a subintegral extension with $\dim(R) = n \geq 2$. Let $J \subset R$ be an ideal of height n such that $\mu(J/J^2) = n$, where $\mu(-)$ stands for the minimal number of generators. Assume that we are given: $J = (a_1, \cdots, a_n) + J^2$. Taking a cue from Bhatwadekar's (unstated) result in [B 1], and the discussion above, we may ask the following question.

Question 1.1. Assume further that $JS = (\beta_1, \cdots, \beta_n)$ such that $\beta_i - a_i \in J^2S$ for $i = 1, \cdots, n$. Then, can we find $b_1, \cdots, b_n \in J$ such that $J = (b_1, \cdots, b_n)$ with $b_i - a_i \in J^2$ for $i = 1, \cdots, n$?

Note that the generators of J/J^2 may not have been induced by a surjection from a projective R-module as there are examples of rings R of dimension n and ideals J of height n such that $\mu(J/J^2) = n$ but J is not even surjective image of a projective R-module. Therefore a combination of [B 1] and [B-RS 3, 4.4] would not work, whereas, an affirmative answer to Question 1.1 would imply Bhatwadekar's (unstated) result in [B 1] (provided $\mathbb{Q} \subset R$). A reader familiar with the Euler class groups will readily understand that we are essentially asking if the natural map from the n-th Euler class group $E^n(R)$ to the n-th Euler class group $E^n(S)$ is injective or not. It requires some arguments (see (3.8)) to ascertain that there is a natural map $\Phi : E^n(R) \to E^n(S)$. For brevity, we write $E(R)$ for $E^n(R)$ and $E(S)$ for $E^n(S)$. In this paper we answer Question 1.1 in the affirmative. In fact, we prove the following result (see (3.11, 4.4) below), thus settling a query of Gubeladze expressed in [G, Remark 5].

Theorem 1.2. *The natural map $\Phi : E(R) \longrightarrow E(S)$ is an isomorphism.*

Before describing our other results on subintegral extensions, a crucial remark is in order. One may wonder if the above theorem can be extended to the case when $R \hookrightarrow S$ is an integral extension. In Section 4 we first show that if $R \hookrightarrow S$ is integral, then there is a group homomorphism $\widetilde{\Phi} : E(R) \longrightarrow E(S)$. We then prove in (4.4) that if R_{red} and S_{red} are birational, then $\widetilde{\Phi}$ is surjective. Further, if $R_{\text{red}} \hookrightarrow S_{\text{red}}$ is subintegral then $\widetilde{\Phi}$ is an isomorphism. Moreover, in (4.5), an example of a finite birational extension $R \hookrightarrow S$ is given, which is not subintegral, such that $E(R)$ is not isomorphic to $E(S)$. Therefore, (1.2) does not extend to integral extensions of rings.

There is this notion of the Euler class group $E(R, L)$ of R with respect to a line bundle L, defined in [B-RS 3]. Again assume that $R \hookrightarrow S$ is subintegral. In (3.12) we prove that $E(R, L)$ is isomorphic to $E(S, L \otimes_R S)$ as well. We prove further that the natural map $\Phi_0 : E_0(R) \longrightarrow E_0(S)$ is an isomorphism if $\dim(R)$ is even (3.25), or if R is an affine algebra over a C_1-field of characteristic zero (3.28), where $E_0(R)$ and $E_0(S)$ are the weak Euler class groups of R and S, respectively (see Section 2 for the definition of the weak Euler class group).

An interesting offshoot of (1.2) is that if R is an affine algebra over a C_1-field of characteristic zero, and if $J \subset R$ is an ideal of height n with $\mu(J/J^2) = n$, then $\mu(J) = n$ if and only if $\mu(JS) = n$. We prove this result in (3.28).

Recall that for a reduced ring A, there is a maximal subintegral extension contained in the total ring of fractions of A. This is called the *seminormalization* of A and is denoted by ${}^+A$ (see [Sw1] for details). From the category of commutative Noetherian rings we have the seminormalization functor, $R \mapsto {}^+(R_{\text{red}})$, to the category of seminormal rings. As a consequence of (1.2), we conclude that the Euler class group behaves well with respect to this functor in the sense that $E(R) \simeq E({}^+(R_{\text{red}}))$.

In Section 5 we answer a question of Ischebeck from [I] for a subintegral extension $R \hookrightarrow S$ for $\dim(R) = 2$, which says that if P, Q are projective R-modules of rank 2 with an isomorphism $\chi : \wedge^2(P) \xrightarrow{\sim} \wedge^2(Q)$, and if there is an isomorphism $\theta : P \otimes_R S \xrightarrow{\sim} Q \otimes_R S$ such that $\wedge^2(\theta) = \chi \otimes S$, then $P \simeq Q$.

The main results constitute Section 3. Although the results are presented in the language of the Euler class group theory, but to appreciate them and their proofs a prior knowledge of the Euler class groups is not really necessary. For readers unfamiliar with this language, a quick look at the latter half of Section 2 should be just enough.

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2. Preliminaries

All the rings considered in this paper are commutative and Noetherian. By dimension of a ring we mean its Krull dimension. Modules are assumed to be finitely generated. Projective modules are assumed to have constant rank.

We start with the following definition.

Definition 2.1. Let R be a ring and P be a projective R-module. An element $p \in P$ is called *unimodular* if there is a surjective R-linear map $\phi : P \twoheadrightarrow R$ such that $\phi(p) = 1$. The set of all unimodular elements of P is denoted by $\mathrm{Um}(P)$. If $P = R^n$, then we write $\mathrm{Um}_n(R)$ for $\mathrm{Um}(R^n)$.

Remark 2.2. It is easy to see that if a projective R-module P has a unimodular element, then $P \simeq Q \oplus R$ for some R-module Q. We describe this phenomenon by saying that P splits off a free summand of rank one.

The following result is due to Serre [Se]. We shall refer to this result as "Serre's splitting theorem".

Theorem 2.3. *Let R be a ring and P be a projective R-module. If rank$(P) \geq$ dim$(R)+1$, then P splits off a free summand of rank one, i.e., $P \simeq Q \oplus R$ for some R-module Q.*

Let P be a projective R-module of rank n and ϕ be an R-linear endomorphism of P. Then ϕ induces an endomorphism $\wedge^n(\phi)$ of $\wedge^n(P)$ in a natural way. We call $\wedge^n(P)$ the *determinant* of P. We call the endomorphism $\wedge^n(\phi)$ the determinant of ϕ and denote it by $\det(\phi)$. As $\wedge^n(P)$ is a projective R-module of rank one, $\det(\phi) \in R$. It can be easily checked that ϕ is an automorphism if and only if $\det(\phi)$ is a unit of R.

Definition 2.4. Let P be a projective R-module. We define $SL(P)$ to be the group of automorphisms of P of determinant one. If $P = R^n$, then we write $SL_n(R)$ for $SL(R^n)$.

We now recall the definition of a subgroup of $SL(P)$. Given a $\varphi \in P^*(= \mathrm{Hom}_R(P, R))$ and a $p \in P$, we define an endomorphism φ_p of P as the composite $P \xrightarrow{\varphi} R \xrightarrow{p} P$. If $\varphi(p) = 0$, then $\varphi_p^2 = 0$ and $1 + \varphi_p$ is an automorphism of P.

Definition 2.5. An automorphism of P is called a *transvection* if it is of the form $1 + \varphi_p$ where $\varphi(p) = 0$ and either φ is unimodular in P^* or p is unimodular in P. The subgroup of $SL(P)$ generated by all transvections will be denoted by $\mathcal{E}(P)$. If $n \geq 3$ and $P = R^n$, then, by a result of Suslin [Su 1, 1.4], $\mathcal{E}(R^n)$ can be identified with $\mathcal{E}_n(R)$, the group of $n \times n$ elementary matrices.

Proposition 2.6. *Let R be a ring and P be a projective R-module such that P has a unimodular element. Let $\alpha, \beta \in Um(P^*)$ be such that $\alpha \equiv \beta$ modulo the nil radical $\mathfrak{n}$ of R. Then there is $\theta \in \mathcal{E}(P)$ such that $\beta = \alpha\theta$.*

Proof. Applying [MK-M-R, 2.3] it follows that there is $\Theta \in \mathcal{E}(P^*)$ such that $\Theta(\alpha) = \beta$. Therefore, Θ is a finite product of transvections of the projective module P^*. For simplicity, we prove this proposition by assuming that Θ itself is a transvection. The general case can be worked out in a similar manner.

Let $\Theta = 1 + \psi_\phi$, where $\psi \in P^{**}$ and $\phi \in P^*$ such that $\psi(\phi) = 0$. Since P is a projective module, P^{**} can be identified with P. Therefore, we may assume $\psi = p$ for some $p \in P$. With this identification we have $\psi(\phi) = \phi(p) = 0$. Now from the definition of a transvection, we have, either $\psi \in Um(P^{**})$ or $\phi \in Um(P^*)$. If $\psi \in Um(P^{**})$, then note that $p \in \mathrm{Um}(P)$. Therefore $1 + \phi_p$ is a transvection of P (as $\phi(p) = 0$).

Now we have $\Theta(\alpha) = (1 + \psi_\phi)(\alpha) = \beta$. Therefore, for any $q \in P$, we have $(1 + \psi_\phi)(\alpha)(q) = \beta(q)$. But $(1 + \psi_\phi)(\alpha)(q) = \alpha(q) + (\phi\psi)(\alpha)(q) = \alpha(q) + (\phi\alpha(p))(q) = \alpha(q) + \alpha(p)\phi(q)$.

On the other hand, $\alpha(1 + \phi_p)(q) = \alpha(q + p\phi(q)) = \alpha(q) + \alpha(p)\phi(q)$ and hence $\alpha(1 + \phi_p)(q) = \beta(q)$ for all $q \in P$. Therefore, if we write $\theta = 1 + \phi_p$, then θ is a transvection of P and $\alpha\theta = \beta$. □

Let S be a ring and C be an ideal of S. Let P be a projective S-module. A result of Bhatwadekar and Roy [B-R, 4.1] asserts that any transvection σ of P/CP can be lifted to a unipotent automorphism of P. We need a variant of their result in the following form, which was told to us by Bhatwadekar (private communication).

Proposition 2.7. *Let S be a ring and J, C be ideals of S such that $J + C = S$. Let P be a projective S-module and σ be a transvection of P/CP. Then σ can be lifted to $\tau \in Aut(P)$ with the property that τ is identity modulo J.*

Proof. Let $\sigma = 1+\psi_q$, where $\psi \in (P/CP)^*$ and $q \in P/CP$ such that $\psi(q) = 0$. Let $p \in P$ and $\theta \in P^*$ be lifts of q and ψ, respectively. Then we have $\theta(p) = c$, for some $c \in C$.

We first consider the case when q is a unimodular element of P/CP. Then there exists $\varphi \in P^*$ such that $\varphi(p) = 1 + d$, for some $d \in C$.

Set $\phi' = (1 + d)\theta - c\varphi$. Then $\phi'(p) = 0$ and ϕ' is a lift of ψ. Therefore $1 + \phi'_p \in \mathrm{Aut}\,(P)$ and it lifts σ. We have $J + C = S$. Therefore, there exist $a \in J$ and $b \in C$ such that $a + b = 1$. Finally we consider $\tau = 1 + a\phi'_p$. Then again $\tau \in \mathrm{Aut}\,(P)$, $\tau = \mathrm{Id}$ modulo J and τ is a lift of σ.

Next we consider the case when $\psi \in \mathrm{Um}((P/CP)^*)$. Then there exists $p' \in P$ such that $\theta(p') = 1 + e$, for some $e \in C$. Consider the element $q' = (1 + e)p + cp'$. Then $\theta(q') = 0$. Therefore, $\tau = 1 + a\theta_{q'}$ will work. □

Lemma 2.8. *Let $R \hookrightarrow S$ be an extension of rings and C be the conductor of R in S. Let $J \subset R$ be an ideal such that $J + C = R$ and $JS \neq S$. Then the natural map $f : J \otimes_R S \twoheadrightarrow JS$ is an isomorphism (of S-modules).*

Proof. We use a local-global argument to prove that f is an isomorphism. To see this, let $\mathfrak{m}$ be a maximal ideal of S and let $\mathfrak{p} = \mathfrak{m} \cap R$. First note that

$$(J \otimes_R S)_{\mathfrak{m}} = J \otimes_R S_{\mathfrak{m}} = (J \otimes_R R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{m}} = J_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{m}}.$$

If $C \not\subset \mathfrak{m}$, then $C \not\subset \mathfrak{p}$ as well, and $R_{\mathfrak{p}} = S_{\mathfrak{p}} = S_{\mathfrak{m}}$ and in this case $J_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{m}}$ and $J_{\mathfrak{p}}S_{\mathfrak{m}}$ are both isomorphic to $J_{\mathfrak{p}}$, and the isomorphism is induced by f. If $C \subset \mathfrak{m}$, then $JS \not\subset \mathfrak{m}$, $J \not\subset \mathfrak{p}$, and therefore $J_{\mathfrak{p}} = R_{\mathfrak{p}}$, $(JS)_{\mathfrak{m}} = S_{\mathfrak{m}}$. Again, it can be easily seen that f induces the isomorphism. □

The proof of the following lemma can be found in [B-RS 3, 2.13]. This is a consequence of a result of Eisenbud-Evans [E-E], as stated in [P, p. 1420]. .

Lemma 2.9. *Let A be a ring and P be a projective A-module of rank n. Let $(\alpha, a) \in (P^* \oplus A)$. Then there exists an element $\beta \in P^*$ such that* $\operatorname{ht}(I_a) \geq n$*, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq n$ then* $\operatorname{ht} I \geq n$*. Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and I is a proper ideal of A, then* $\operatorname{ht} I = n$*.*

The following lemma is standard. For a proof the reader may consult [B-RS 3, 2.11].

Lemma 2.10. *Let R be a Noetherian ring and $J \subset R$ be an ideal of R. Let $K \subset J$ and $L \subset J^2$ be two ideals of R such that $K + L = J$. Then $J = K + (e)$ for some $e \in L$ with $e(1-e) \in K$ and $K = J \cap J'$ where $J' + L = R$.*

The next lemma, which can be proved using (2.10) and (2.9), is a synthesis of [B-RS 3, 2.14] and [B-RS 5, 2.4]. We shall call this lemma as the "moving lemma".

Lemma 2.11. (Moving Lemma) *Let R be a Noetherian ring of dimension d and let P be a projective R-module of rank n, where $2n \geq d+2$. Let $J \subset R$ be an ideal of height n and let $\overline{\alpha} : P/JP \twoheadrightarrow J/J^2$ be a surjection. Then there exists an ideal $J' \subset R$ and a surjection $\beta : P \twoheadrightarrow J \cap J'$ such that:*

(1) *$J + J' = R$.*
(2) *$\beta \otimes R/J = \overline{\alpha}$.*
(3) *$ht(J') \geq n$.*
(4) *Given finitely many ideals $J_1, \cdots, J_r$ of R, each of height $\geq d-n+1$, the ideal J' can be chosen with the additional property that it is comaximal with J_i for $i = 1, \cdots, r$.*

The following proposition is implicit in the proof of [B-RS 2, 2.5].

Proposition 2.12. *Let A be a commutative Noetherian ring of dimension $d \geq 1$ and I be an ideal of $A[T]$ of height ≥ 2. Assume that $I = (f_1, \cdots, f_n) + I^2$, where $n \geq d+1$. Then there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$ for $i = 1, \cdots, n$.*

We shall need the following "subtraction principle", which is a simplified version of [B-RS 3, 3.3].

Proposition 2.13. *Let A be a ring of dimension $n \geq 2$ and J, J' be two comaximal ideals of height n. Let $P = Q \oplus A$ be a projective A-module of rank n. Let $\alpha : P \twoheadrightarrow J \cap J'$ and $\beta : P \twoheadrightarrow J'$ be two surjections such that $\alpha \otimes A/J' = \beta \otimes A/J'$. Then there exists a surjection $\theta : P \twoheadrightarrow J$ such that $\theta \otimes A/J = \alpha \otimes A/J$.*

We quickly recall the generalities of the Euler class group theory. We first accumulate some basic definitions, namely, the definitions of the Euler class group, the Euler class of a projective module, and then state some results which are relevant to this paper. Detailed accounts of these topics can be found in [B-RS 3, D 1]. A reader familiar with these topics can safely go to the next section.

We start with the definition (from [B-RS 3]) of the n-th Euler class group $E^n(R,L)$ of a commutative Noetherian ring of dimension n with respect to a projective R-module L of rank one. For brevity of notation, we shall denote $E^n(R,L)$ by $E(R,L)$.

Definition 2.14. (*The Euler class group* $E(R,L)$): Write $F = L \oplus R^{n-1}$. Let $J \subset R$ be an ideal of height n such that J/J^2 is generated by n elements. Two surjections α, β from F/JF to J/J^2 are said to be *related* if there exists $\sigma \in SL(F/JF)$ such that $\alpha\sigma = \beta$. Clearly this is an equivalence relation on the set of surjections from F/JF to J/J^2. Let $[\alpha]$ denote the equivalence class of α. Such an equivalence class $[\alpha]$ is called a *local L-orientation* of J. By abuse of notation, we shall identify an equivalence class $[\alpha]$ with α. A local L-orientation α is called a *global L-orientation* if $\alpha : F/JF \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : F \twoheadrightarrow J$.

Let G be the free abelian group on the set of pairs $(\mathfrak{n}, \omega_{\mathfrak{n}})$ where $\mathfrak{n}$ is an $\mathfrak{m}$-primary ideal for some maximal ideal $\mathfrak{m}$ of height n such that $\mathfrak{n}/\mathfrak{n}^2$ is generated by n elements and $\omega_{\mathfrak{n}}$ is a local L-orientation of $\mathfrak{n}$. Let $J \subset R$ be an ideal of height n such that J/J^2 is generated by n elements and ω_J be a local L-orientation of J. Let $J = \cap_i \mathfrak{n}_i$ be the (irredundant) primary decomposition of J. We associate to the pair (J, ω_J), the element $\sum_i (\mathfrak{n}_i, \omega_{\mathfrak{n}_i})$ of G where $\omega_{\mathfrak{n}_i}$ is the local L-orientation of $\mathfrak{n}_i$ induced by ω_J. By abuse of notation, we denote $\sum_i (\mathfrak{n}_i, \omega_{\mathfrak{n}_i})$ by (J, ω_J). Let H be the subgroup of G generated by set of pairs (J, ω_J), where J is an ideal of height n and ω_J is a global L-orientation of J. The Euler class group of R with respect to L is $E(R,L) \stackrel{\text{def}}{=} G/H$.

Remark 2.15. When $L \simeq R$, the Euler class group $E(R,R)$ is simply denoted by $E(R)$.

Remark 2.16. In [M-Y2, Section 3] Mandal-Yang proved certain interesting functorial properties of the Euler class groups. Here we quote one of their results which is most relevant to this paper. Let A, B be commutative Noetherian rings, each of dimension $n \geq 2$. Let $f : A \longrightarrow B$ be a morphism of rings which satisfies a special property: *for any ideal I of A with ht$(I) = n$ and $\mu(I/I^2) = n$, the ideal $IB := f(I)B$ has height $\geq n$.* Let L be a projective A-module of rank one. Then they show that [M-Y2, 3.3] there is a group homomorphism $E(f) : E(A,L) \longrightarrow E(B, L \otimes_A B)$. Further, if $g : B \longrightarrow C$ is another morphism of rings satisfying the same property as above, one has the following commutative diagram (see [M-Y2, 3.4]):

$$\begin{tikzcd} E(A,L) \arrow[rr, "E(f)"] \arrow[dr, "E(gf)"'] & & E(B, L\otimes B) \arrow[dl, "E(g)"] \\ & E(C, L\otimes C) & \end{tikzcd}$$

For example, if $f : A \longrightarrow B$ is a flat extension of rings, then f satisfies the property specified above. In the next section, after introducing subintegral extension of rings, we shall show that such extensions enjoy the same property (see Remark 3.8).

Definition 2.17. (*The Euler class of a projective module*): Let P be a projective R-module of rank n such that $L \simeq \wedge^n(P)$ and let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism. Let $\varphi : P \twoheadrightarrow J$ be a surjection where J is an ideal of height n. Therefore we obtain an induced surjection $\overline{\varphi} : P/JP \twoheadrightarrow J/J^2$. Let $\overline{\gamma} : F/JF \xrightarrow{\sim} P/JP$ be an isomorphism such that $\wedge^n(\overline{\gamma}) = \overline{\chi}$. Let ω_J be the local L-orientation of J given by the composite surjection $\overline{\varphi}\,\overline{\gamma} : F/JF \twoheadrightarrow J/J^2$. Let $e(P,\chi)$ be the image in $E(R,L)$ of the element (J,ω_J) of G. If $\mathbb{Q} \subset R$, then it is proved in [B-RS 3] that the assignment sending the pair (P,χ) to the element $e(P,\chi)$ of $E(R,L)$ is well defined. The *Euler class* of (P,χ) is defined to be $e(P,\chi)$.

Theorem 2.18. [B-RS 3, 4.2, 4.3, 4.4] *Let R be a ring of dimension $n \geq 2$ and L be a projective R-module of rank 1. Let P be a projective R-module of rank n with $L \simeq \wedge^n(P)$ and let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism. Let $J \subset R$ be an ideal of height n and ω_J be a local L-orientation of J.*

(1) *Suppose that the image of (J,ω_J) is zero in $E(R,L)$. Then there exists a surjection $\alpha : F \twoheadrightarrow J$ such that ω_J is induced by α (in other words, ω_J is a global L-orientation).*
(2) *Assume that $\mathbb{Q} \subset R$. Then, $P \simeq P_1 \oplus R$ for some projective R-module P_1 of rank $n-1$ if and only if $e(P,\chi) = 0$ in $E(R,L)$.*
(3) *Assume that $\mathbb{Q} \subset R$. Let $e(P,\chi) = (J,\omega_J)$ in $E(R,L)$. Then there exists a surjective map $\alpha : P \twoheadrightarrow J$ such that (J,ω_J) is induced by (α,χ).*

Remark 2.19. A few words on the assumption that $\mathbb{Q} \subset R$ is in order. The reader should note that this assumption is necessary only when we talk about the Euler class of a projective R-module. This assumption was needed to prove [B-RS 3, 3.1], which essentially shows that the Euler class of a projective module is well defined. But a careful inspection of the proof of [B-RS 3, 3.1] would reveal that if $\dim(R) = 2$, we do not need to assume that $\mathbb{Q} \subset R$ to define the Euler class. However, in the definition of $E(R[T])$ given below, we need to assume that $\mathbb{Q} \subset R$ to start with.

Let R be a commutative Noetherian ring containing $\mathbb{Q}$ with $\dim(R) = n \geq 2$. The notion of the n-th Euler class group $E^n(R[T])$ has been defined in [D 1]. The reader should note that the definition of $E^n(R[T])$ is different from that of $E^n(R)$ and is not obtained by just replacing R by $R[T]$. Further, for a commutative Noetherian ring A of dimension d and a projective A-module L of rank one, Mandal-Yang [M-Y1] defined the r-th Euler class groups $E^r(A, L)$ for $1 \leq r \leq n$. The definition of $E^n(R[T])$ given below from [D 1] is not obtained from their definition either (by taking $A = R[T]$, $d = n+1$, $L = R[T]$ and $r = n$). Let us point out the difference. For an ideal I of $R[T]$ of height n with $\mu(I/I^2) = n$, a local orientation of I is defined as an $SL_n(R[T]/I)$-equivalence class of surjections in [D 1] whereas in [M-Y1] a local orientation of I is defined as an $E_n(R[T]/I)$-equivalence class of surjections. Therefore, a priori the definitions are different and it will be interesting to know whether the groups thus obtained are isomorphic or not.

For brevity we denote $E^n(R[T])$ as $E(R[T])$ and recall its definition from [D 1].

Definition 2.20. (**The Euler class group** $E(R[T])$) Let R be a Noetherian ring of dimension $n \geq 3$ containing $\mathbb{Q}$. Let $I \subset R[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Two surjections α and β from $(R[T]/I)^n \twoheadrightarrow I/I^2$ are said to be *related* if there exists $\sigma \in SL_n(R[T]/I)$ such that $\alpha\sigma = \beta$. This is an equivalence relation on the set of surjections from $(R[T]/I)^n$ to I/I^2. Let $[\alpha]$ denote the equivalence class of α. We call such an equivalence class $[\alpha]$ a *local orientation of* I. It was shown in [D 1, 4.4], that if $\alpha : (R[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : R[T]^n \twoheadrightarrow I$ then so can any β equivalent to α. We call a local orientation $[\alpha]$ of I a *global orientation of* I if the surjection $\alpha : (R[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : R[T]^n \twoheadrightarrow I$. Let G be the free abelian group on the set of pairs (I, ω_I) where $I \subset R[T]$ is an ideal of height n such that $\operatorname{Spec}(R[T]/I)$ is connected, I/I^2 is generated by n elements and $\omega_I : (R[T]/I)^n \twoheadrightarrow I/I^2$ is a local orientation of I. Let $I \subset R[T]$ be an ideal of height n and ω_I a local orientation of I. Now I can be decomposed uniquely as $I = I_1 \cap \cdots \cap I_r$, where the I_k's are ideals of $R[T]$ of height n, pairwise comaximal, such that $\operatorname{Spec}(R[T]/I_k)$ is connected for each k. Clearly ω_I induces local orientations ω_{I_k} of I_k for $1 \leq k \leq r$. By (I, ω_I) we mean the element $\Sigma(I_k, \omega_{I_k})$ of G. Let H be the subgroup of G generated by set of pairs (I, ω_I), where I is an ideal of $R[T]$ of height n generated by n elements and ω_I is a global orientation of I given by the set of generators of I. We define the Euler class group of $R[T]$, denoted by $E(R[T])$, to be G/H.

Definition 2.21. (**The Euler class of a projective** $R[T]$**-module**) Let R be as in (2.20). Let P be a projective $R[T]$-module of rank n with trivial determinant. Fix a trivialization $\chi : R[T] \xrightarrow{\sim} \wedge^n(P)$. Let $\alpha : P \twoheadrightarrow I$ be a surjection such that I is an ideal of height n. Note that, since P has trivial determinant and $\dim(R[T]/I) \leq 1$, P/IP is a free $R[T]/I$-module. Composing $\alpha \otimes R[T]/I$ with an isomorphism $\gamma : (R[T]/I)^n \xrightarrow{\sim} P/IP$ with the property $\wedge^n(\gamma) = \chi \otimes R[T]/I$ we get a local orientation, say ω_I, of I. Let $e(P, \chi)$ be the image in $E(R[T])$ of the element (I, ω_I) of G. (We say that (I, ω_I) is obtained from the pair (α,χ)). It can be proved that the assignment sending the pair (P,χ) to $e(P,\chi)$ is well defined (see [D 1, 4.6]). We define the *Euler class* of P to be $e(P,\chi)$.

Theorem 2.22. [D 1] *Let R be a Noetherian ring (containing $\mathbb{Q}$) of dimension $n \geq 3$. Let $I \subset R[T]$ be an ideal of $R[T]$ of height n such that I/I^2 is generated by n elements and ω_I be a local orientation of I. Let P be a rank n projective $R[T]$-module with trivial determinant with a trivialization $\chi : R[T] \simeq \wedge^n(P)$.*

(a) Suppose that the image of (I,ω_I) is zero in $E(R[T])$. Then ω_I is a global orientation of I.
(b) Suppose that $e(P,\chi) = (I,\omega_I)$ in $E(R[T])$. Then there exists a surjection $\alpha : P \twoheadrightarrow I$ such that ω_I is induced by α and χ (as described above).
(c) P has a unimodular element if and only if $e(P,\chi) = 0$ in $E(R[T])$.

The following results will be very useful in subsequent sections.

Proposition 2.23. *Let R be a ring of dimension n and let $R_{red} = R/\mathfrak{n}$, where $\mathfrak{n}$ denotes the nil radical of R. Let L be a projective R-module of rank one.*

(1) *The groups $E(R,L)$ and $E(R_{red}, L \otimes R_{red})$ are canonically isomorphic* [B-RS 3, 4.6].
(2) *Let $\mathbb{Q} \subset R$. Then $E(R_{red}[T])$ and $E(R[T])$ are canonically isomorphic* [D 3, 2.15].

We now recall the definition of the weak Euler class group of a ring from [B-RS 3].
Definition 2.24. (*The weak Euler class group $E_0(R,L)$*): Let R be a ring of dimension $n \geq 2$. Let G_0 be the free abelian group on the set of all ideals $\mathfrak{n}$, where $\mathfrak{n}$ is $\mathfrak{m}$-primary for some maximal ideal $\mathfrak{m}$ of height n such that there is a surjection $F \twoheadrightarrow \mathfrak{n}/\mathfrak{n}^2$. Given any ideal J of height n, we take the (irredundant) primary decomposition $J = \cap_i \mathfrak{n}_i$ and associate to J, the element $\sum_i \mathfrak{n}_i$ of G_0. We denote this element by (J). Let H_0 be the subgroup of G_0 generated by all (J) such that J is a surjective image of F. The weak Euler class group of R with respect to L is defined as $E_0(R,L) = G_0/H_0$.
Remark 2.25. It is clear from the above definitions that there is an obvious canonical surjective group homomorphism $\Theta_L : E(R,L) \twoheadrightarrow E_0(R,L)$ which sends an element (J,ω_J) of $E(R,L)$ to (J) in $E_0(R,L)$.

We shall need the following result on $E_0(R,L)$.

Proposition 2.26. [B-RS 3, 6.2] *Let R be a ring (containing $\mathbb{Q}$) of even dimension n and $J \subset R$ be an ideal of height n. Then $(J) = 0$ in $E_0(R,L)$ if and only if J is a surjective image of a projective R-module of rank n which is stably isomorphic to $L \oplus R^{n-1}$.*

Remark 2.27. It has been proved in [B-RS 3, Theorem 6.8] that the groups $E_0(R,L)$ and $E_0(R,R)$ are canonically isomorphic. Therefore, from now on we shall simply denote the weak Euler class group as $E_0(R)$.

3. The main results

We first recall the definitions and some basic properties of subintegral extensions. The reader may refer to [Sw1] and [I] for further details.

Definition 3.1. An extension $R \hookrightarrow S$ of rings is called *subintegral* if: (1) it is integral, (2) the induced map $\text{Spec}(S) \longrightarrow \text{Spec}(R)$ is bijective, and (3) for each $\mathfrak{P} \in \text{Spec}(S)$ the induced field extension $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow S_{\mathfrak{P}}/\mathfrak{P}S_{\mathfrak{P}}$ is trivial, where $\mathfrak{p} = \mathfrak{P} \cap R$.

The following characterization of subintegral extensions is from [Sw1].

Lemma 3.2. *$R \hookrightarrow S$ is subintegral if and only if S is integral over R and for any field F and any homomorphism ϕ, the diagram*

$$\begin{tikzcd} R \arrow[rr] \arrow[dr, "\phi"'] & & S \arrow[dl] \\ & F & \end{tikzcd}$$

can be filled in uniquely.

Definition 3.3. An extension of the form $R \hookrightarrow R[b]$ with $b^2, b^3 \in R$ is subintegral. It is called *elementarily subintegral*.

Remark 3.4. We record the following fundamental facts about subintegral extensions.

(1) $R \hookrightarrow S$ is subintegral if and only if S is the filtered union of subrings which can be obtained from R by a finite number of elementarily subintegral extensions.
(2) If R is a reduced Noetherian ring then any subintegral extension of R is contained in $\overline{R}$, the integral closure of R in its total ring of fractions.
(3) Let $R \hookrightarrow S$ be an extension of rings and $R \hookrightarrow R'$ be a faithfully flat extension. Write $S' = S \otimes_R R'$. Then $R \hookrightarrow S$ is subintegral if and only if $R' \hookrightarrow S'$ is subintegral. (See [Sw1, Page 215] for details).

In [I] Ischebeck studied the behaviour of certain K-theoretical functors under subintegral extensions. In particular, it is proved in [I, Proposition 6] that if $R \hookrightarrow S$ is a (finite) subintegral extension, then the Chow groups $CH_i(S)$ and $CH_i(R)$ are isomorphic. Along the same line, in [G], Gubeladze's object of study is the orbit space of unimodular rows under the natural action of elementary matrices. This orbit space carries a group structure (thanks to the work of Van der Kallen [VK1]), and as shown in [B-RS 3], is intimately related to the Euler class group and the weak Euler class group. Therefore, it is natural to ask the following questions.

Question 3.5. Let $R \hookrightarrow S$ be a subintegral extension of Noetherian rings with $\dim(R) = n \geq 2$. Let L be a projective R-module of rank one. Is $E(R, L) \simeq E(S, L \otimes_R S)$? Also, is $E_0(R, L) \simeq E_0(S, L \otimes_R S)$?

We have consciously decided to tackle the above questions in the case $L = R$ first. The proofs in this case are much more comprehensible as one is working with generators. We prove that $E(R)$ is isomorphic to $E(S)$. The general case for arbitrary L is proved in (3.12). If the dimension of R is even, then we prove that $E_0(R)$ is isomorphic to $E_0(S)$.

Before proceeding we need the following lemmas. These two lemmas must be well known. However, due to the lack of an appropriate reference, we prove them here.

Lemma 3.6. *Let $R \hookrightarrow S$ be a subintegral extension. Then $R_{red} \hookrightarrow S_{red}$ is also subintegral.*

Proof. We shall use (3.2). First of all, it is easy to check that S_{red} is integral over R_{red}.

Let $i : R \hookrightarrow S$ and $i' : R_{\text{red}} \hookrightarrow S_{\text{red}}$ be the inclusion maps. Let $\pi : R \twoheadrightarrow R_{\text{red}}$ and $\pi' : S \twoheadrightarrow S_{\text{red}}$ be the natural surjections. Let F be a field and $\theta : R_{\text{red}} \longrightarrow F$ be any homomorphism. Then we have the composite $\varphi = \theta\pi : R \longrightarrow F$. Since $R \hookrightarrow S$ is subintegral, by (3.2) there exists a unique homomorphism $\psi : S \longrightarrow F$ such that $\psi i = \theta\pi$.

Now we define a map $\Psi : S_{\text{red}} \longrightarrow F$ by $\Psi([s]) = \psi(s)$. To see that Ψ is well defined, let $s \in S$ be such that $s^r = 0$ for some positive power r. Then $\psi(s^r) = 0$ in F. Since F is a field, $\psi(s) = 0$, implying that $\Psi([s]) = 0$.

It is quite clear that the diagram:

$$\begin{tikzcd} R_{\text{red}} \arrow[rr, "i'"] \arrow[dr, "\theta"'] & & S_{\text{red}} \arrow[dl, "\Psi"] \\ & F & \end{tikzcd}$$

is commutative. Uniqueness of Ψ can be easily checked using the uniqueness of ψ. $\square$

Lemma 3.7. *Let $R \hookrightarrow S$ be an elementarily subintegral extension. Let C be the conductor ideal of R in S. Then $(R/C)_{red} = (S/C)_{red}$.*

Proof. Let $R \hookrightarrow R[b]$ with $b^2, b^3 \in R$, be an elementarily subintegral extension. Let $K = \sqrt{C}$, radical of C in $R[b]$. Then it follows that $(R[b]/C)_{red} = R[b]/K$ and $(R/C)_{red} = R/K \cap R$. Now we will show that $R[b]/K = R/K \cap R$. Note that $b \in K$. Therefore, $R[b]/K = R + Rb/K = R + K/K = R/K \cap R$. $\square$

The following remark ensures that for a subintegral extension $R \hookrightarrow S$, there are natural maps from $E(R)$ to $E(S)$ and from $E_0(R)$ to $E_0(S)$.

Remark 3.8. Let $R \hookrightarrow S$ be a subintegral extension and let $\dim(R) = n$. Then $\dim(S) = n$. The definition of a subintegral extension asserts that the inclusion $i : R \hookrightarrow S$ induces a bijection $i^* : \text{Spec}(S) \to \text{Spec}(R)$. As S is integral over R, the *going up theorem* holds for this extension. As R, S are both Noetherian, this implies that i^* is a closed map. But since i^* is bijective, it is therefore an open map and as a consequence, the *going down theorem* also holds for this extension. As S is integral over R, the *lying over theorem* is already there. These two theorems imply that for an ideal I of R, we have $\text{ht}(I) = \text{ht}(IS)$. It now follows from [M-Y2, 3.3] that there is a morphism from $E(R)$ to $E(S)$ (see also (2.16) above). However, we give the details for the reader. Let (I, ω_I) be a pair, where I is an ideal of R of height n and $\omega_I : (R/I)^n \twoheadrightarrow I/I^2$ is a local orientation of I. Then by the above discussion, we have $\text{ht}(IS) = n$. Although S may not be flat over R, note that the local orientation ω_I induces a local orientation $\omega_I^* : (S/IS)^n \twoheadrightarrow IS/(IS)^2$ of IS in a natural way. Clearly, if ω_I is a global orientation, then so is ω_I^*. Thus we have a canonical morphism $\Phi : E(R) \to E(S)$ which maps (I, ω_I) to (IS, ω_I^*). It is now easy to observe from the above discussion that there is also a canonical morphism $\Phi_0 : E_0(R) \to E_0(S)$ which sends (I) to (IS).

To answer the question raised at the beginning, we first prove the following result.

Theorem 3.9. *Let R be a Noetherian ring of dimension $n \geq 2$. Let $R \hookrightarrow S$ be a finite subintegral extension. Then the induced homomorphism $\Phi : E(R) \longrightarrow E(S)$ is an isomorphism.*

Proof. By (3.6) the extension $R_{\text{red}} \hookrightarrow S_{\text{red}}$ is subintegral. We know that $E(R) \simeq E(R_{\text{red}})$ and $E(S) \simeq E(S_{\text{red}})$ by (2.23), and therefore without loss of generality we assume that R, S are both reduced. Further, we may assume that the extension $R \hookrightarrow S$ is elementarily subintegral. From (3.7), we have $(R/C)_{\text{red}} = (S/C)_{\text{red}}$, where C is the conductor of R in S. Since the extension $R \hookrightarrow S$ is finite and R, S are both reduced rings with the same total ring of fractions, it follows that $\text{ht}(C) \geq 1$.

Step 1. In this step we prove that the induced map $\Phi : E(R) \to E(S)$ is injective.

Let $(I, \omega_I) \in E(R)$ such that $(IS, \omega_I^*) = 0$ in $E(S)$. Here $I \subset R$ is an ideal of height n and ω_I is a local orientation of I represented by, say, $I = (a_1, \cdots, a_n) + I^2$. To prove that $(I, \omega_I) = 0$ in $E(R)$, we have to find $v_1, \cdots, v_n$ such that $I = (v_1, \cdots, v_n)$ with $v_i - a_i \in I^2$ for $i = 1, \cdots, n$.

Since $(IS, \omega_I^*) = 0$ in $E(S)$, there exist $\alpha_1, \cdots, \alpha_n \in IS$ such that $IS = (\alpha_1, \cdots, \alpha_n)$ where $a_i - \alpha_i \in (IS)^2$.

Since $(a_1, \cdots, a_n) + I^2 = I$, applying the moving lemma (2.11), we can find $b_1, \cdots, b_n \in I$ and an ideal I' of R such that

(1) $I \cap I' = (b_1, \cdots, b_n)$ with $a_i - b_i \in I^2$.
(2) $I' + I \cap C = R$.
(3) $\text{ht}(I') \geq n$.

If $\text{ht}(I') > n$, then $I' = R$ and we are done by (1). Therefore we assume that $\text{ht}(I') = n$. Now consider the equation $IS \cap I'S = (b_1, \cdots, b_n)$ in S. On the other hand we have $IS = (\alpha_1, \cdots, \alpha_n)$ such that $a_i - \alpha_i \in (IS)^2$. Therefore, $\alpha_i - b_i \in I^2S$. Using the subtraction principle (2.13), we have $\beta_1, \cdots, \beta_n \in I'S$ such that $I'S = (\beta_1, \cdots, \beta_n)$ with $\beta_i - b_i \in (I'S)^2$.

As $I' + C = R$, we have $I' \otimes S/C \simeq S/C$. Further note that the image of I' in R/C is R/C and the image of $I'S$ in S/C is S/C. This implies that $(\overline{\beta_1}, \cdots, \overline{\beta_n}) \in \text{Um}_n(S/C)$ where bar denotes modulo C in S. Therefore, $(\overline{\beta_1}, \cdots, \overline{\beta_n}) \in \text{Um}_n((S/C)_{\text{red}})$. Note that we have $(R/C)_{\text{red}} = (S/C)_{\text{red}}$. As the canonical map : $\text{Um}_n(R/C) \longrightarrow \text{Um}_n((R/C)_{\text{red}})$ is surjective, there are elements $f_1, \cdots, f_n \in R$ such that $(\widetilde{f_1}, \cdots, \widetilde{f_n}) \in \text{Um}_n(R/C)$ (where tilde denotes reduction modulo C in R). Moreover, for each $i \in \{1, \cdots, n\}$, we have $\overline{f_i} - \overline{\beta_i} \in \mathfrak{n}(S/C)$, where $\mathfrak{n}(S/C)$ is the nil-radical of S/C.

Now we have two unimodular rows $(\overline{\beta_1}, \cdots, \overline{\beta_n})$ and $(\overline{f_1}, \cdots, \overline{f_n})$ in S/C such that $\overline{f_i} - \overline{\beta_i} \in \mathfrak{n}(S/C)$. Therefore, by [MK-M-R, 2.3], there exists a transvection $\overline{\sigma}$ of $(S/C)^n$ such that $(\overline{\beta_1}, \cdots, \overline{\beta_n})\overline{\sigma} = (\overline{f_1}, \cdots, \overline{f_n})$. By (2.7), $\overline{\sigma}$ can be lifted to an automorphism σ of S^n such that $\sigma = \text{Id}$ modulo $I'S$. Let $(\beta_1, \cdots, \beta_n)\sigma = (g_1, \cdots, g_n)$. Then we have $(\overline{g_1}, \cdots, \overline{g_n}) = (\overline{\beta}_1, \cdots, \overline{\beta}_n)\overline{\sigma} = (\overline{f_1}, \cdots, \overline{f_n})$ in S/C. Therefore $f_i - g_i \in C$. Since $f_i \in R$, we have $g_i \in R$.

We now claim that $I' = (g_1, \cdots, g_n)$ and $g_i - b_i \in I'^2$.

Proof of the claim : We first note that as I' is comaximal with C and C is the conductor ideal, we have $I'S \cap C = (I'S)C = I'C = I' \cap C$. Further, $S/I'S = (I'S + C)/I'S \simeq C/(I'S \cap C) = C/(I' \cap C) \simeq (I' + C)/I' = R/I'$. Therefore, $I'S \cap R = I'$.

It now follows from above that $(g_1, \cdots, g_n) \subseteq I'$. As $\sigma = \text{Id}$ modulo $I'S$, it is easy to see that $g_i - b_i \in I'^2S \cap R = I'^2$. As $(\widetilde{f_1}, \cdots, \widetilde{f_n}) \in \text{Um}_n(R/C)$ and $f_i - g_i \in C$, it follows that $(g_1, \cdots, g_n) + C = R$. Recall that we also have $I'S = (g_1, \cdots, g_n)S$.

Let $\mathfrak{m}$ be any maximal ideal of R. If $C \subset \mathfrak{m}$, then $I'R_\mathfrak{m} = R_\mathfrak{m} = (g_1, \cdots, g_n)R_\mathfrak{m}$. On the other hand, if $C \not\subset \mathfrak{m}$, then $R_\mathfrak{m} = T^{-1}R = T^{-1}S$, where $T = R \setminus \mathfrak{m}$ (this is true because for any $t \in C$, $R_t = S_t$). In this case,

$$I'R_\mathfrak{m} = I'T^{-1}R = I'T^{-1}S = (g_1, \cdots, g_n)T^{-1}S = (g_1, \cdots, g_n)R_\mathfrak{m}.$$

Therefore, $I' = (g_1, \cdots, g_n)$ with $g_i - b_i \in I'^2$. This proves the claim.

We have : (i) $I \cap I' = (b_1, \cdots, b_n)$, (ii) $I' = (g_1, \cdots, g_n)$ with $g_i - b_i \in I'^2$. We can now apply the subtraction principle (2.13) to obtain $v_1, \cdots, v_n \in I$ such that $I = (v_1, \cdots, v_n)$ with $v_i - b_i \in I^2$. As $a_i - b_i \in I^2$, it follows that $v_i - a_i \in I^2$. This proves that $(I, \omega_I) = 0$ in $E(R)$ and that Φ is injective.

Step 2. In this step we show that $\Phi : E(R) \to E(S)$ is surjective.

Let $(I, \omega_I) \in E(S)$. Suppose that ω_I is induced by : $I = (f_1, \cdots, f_n) + J^2$. By using the moving lemma (2.11), we can find $g_1, \cdots, g_n \in I$ and an ideal $K \subseteq S$ such that

(a) $(g_1, \cdots, g_n) = I \cap K$ where $g_i - f_i \in I^2$
(b) $K + I \cap C = S$ where $\text{ht}(K) \geq n$.

If ht$(K) > n$, then $K = S$ and $I = (g_1, \cdots, g_n)$ implying that $(I, \omega_I) = 0$ in $E(S)$ and there is nothing to prove. Therefore we assume that ht$(K) = n$. Let ω_K be the local orientation induced by $g_1, \cdots, g_n$. Then from (a) we have $(I, \omega_I) + (K, \omega_K) = 0$ in $E(S)$. In order to prove that (I, ω_I) has a preimage in $E(R)$ it is enough to prove that (K, ω_K) has a preimage.

Let $K \cap R = J$. As $K + C = S$, we have $J + C = R$, and hence there exists $c \in C$ such that $l = 1 - c \in J$. We can assume that ht$(l) = 1$. (If ht$(l) = 0$, choose $l' \in J$ such that l' does not belong to any minimal prime ideal of R. Then ht$(l + l' - ll') = 1$. Now $(1 - l)(1 - l') = 1 - l - l' + ll'$. If we write $l'' = l + l' - ll'$, then we have $1 - l'' \in C$ and ht$(l'') = 1$ and we can work with l''). Since $c \in C$, we have $R_c = S_c$. Therefore $R/(1 - c) = S/(1 - c)$ and $R \hookrightarrow S$ is an analytic isomorphism along $l \in J$. Therefore using [N, 1.3], we have

(c) $R/J \simeq S/K$.
(d) $K = JS$
(e) As $l \in J$, we have $J/J^2 \simeq K/K^2$.

Note that we have $K = (g_1, \cdots, g_n) + K^2$. As $J/J^2 \simeq K/K^2$, corresponding to this set of generators of K/K^2 we have a set of generators of J/J^2. Calling them $a_1, \cdots, a_n$ we have $J = (a_1, \cdots, a_n) + J^2$. Let ω_J be the associated local orientation of J. Then $(J, \omega_J) \in E(R)$ and clearly $\Phi((J, \omega_J)) = (K, \omega_K)$. $\square$

To extend the above theorem to all subintegral extensions we first prove that the Euler class group commutes with direct limit in the following sense. Let S be a Noetherian ring such that S is the filtered direct limit of a direct system of Noetherian subrings $\{S_\alpha, \mu_{\alpha\beta}\}$ indexed by Ω. Here, for $\alpha \leq \beta$ the map $\mu_{\alpha\beta} : S_\alpha \hookrightarrow S_\beta$ is the inclusion map and for each $\alpha \in \Omega$, let $\mu_\alpha : S_\alpha \hookrightarrow S$ be the inclusion. Assume that the following conditions hold: (1) $\dim(S) = n = \dim(S_\alpha)$ for each $\alpha \in \Omega$, and (2) for any ideal $I_\alpha \subset S_\alpha$ with ht$(I_\alpha) = n$ and $\mu(I_\alpha/I_\alpha^2) = n$, one has ht$(I_\alpha S_\beta) \geq n$ for $\alpha \leq \beta$ and ht$(I_\alpha S) \geq n$. It is now easy to see that for all $\alpha, \beta \in \Omega$, the map $\mu_{\alpha\beta} : S_\alpha \longrightarrow S_\beta$ induces $\theta_{\alpha\beta} : E(S_\alpha) \longrightarrow E(S_\beta)$ and $\mu_\alpha : S_\alpha \longrightarrow S$ induces $\phi_\alpha : E(S_\alpha) \longrightarrow E(S)$ so that $\{E(S_\alpha), \theta_{\alpha\beta}\}$ is a direct system of groups and $\phi_\beta\theta_{\alpha\beta} = \phi_\alpha$. Then we show below that the Euler class group $E(S)$ is isomorphic to the direct limit $\{\varinjlim E(S_\alpha), \theta_\alpha\}$. A situation as above will occur when, for example, the ring morphisms are all flat extensions (see (2.16)). In (3.11) we shall soon encounter another set up where it takes place naturally.

Theorem 3.10. *With notations as above,* $E(S) = E(\varinjlim S_\alpha) \simeq \varinjlim E(S_\alpha)$.

Proof. As for each α there is a group homomorphism $\phi_\alpha : E(S_\alpha) \longrightarrow E(S)$ with $\phi_\beta\theta_{\alpha\beta} = \phi_\alpha$, by the properties of direct limit there is a group homomorphism $\psi : \varinjlim E(S_\alpha) \longrightarrow E(S)$. We prove that ψ is an isomorphism.

An element x of $\varinjlim E(S_\alpha)$ is of the form $x = \theta_\alpha(x_\alpha)$ for some $\alpha \in \Omega$ and $x_\alpha \in E(S_\alpha)$. Let $x_\alpha = (J_\alpha, \omega_\alpha) \in E(S_\alpha)$, where J_α is an ideal of S_α of height n and ω_α is a local orientation of J_α induced by, say, $J_\alpha = (a_1, \cdots, a_n) + J_\alpha^2$. Assume that $\psi(x) = 0$ in $E(S)$. This implies that $\phi_\alpha((J_\alpha, \omega_\alpha)) = (J_\alpha S, \omega_\alpha^*) = 0$ in $E(S)$, where ω_α^* is the local orientation of $J_\alpha S$ induced by ω_α. Applying (2.18) we obtain $b_1, \cdots, b_n \in J_\alpha S$ such that $J_\alpha S = (b_1, \cdots, b_n)$ with $b_i - a_i = \lambda_i \in J_\alpha^2 S$. Since S is the filtered direct limit of the subrings S_α, it is easy to see that there exists $\beta \in \Omega$ such that $a_1, \cdots, a_n, b_1, \cdots, b_n, \lambda_1, \cdots, \lambda_n \in S_\beta$. Now there is $\gamma \in \Omega$ such that $S_\alpha \hookrightarrow S_\gamma$ and $S_\beta \hookrightarrow S_\gamma$. It follows that in S_γ, we have $J_\alpha S_\gamma = (b_1, \cdots, b_n)$ with $b_i - a_i = \lambda_i \in J_\alpha^2 S_\gamma$. This implies that $(J_\alpha S_\gamma, \omega_\alpha^*) = 0$ in $E(S_\gamma)$ and therefore $x = 0$ in $\varinjlim E(S_\alpha)$, proving that $\psi : \varinjlim E(S_\alpha) \longrightarrow E(S)$ is injective.

Next we prove that ψ is surjective. Let $(I, \omega) \in E(S)$. Then I is an ideal of S of height n and ω is a local orientation of I induced by, say, $I = (f_1, \cdots, f_n) + I^2$. Then by (2.10), there exists $e \in I$ such that $I = (f_1, \cdots, f_n, e)$ where $e(1-e) \in (f_1, \cdots, f_n)$. Suppose that $e(1-e) = k_1 f_1 + \cdots k_n f_n$ where $k_i \in S$. Since S is the filtered direct limit of the subrings S_α, it is easy to see that there exits $\alpha \in \Omega$ such that $f_1, \cdots, f_n, e, k_1, \cdots, k_n \in S_\alpha$. Let $I' = (f_1, \cdots, f_n, e)S_\alpha$. Then $I' = (f_1, \cdots, f_n, e - k_1 f_1 - \cdots - k_n f_n) = (f_1, \cdots, f_n, e^2)$ implying that $I' = (f_1, \cdots, f_n) + I'^2$. If ω' denotes the local orientation of I' induced by this set of generators of I'/I'^2, then $(I', \omega') \in E(S_\alpha)$. Then clearly $\phi_\alpha((I', \omega')) = (I, \omega)$ and it proves that $\psi(\theta_\alpha((I', \omega'))) = (I, \omega)$, implying the surjectivity of ψ. $\square$

Corollary 3.11. *Let R be a Noetherian ring of dimension $n \geq 2$. Let $R \hookrightarrow S$ be any subintegral extension. Then the induced homomorphism $\Phi : E(R) \longrightarrow E(S)$ is an isomorphism.*

Proof. Recall that S is the filtered union of subrings S_α where each S_α is obtained from R by a finite sequence of elementarily subintegral extensions. This means that given two subrings S_α, S_β of the above type, there is a subring S_γ of the above type such that $R \hookrightarrow S_\alpha \hookrightarrow S_\gamma$ and $R \hookrightarrow S_\beta \hookrightarrow S_\gamma$. Let $S = \cup_{\alpha \in \Omega} S_\alpha$.

For elements $\alpha, \beta \in \Omega$ define $\alpha \leq \beta$ to mean $S_\alpha \subseteq S_\beta$ and let $\mu_{\alpha\beta} : S_\alpha \hookrightarrow S_\beta$ be the inclusion map. Then S is the filtered direct limit of $\{S_\alpha\}_{\alpha \in \Omega}$, i.e.,

$$S = \varinjlim_{\alpha \in \Omega} S_\alpha.$$

For $\alpha \leq \beta (\in \Omega)$ we have a group homomorphism $\theta_{\alpha\beta} : E(S_\alpha) \longrightarrow E(S_\beta)$ induced by the inclusion map $S_\alpha \hookrightarrow S_\beta$. Note that $S_\alpha \hookrightarrow S_\beta$ is subintegral.

We then have the direct limit $\varinjlim E(S_\alpha)$ of the direct system of groups $\{E(S_\alpha)\}_{\alpha \in \Omega}$ and group homomorphisms $\theta_\alpha : E(S_\alpha) \longrightarrow \varinjlim E(S_\alpha)$.

By (3.10), $E(S) = E(\varinjlim S_\alpha) \simeq \varinjlim E(S_\alpha)$. Since by (3.9), $E(R) \simeq E(S_\alpha)$ for each α, it follows that $E(R) \simeq E(S)$. $\square$

We now proceed to prove that if $R \hookrightarrow S$ is subintegral, then $E(R, L)$ is isomorphic to $E(S, L \otimes_R S)$, where L is a projective R-module of rank one. Again we need to ensure that there is a natural morphism from $\Phi : E(R, L) \longrightarrow E(S, L \otimes_R S)$. As for any ideal $J \subset R$ of height n, by (3.8) we have $\mathrm{ht}(JS) = n$, the existence of Φ is ensured by [M-Y2, 3.3]. However, we present the explicit description of Φ below for the convenience of the reader.

We write $F = L \oplus R^{n-1}$. Let J be any ideal of R of height n and $\omega_J : F/JF \twoheadrightarrow J/J^2$ be any surjection. Tensoring with S/JS over R/J we obtain the induced surjection

$$\widetilde{\omega_J} : \frac{(F \otimes_R S)}{JS(F \otimes_R S)} \twoheadrightarrow \frac{(J \otimes_R S)}{JS(J \otimes_R S)}.$$

Now composing $\widetilde{\omega_J}$ with the surjective map $\widetilde{f}$ induced by the natural surjection $f : J \otimes_R S \twoheadrightarrow JS$ we obtain a local orientation of JS. We call it ω_J^*. Thus

$$\omega_J^* : \frac{(F \otimes_R S)}{JS(F \otimes_R S)} \overset{\widetilde{\omega_J}}{\twoheadrightarrow} \frac{(J \otimes_R S)}{JS(J \otimes_R S)} \overset{\widetilde{f}}{\twoheadrightarrow} \frac{JS}{J^2S}.$$

Note that if ω_J can be lifted to a surjection $\theta : F \twoheadrightarrow J$, then so can be ω_J^*. Therefore, we have a well defined group homomorphism $\Phi : E(R, L) \longrightarrow E(S, L \otimes_R S)$ which takes an element (J, ω_J) of $E(R, L)$ to (JS, ω_J^*) of $E(S, L \otimes_R S)$.

We now prove the following theorem. We shall not give a detailed proof as it is along the same line as the last two theorems. We shall only highlight the crucial deviations.

Theorem 3.12. *Let $R \hookrightarrow S$ be a subintegral extension and L be a projective R-module of rank one. Then the map $\Phi : E(R, L) \longrightarrow E(S, L \otimes_R S)$, described above, is an isomorphism.*

Proof. As the direct limit argument of (3.11) works in this case too, we may assume that S is obtained from R by a finite number of subintegral extensions. Therefore, as before, we may assume that R, S are both reduced and $R \hookrightarrow S$ is elementarily subintegral. If C is the conductor of R in S, then $\mathrm{ht}(C) \geq 1$ and $(R/C)_{\mathrm{red}} = (S/C)_{\mathrm{red}}$.

Let $(J, \omega_J) \in E(R, L)$ be such that $(JS, \omega_J^*) = 0$ in $E(S, L \otimes_R S)$. Applying the moving lemma (2.11) and the subtraction principle (2.13), we may assume that $J + C = R$. As $(JS, \omega_J^*) = 0$ in $E(S, L \otimes_R S)$, there exists a surjection $\beta' : F \otimes_R S \twoheadrightarrow JS$ such that β' lifts ω_J^*. As $JS + C = S$, it follows that $\beta'' = \beta' \otimes_S S/C \in \mathrm{Um}((F \otimes_R S/C)^*)$. Now β'' will induce a unimodular element of $(F \otimes_R (S/C)_{\mathrm{red}})^*$ and also, as $(R/C)_{\mathrm{red}} = (S/C)_{\mathrm{red}}$, a unimodular element of $(F \otimes_R (R/C)_{\mathrm{red}})^*$. We lift the latter one to $\delta \in \mathrm{Um}((F \otimes_R R/C)^*)$. Now note that $\delta \otimes_{R/C} S/C$ and β'' are the same modulo the nil radical of S/C. Therefore, applying (2.6), we obtain $\overline{\sigma} \in \mathcal{E}(F \otimes_R S/C)$ such that $\delta \otimes_{R/C} S/C = \beta\overline{\sigma}$. Applying (2.7) we can lift $\overline{\sigma}$ to an automorphism σ of $F \otimes_R S$ such that $\sigma \equiv \mathrm{id}$ modulo JS.

Write $\beta = \beta'\sigma$. Then $\beta : F \otimes_R S \twoheadrightarrow JS$ and β lifts ω_J^* (as σ is identity modulo JS). As $J + C = R$, we have $J \otimes_R S \simeq JS$ (see (2.8)), and $J \otimes_R R/C \simeq R/C$, and $J \otimes_R S/C \simeq S/C$. Up to these identifications, the following diagram is Cartesian

$$\begin{tikzcd}
J \arrow[r] \arrow[d] & J \otimes S \simeq JS \arrow[d] \\
(R/C) \arrow[r] & (S/C)
\end{tikzcd}$$

As β and δ agree over S/C, they will patch to yield a surjection $\alpha : F \twoheadrightarrow J$. Here is the patching diagram:

$$\begin{tikzcd}
F \arrow[rr] \arrow[dr, "\alpha"] \arrow[dd] & & F \otimes S \arrow[dr, "\beta"] \arrow[dd] & \\
& J \arrow[rr] \arrow[dd] & & J \otimes S \simeq JS \arrow[dd] \\
F \otimes R/C \arrow[rr] \arrow[dr, "\delta"] & & F \otimes S/C \arrow[dr] & \\
& R/C \arrow[rr] & & S/C
\end{tikzcd}$$

Identifying $J \otimes_R S$ with JS and using the isomorphism $S/JS \simeq R/J$, we have:

$$\alpha \underset{R}{\otimes} (R/J) = \alpha \underset{R}{\otimes} (S/JS) = (\alpha \underset{R}{\otimes} S) \underset{S}{\otimes} (S/JS) =$$

$$\beta \underset{S}{\otimes} (S/JS) = \omega_J \underset{R/J}{\otimes} (S/JS) = \omega_J \underset{R/J}{\otimes} (R/J) = \omega_J.$$

Thus α lifts ω_J, implying that $(J, \omega_J) = 0$ in $E(R, L)$. This proves that Φ is injective.

The proof that Φ is surjective is similar to the proof in Step 2 of (3.9). $\square$

Applying the above theorem we have the following corollaries.

Corollary 3.13. *Let R be a ring of dimension $n \geq 2$ and ${}^+(R_{red})$ be the seminormalization of R_{red}. Let L be a projective R-module of rank one and write $\widetilde{L} = L \otimes {}^+(R_{red})$. Then $E(R, L) \simeq E({}^+(R_{red}), \widetilde{L})$.*

Proof. Recall that if A is a reduced ring then its seminormalization is the subintegral closure of A in its total ring of fractions.

Here we have $E(R, L) \simeq E(R_{\text{red}}, L \otimes R_{\text{red}})$ by (2.23), and the group $E(R_{\text{red}}, L \otimes R_{\text{red}})$ is isomorphic to $E({}^+(R_{\text{red}}), \widetilde{L})$ by (3.12). $\square$

The (unstated) result of Bhatwadekar from [B 1], as mentioned in the introduction, can now be deduced (although we have the restriction that $\mathbb{Q} \subset R$).

Corollary 3.14. *Let $R \hookrightarrow S$ be a subintegral extension of $\mathbb{Q}$-algebras with dim$(R) = n \geq 2$. Let P be a projective R-module of rank n. Then P has a unimodular element if and only if the projective S-module $P \otimes_R S$ has a unimodular element.*

Remark 3.15. The above corollary is also true if we take $S = {}^+(R_{\text{red}})$. Further, if $\dim(R) = 2$, then we do not need the assumption that $\mathbb{Q} \subset R$ (see (2.19)).

Adapting the same method as above, one can similarly prove the following result. The reader may also note that by (3.4), as $R[T]$ is faithfully flat over R, the extension $R \hookrightarrow S$ is subintegral if and only if so is the extension $R[T] \hookrightarrow S[T]$.

Theorem 3.16. *Let R be a ring (containing $\mathbb{Q}$) of dimension $n \geq 3$. Let $R \hookrightarrow S$ be a subintegral extension. Then $E(R[T]) \simeq E(S[T])$. In particular, if R is reduced and if S is the seminormalization of R, then $E(R[T]) \simeq E(S[T])$. Therefore, for an arbitrary R, the groups $E(R[T])$ and $E({}^+(R_{red})[T])$ are isomorphic.*

Proof. Instead of writing the whole proof, we just work out one key step. The rest can be easily worked out. We can assume that the rings are reduced and the extension $R \hookrightarrow S$ is elementarily subintegral. Let C be the conductor of R in S. We have ht$(C) \geq 1$.

Let $I = (f_1, \cdots, f_n) + I^2$. We just show how to 'move away' from I to obtain a suitable residual ideal I' of height n so that I' is comaximal with both I and $C[T]$. Let $J = I^2 \cap C \subset R$. Then ht$(J) \geq 1$. Let $b \in J$ be such that ht$(b) = 1$. Let bar denote reduction modulo (b). We have, $\overline{I} = (\overline{a_1}, \cdots, \overline{a_n}) + \overline{I}^2$ in $\overline{R}[T]$. As dim$(\overline{R}) \leq n-1$, it follows from (2.12) that there exist $g_1, \cdots, g_n \in I$ such that $\overline{I} = (\overline{g_1}, \cdots, \overline{g_n})$, where $\overline{g_i} - \overline{f_i} \in \overline{I}^2$. Therefore, $I = (g_1, \cdots, g_n, b)$ such that $g_i = f_i \in I^2$. One can now apply (2.9) and (2.10) to find an ideal I' such that (possibly after some renaming the g_i's): (1) $I \cap I' = (g_1, \cdots, g_n)$, (2) $I' + I \cap C[T] = R[T]$, (3) ht$(I') = n$.

Note that $I' = (g_1, \cdots, g_n) + I'^2$. One can work with I' and apply the subtraction principle [D 1, 4.3] at appropriate places to prove the results. $\square$

Remark 3.17. In our forthcoming paper [D-Z], we have been able to define $E(R[T], L)$, where L is a line bundle over $R[T]$, and prove results analogous to [D 1]. Further, in [D-Z] we prove that if $R \hookrightarrow S$ is subintegral then $E(R[T], L) \simeq E(S[T], L \otimes S[T])$.

Let R be a ring of dimension $n \geq 2$. Given a pair (J, ω_J), where $J \subset R$ is an ideal of height ≥ 2 and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ a surjection, the *Segre class* $s(J, \omega_J)$ has been defined in [D-RS] in the following way:

Suppose that ω_J induces $J = (a_1, \cdots, a_n) + J^2$. Applying a variant of the moving lemma [D-RS, 2.7], we can find $c_1, \cdots, c_n \in J$ such that $(c_1, \cdots, c_n) = J \cap J_1$ where ht $J_1 \geq n$, $J_1 + J = R$ and $c_i = a_i$ modulo J^2. If J_1 is a proper ideal then $J_1 = (c_1, \cdots, c_n) + J_1^2$ and it induces a local orientation $\omega_{J_1} : (R/J_1)^n \twoheadrightarrow J_1/J_1^2$. The Segre class of the pair (J, ω_J) is defined as: $s(J, \omega_J) = -(J_1, \omega_{J_1}) \in E(R)$. If $J_1 = R$ then $J = (c_1, \cdots, c_n)$ and the Segre class is defined to be zero. It is proved that the definition of the Segre class does not depend on the choice of J_1. Further, when $\text{ht}(J) = n$, the Segre class coincides with the Euler class of (J, ω_J). We now recall the following result on Segre classes.

Theorem 3.18. [D-RS, 3.3] *Let R be a ring of dimension $n \geq 2$. Let $J \subset R$ be an ideal of height ≥ 2 and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. Suppose that $s(J, \omega_J) = 0$ in $E(R)$. Then ω_J can be lifted to a surjection $\theta : R^n \twoheadrightarrow J$.*

We now have

Theorem 3.19. *Let R be a ring of dimension $n \geq 2$ and $R \hookrightarrow S$ be a subintegral extension. Let $J \subset R$ be an ideal of height ≥ 2 and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be a surjection. Assume that the induced surjection $\omega_J^* : (S/JS)^n \twoheadrightarrow JS/J^2S$ has a lift to a surjection $\theta : S^n \twoheadrightarrow JS$. Then ω_J can be lifted to a surjection $\alpha : R^n \twoheadrightarrow J$.*

Proof. The hypothesis tells that the Segre class $s(JS, \omega_J^*) = 0$ in $E(S)$. It is now obvious from the definition of the Segre class and (3.11) that $s(J, \omega_J) = 0$ in $E(R)$. Therefore, by the above theorem, ω_J can be lifted to a surjection $\theta : R^n \twoheadrightarrow J$. □

We now mention another immediate consequence of (3.11). We need to recall some generalities from [B-RS 3, Section 7]. Let A be a ring of dimension 2. Let $\widetilde{K_0}Sp(A)$ be the set of isometry classes of (P, s), where P is a projective A-module of rank 2 and $s : P \times P \to A$ a non-degenerate skew-symmetric bilinear form. In [B-RS 3], a group structure is defined on $\widetilde{K_0}Sp(A)$, where the pair (A^2, h) plays the role of the identity element, where h is the unique (up to isometry) non-degenerate alternating form on A^2. It is then remarked that this group coincides with the usual notion of $\widetilde{K_0}Sp(A)$. Further, it is proved in [B-RS 3, 7.2] that $\widetilde{K_0}Sp(A)$ is isomorphic to the Euler class group $E(A)$.

Corollary 3.20. *Let $R \hookrightarrow S$ be a subintegral extension with dim$(R) = 2 =$ dim(S). Then the groups $\widetilde{K_0}Sp(R)$ and $\widetilde{K_0}Sp(S)$ are isomorphic.*

Proof. We have $\widetilde{K_0}Sp(R) \simeq E(R)$ and $\widetilde{K_0}Sp(S) \simeq E(S)$, the result is obvious from (3.11). □

We now consider the weak Euler class groups. We shall first prove the invariance of the weak Euler class group under finite subintegral extension (for even dimensional rings) and then generalize it to arbitrary subintegral extension by a direct limit argument. Before proceeding we first clarify a notation.

Remark 3.21. Let R be a ring of dimension n and take $(I, \omega_I) \in E(R)$. Let $u \in R$ be a unit modulo I. By $\overline{u}\omega_I$ we mean the local orientation obtained from the composition

$$(R/I)^n \overset{\delta}{\simeq} (R/I)^n \overset{\omega_I}{\twoheadrightarrow} I/I^2,$$

where $\delta \in GL_n(R/I)$ has determinant $\overline{u}$ (here "bar" means modulo I). It follows from [B 2, 2.2] that if ω_1 and ω_2 are two local orientations of I, then there exists $v \in R$ such that v is a unit modulo I and $\omega_2 = \overline{v}\omega_1$.

Theorem 3.22. *Let R be a ring (containing $\mathbb{Q}$) with dim$(R) = n$, where n is even. Let $R \hookrightarrow S$ be a finite subintegral extension. Then $E_0(R) \simeq E_0(S)$.*

Proof. Let $\Phi_0 : E_0(R) \longrightarrow E_0(S)$ be the group homomorphism induced by the inclusion $R \hookrightarrow S$. We have already proved that there is an isomorphism $\Phi : E(R) \overset{\sim}{\to} E(S)$. Recall that there are canonical surjective morphisms $\psi : E(R) \twoheadrightarrow E_0(R)$ and $\psi' : E(S) \twoheadrightarrow E_0(S)$. Furthermore, $\psi'\Phi = \Phi_0\psi$. Therefore it easily follows that Φ_0 is surjective. Note that this is true even if we do not assume that n is even.

We may assume that R is reduced. Let C be the conductor of R in S. Then ht$(C) \geq 1$. To prove that Φ_0 is injective, let $(I) \in E_0(R)$ be such that $\Phi_0((I)) = (IS) = 0$ in $E_0(S)$. Let ω be any local orientation of I and ω^* be the local orientation of IS induced by ω. Then $(I, \omega) \in E(R)$, $(IS, \omega^*) \in E(S)$ and $\psi((I, \omega)) = (I)$, $\psi'((IS, \omega^*)) = (IS)$. Applying moving lemma (2.11) we can find $(J, \omega_J) \in E(R)$ such that $(I, \omega) + (J, \omega_J) = 0$ in $E(R)$ and $J + I \cap C = R$. Then $(JS) = 0$ in $E_0(S)$ and if we can prove that $(J) = 0$ in $E_0(R)$ then it implies that $(I) = 0$ in $E_0(R)$. Therefore, without any loss of generality we may assume that I is comaximal with C. Consequently, $S/IS \simeq R/I$. We will need this information in the latter part.

Since $(IS) = 0$ in $E_0(S)$, by (2.26) there exists a stably free S-module P' of rank n together with an isomorphism $\chi' : S \overset{\sim}{\to} \wedge^n P'$ such that $e(P', \chi') = (IS, \omega \otimes S)$ in $E(S)$.

As P' is a stably free S-module of rank $n = \dim(S)$, there is a unimodular row $(a_0, a_1, \cdots, a_n) \in \text{Um}_{n+1}(S)$ corresponding to P'. It is an easy exercise (for a solution see the proof of (6.1), last paragraph) to show that there is a stably free R-module P of rank n such that $P \otimes S \simeq P'$. Let $\chi : R \overset{\sim}{\to} \wedge^n P$ be an isomorphism. Consider the Euler class $e(P \otimes S, \chi \otimes S) = e(P', \chi \otimes S)$. Now χ' and $\chi \otimes S$ differ by a unit of S, say, u. Therefore, $e(P \otimes S, \chi \otimes S) = (IS, u\omega)$ in $E(S)$. As $S/IS \simeq R/I$, the image of u in S/IS has a lift to $\overline{v} \in (R/I)^*$ (where bar denotes reduction modulo I). We then have $\Phi(e(P, \chi)) = e(P \otimes S, \chi \otimes S) = (IS, u\omega) = (IS, \overline{u}\omega)$. On the other hand, $\Phi((I, \overline{v}\omega)) = (IS, \overline{u}\omega)$. As Φ is injective, it follows that $e(P, \chi) = (I, \overline{v}\omega)$. By (2.18) there is a surjection $\alpha : P \twoheadrightarrow I$. As P is stably free, it follows from (2.26) that $(I) = 0$ in $E_0(R)$. This proves that $\Phi_0 : E_0(R) \to E_0(S)$ is injective and completes the proof. $\square$

Remark 3.23. If $\dim(R) = 2$, we do not need to assume that $\mathbb{Q} \subset R$ in the above theorem.

We now prove that the weak Euler class group also commutes with direct limit. Let S be a Noetherian ring which is the filtered direct limit of a direct system of Noetherian subrings $\{S_\alpha, \mu_{\alpha\beta}\}$ and the set up be exactly as in (3.10). Then it is easy to see that the weak Euler class groups form a direct system $\{E_0(S_\alpha), f_{\alpha\beta}\}$. Let $\{\varinjlim E_0(S_\alpha), f_\alpha\}$ be its direct limit. For each $\alpha \in \Omega$, we also have group homomorphism $h_\alpha : E_0(S_\alpha) \longrightarrow E_0(S)$ induced by the inclusion $\mu_\alpha : S_\alpha \hookrightarrow S$ with the property that $h_\beta f_{\alpha\beta} = h_\alpha$ for $\alpha \leq \beta$. We prove below that $E_0(S)$ is isomorphic to $\varinjlim E_0(S_\alpha)$. Note that we do not put any restriction on the dimension of the ring.

Theorem 3.24. *With notations as above,* $E_0(S) = E_0(\varinjlim S_\alpha) \simeq \varinjlim E_0(S_\alpha)$.

Proof. In this proof we shall apply (3.10) and freely use the notations from there.

As for each α there is a group homomorphism $h_\alpha : E_0(S_\alpha) \longrightarrow E_0(S)$ with $h_\beta f_{\alpha\beta} = h_\alpha$ for $\alpha \leq \beta$, by the properties of direct limit there is a map $g : \varinjlim E_0(S_\alpha) \longrightarrow E_0(S)$. We prove that g is an isomorphism.

By (2.25) there is a canonical surjective morphism $E(S_\alpha) \twoheadrightarrow E_0(S_\alpha)$ for each $\alpha \in \Omega$. They will induce a surjection $\Phi : \varinjlim E(S_\alpha) \twoheadrightarrow \varinjlim E_0(S_\alpha)$. We also have a canonical surjection $\Psi : E(S) \twoheadrightarrow E_0(S)$. We then have the following commutative diagram.

$$\begin{tikzcd}
\varinjlim E(S_\alpha) \arrow[r, "\psi"] \arrow[d, "\Phi"'] & E(S) \arrow[d, "\Psi"] \\
\varinjlim E_0(S_\alpha) \arrow[r, "g"] & E_0(S)
\end{tikzcd}$$

Therefore, g is surjective. To prove that g is injective, note that an element x of $\varinjlim E_0(S_\alpha)$ is of the form $x = f_\alpha(x_\alpha)$ for some $\alpha \in \Omega$ and $x_\alpha \in E_0(S_\alpha)$. Let $x_\alpha = (J_\alpha) \in E_0(S_\alpha)$, where J_α is an ideal of S_α of height n. Assume that $g(x) = 0$ in $E_0(S)$. This implies that $h_\alpha((J_\alpha)) = (J_\alpha S) = 0$ in $E_0(S)$.

Let $(J_\alpha, \omega_\alpha) \in E(S_\alpha)$ be a preimage of (J_α). By a slight abuse of notations let us view $(J_\alpha, \omega_\alpha)$ as an element of $\varinjlim E(S_\alpha)$ and write $\psi((J_\alpha, \omega_\alpha)) = (J_\alpha S, \omega_\alpha^*)$, where ω_α^* is induced by ω_α. By the commutativity of the above diagram, $\Psi((J_\alpha S, \omega_\alpha^*)) = (J_\alpha S) = 0$ in $E_0(S)$. Applying [B-RS 2, 3.3] (which works for a commutative Noetherian ring) it follows that

$$(J_\alpha S, \omega_\alpha^*) + \sum_{i=1}^{k} (I_i, \omega_i) = \sum_{j=k+1}^{l} (I_j, \omega_j)$$

in $E(S)$, where each of $I_1, \cdots, I_l$ is generated by n elements.

We now want to lift the above equation in $\varinjlim E(S_\alpha)$. Let us describe the process with one element, say, (I_1, ω_1). Let $I_1 = (a_1, \cdots, a_n)$ and let ω denote the global orientation of I_1 induced by these generators. Then by (3.21), $(I_1, \omega_1) = (I_1, \overline{u}\omega)$ for some $u \in S$ which is unit modulo I_1. There exists $v \in S$ such that $uv - 1 = a_1b_1 + \cdots + a_nb_n$. As S is the filtered direct limit of $\{S_\alpha\}_{\alpha\in\Omega}$, we can find some $\beta_1 \in \Omega$ such that $a_1, \cdots, a_n, b_1 \cdots, b_n, u, v \in S_{\beta_1}$. Let $K_1 = (a_1, \cdots, a_n)S_{\beta_1}$ and let σ denote the global orientation of K_1 induced by these generators. Composing σ with an automorphism of $(S_{\beta_1}/K_1)^n$ with determinant u modulo K_1 we get a local orientation, say, σ_1 of K_1. It is then clear that $\phi_{\beta_1}((K_1, \sigma_1)) = (I_1, \omega_1)$.

Applying the above process for each of $I_1, \cdots, I_l$ we can find a suitable $\beta \in \Omega$ and elements $(K_i, \sigma_i) \in E(S_\beta)$, $1 \leq i \leq l$ such that $\phi_\beta((K_i, \sigma_i)) = (I_i, \omega_i)$ for each i. Moreover, applying (3.10) it is easy to see that the following equation holds in $E(S_\beta)$.

$$(J_\alpha S_\beta, \omega_\alpha \otimes S_\beta) + \sum_{i=1}^{k}(K_i, \sigma_i) = \sum_{j=k+1}^{l}(K_j, \sigma_j)$$

As each K_i is generated by n elements, it follows that $(J_\alpha S_\beta) = 0$ in $E_0(S_\beta)$ and as a consequence, $x = 0$ in $\varinjlim E_0(S_\alpha)$. □

Corollary 3.25. *Let R be a ring (containing $\mathbb{Q}$) with dim$(R) = n$, where n is even. Let $R \hookrightarrow S$ be a subintegral extension. Then $E_0(R) \simeq E_0(S)$.*

Proof. The proof is along the same line as in (3.11) and is obtained by using (3.22) and (3.24). □

Remark 3.26. Similarly one can prove that if $R \hookrightarrow S$ is a subintegral extension of even dimensional $\mathbb{Q}$-algebras, then $E_0(R[T]) \simeq E_0(S[T])$.

Remark 3.27. Let $R \hookrightarrow S$ be a subintegral extension of even dimensional rings and L be a projective R-module of rank one. By (2.27) we know that the weak Euler class group does not depend on L. Therefore, applying (2.27) and (3.25) above, we have $E_0(R, L) \simeq E_0(R) \simeq E_0(S) \simeq E_0(S, L \otimes S)$.

When dim(R) is not necessarily even, we have the following affirmative result. Recall that for a module M, the notation $\mu(M)$ stands for the minimal number of generators of M.

Theorem 3.28. *Let R be an affine algebra over a C_1-field k of characteristic zero and $R \hookrightarrow S$ be a subintegral extension with dim$(R) = n \geq 2$. Then $E_0(R) \simeq E_0(S)$. In particular, if J is an ideal of R of height n such that $\mu(J/J^2) = n$, then $\mu(J) = n$ if and only if $\mu(JS) = n$.*

Proof. For any affine algebra of dimension $n \geq 2$ over a C_1-field of characteristic zero, the Euler class group is isomorphic to the weak Euler class group. A proof for $n \geq 3$

is given in [D 2, 5.2], whereas the case $n = 2$ can be worked out easily. Therefore, we have $E(R) \simeq E_0(R)$ and $E(S) \simeq E_0(S)$ and the first assertion follows from (3.11).

To prove the second part, let $J \subset R$ be an ideal of height n. If J is generated by n elements, then obviously so is JS.

Conversely, let $\mu(J/J^2) = n$ and suppose it is given that $\mu(JS) = n$. Let $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be any surjection and let $\omega_J^* : (S/JS)^n \twoheadrightarrow JS/J^2S$ be the surjection induced by ω_J. As $E(S) \simeq E_0(S)$, we have $(JS, \omega_J^*) = 0$ in $E(S)$ and therefore by (3.11), $(J, \omega_J) = 0$ in $E(R)$ and J is generated by n elements. $\square$

We conclude this section with the following set of questions.

Question 3.29. Let $R \hookrightarrow S$ be a subintegral extension.

(1) Can one extend (3.25) to arbitrary dimension?
(2) Let $\dim(R) = d$ and n be an integer such that $2n \geq d + 3$. Are the n-th Euler class groups $E^n(R)$ and $E^n(S)$ isomorphic? (For the definition of the n-th Euler class group, see [B-RS 4])

4. INTEGRAL EXTENSIONS

The aim of this short section is to explore what happens when $R \hookrightarrow S$ is an integral extension. We shall give an example to show that even if $R \hookrightarrow S$ is a finite birational extension, $E(R)$ may not be isomorphic to $E(S)$. Before giving this example, we engage ourselves in a more delicate investigation and prove a result which generalizes (3.11) and improves the understanding further. This entire section emerged from Bhatwadekar's ideas, which he communicated to us after seeing an earlier version of the paper. We have only checked the details. Once again, we thank him most sincerely.

First, to ensure that there is a group homomorphism from $E(R)$ to $E(S)$ when $R \hookrightarrow S$ is integral, we need to prove some generalities on the Euler class group. In fact we define a group which is very similar to the Euler class group.

Definition 4.1. Let A be a Noetherian ring of dimension $n \geq 2$. Let $\mathcal{G}$ be the free abelian group on the pairs (J, ω_J), where: (1) J is an $\mathfrak{m}$-primary ideal for some maximal ideal $\mathfrak{m}$ of A (*not necessarily of height* n), (2) ω_J is an $SL_n(A/J)$-equivalence class of surjections from $(A/J)^n \twoheadrightarrow J/J^2$ (i.e., a *local orientation* of J). Given any zero dimensional ideal I of A and a surjection $\omega_I : (A/I)^n \twoheadrightarrow I/I^2$, one can associate an element of $\mathcal{G}$ in an obvious manner; we call it (I, ω_I). Let $\mathcal{H}$ be the subgroup of $\mathcal{G}$ generated by all elements of the type (I, ω_I) where $\dim(A/I) = 0$ and ω_I can be lifted to a surjection from A^n to I. We define $\widetilde{E}(A) = \mathcal{G}/\mathcal{H}$.

One can follow the theory of Euler class groups as developed in [B-RS 3] and adapting similar methods can prove the following assertion: *Let J be a zero dimensional ideal* *of A and ω_J be a local orientation of J. Then $(J, \omega_J) = 0$ in $\widetilde{E}(A)$ if and only if ω_J can be lifted to a surjection $\alpha : A^n \twoheadrightarrow J$.* One has to modify the "addition" and "subtraction" principles suitably (we skip the details here). As the reader may have suspected by now, the group defined above is indeed isomorphic to the Euler class group.

Proposition 4.2. *Let A be a Noetherian ring of dimension $n \geq 2$. The canonical map from $E(A)$ to $\widetilde{E}(A)$ is an isomorphism.*

Proof. It is obvious that there is a canonical map, say, $\theta : E(A) \longrightarrow \widetilde{E}(A)$, which takes an element (J, ω_J) of $E(A)$ to (J, ω_J) in $\widetilde{E}(A)$. It is also clear that θ is a group homomorphism.

We now define a map in the reverse direction. Let $J \subset A$ be an ideal such that $\dim(A/J) = 0$ and let ω_J be a local orientation of J given by: $J = (a_1, \cdots, a_n) + J^2$. By (2.10) there is $e \in J^2$ such that $J = (a_1, \cdots, a_n, e)$, where $e(1-e) \in (a_1, \cdots, a_n)$. Using a standard general position argument (see [D-RS, 2.4]) it follows that there are elements $\gamma_1, \cdots, \gamma_n \in A$ such that the ideal $I = (a_1 + \gamma_1 e, \cdots, a_n + \gamma_n e)$ has the property that $\text{ht}\,(I_e) \geq n$. Note that $I + (e) = J$ and $(e) \subset J^2$. Applying (2.10) we see that there is an ideal J' such that

$$(a_1 + \gamma_1 e, \cdots, a_n + \gamma_n e) = J \cap J'$$

where $J' + (e) = A$. Now it is easy to deduce that $\text{ht}\,(J') \geq n$. The case when $J' = A$ being trivial, we assume that $\text{ht}\,(J') = n$. Let us write $b_i = a_i + \gamma_i e$. Clearly $b_1, \cdots, b_n$ induce ω_J. Let $\omega_{J'}$ be the local orientation of J' induced by $b_1, \cdots, b_n$. We then have $(J, \omega_J) + (J', \omega_{J'}) = 0$ in $\widetilde{E}(A)$. One can repeat the above procedure for J' and $\omega_{J'}$ to obtain an ideal J'' of height n and a local orientation $\omega_{J''}$ such that $(J', \omega_{J'}) + (J'', \omega_{J''}) = 0$ in $\widetilde{E}(A)$. Therefore, $(J, \omega_J) = (J'', \omega_{J''})$ in $\widetilde{E}(A)$.

We define $\eta : \widetilde{E}(A) \longrightarrow E(A)$ by sending (J, ω_J) to $(J'', \omega_{J''})$ in $E(A)$. It can be easily checked that η is well-defined. Clearly η is a group homomorphism. Further, θ and η are inverses of each other. $\square$

Remark 4.3. Clearly $\widetilde{E}(A) \simeq \widetilde{E}(A_{\text{red}})$.

We now assume that $R \hookrightarrow S$ is an integral extension with $\dim(R) = n \geq 2$. Let $(J, \omega_J) \in \widetilde{E}(R)$, where $\dim(R/J) = 0$ and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ is a local orientation of J. As $R \hookrightarrow S$ is integral, we have $\dim(S/JS) = 0$. Further, ω_J induces $\omega_J^* : (S/JS)^n \twoheadrightarrow JS/(JS)^2$, a local orientation of JS. Therefore, $(JS, \omega_J^*) \in \widetilde{E}(S)$. It is now easy to see that there is a group homomorphism, say, $\Phi : \widetilde{E}(R) \longrightarrow \widetilde{E}(S)$, which takes (J, ω_J) to (JS, ω_J^*). Using (4.2), we have a group homomorphism from $E(R)$ to $E(S)$.

We now prove the following theorem which improves (3.11).

Theorem 4.4. *Let $R \hookrightarrow S$ be an integral extension such that the extension $R_{red} \hookrightarrow S_{red}$ is birational. Then the map $\Phi : \widetilde{E}(R) \longrightarrow \widetilde{E}(S)$ is surjective. Further, if $R_{red} \hookrightarrow S_{red}$ is a subintegral extension, then Φ is an isomorphism.*

Proof. We first show that it is enough to prove the result when $R \hookrightarrow S$ is a finite extension. To see this, let $(I, \omega) \in \widetilde{E}(S)$. Then I is an ideal of S with $\dim(S/I) = 0$ and ω is a local orientation of I induced by, say, $I = (f_1, \cdots, f_n) + I^2$. Then by (2.10), there exists $e \in I$ such that $I = (f_1, \cdots, f_n, e)$ where $e(1-e) \in (f_1, \cdots, f_n)$. Suppose that $e(1-e) = k_1f_1 + \cdots k_nf_n$ where $k_i \in S$. Now consider $R_1 = R[f_1, \cdots, f_n, e, k_1, \cdots, k_n] \hookrightarrow S$ and $R \hookrightarrow R_1$ is finite. Let $I' = (f_1, \cdots, f_n, e)R_1$. Then $I' = (f_1, \cdots, f_n, e - k_1f_1 - \cdots - k_nf_n) = (f_1, \cdots, f_n, e^2)$ implying that $I' = (f_1, \cdots, f_n) + I'^2$. If ω' denotes the local orientation of I' induced by this set of generators of I'/I'^2, then $(I', \omega') \in \widetilde{E}(R_1)$. The map from $\widetilde{E}(R_1)$ to $\widetilde{E}(S)$ takes (I', ω') to (I, ω). It is enough to find a preimage of (I', ω') in $\widetilde{E}(R)$. Therefore, we may assume the extension to be finite to start with.

Let us write $\overline{R} = R_{\text{red}}$ and $\overline{S} = S_{\text{red}}$. Let C be the conductor of $\overline{R}$ in $\overline{S}$. As $\overline{R} \hookrightarrow \overline{S}$ is birational, we have $\text{ht}(C) \geq 1$.

Let $(I, \omega_I) \in \widetilde{E}(\overline{S})$. By (4.2) we may assume that $\text{ht}(I) = n$. Exactly the same proof as in Step 2 of (3.9) will show that the map from $\widetilde{E}(\overline{R})$ to $\widetilde{E}(\overline{S})$ is surjective. It follows that the map $\Phi : \widetilde{E}(R) \longrightarrow \widetilde{E}(S)$ is surjective.

We now assume that $\overline{R} \hookrightarrow \overline{S}$ is subintegral. Then, by (3.11) $E(\overline{R}) \simeq E(\overline{S})$, and we have,

$$\widetilde{E}(R) \simeq E(R) \simeq E(\overline{R}) \simeq E(\overline{S}) \simeq E(S) \simeq \widetilde{E}(S).$$

$\square$

Example 4.5. (**Bhatwadekar**) Let $S = \mathbb{C}[X, Y]$ and $f = (X^2 - Y^3)$. Let $S_1 = S/(f)$. Then $(S_1)^* = \mathbb{C}^*$. Then we have, $SK_1(S_1) \neq 0$ (see [Kr, Section 12], [Mu-Pe]). Let $R = \mathbb{C}[X] + fS$. Then S is a finite (birational) extension of R and fS is the conductor of R in S. Moreover, $R/fS \simeq \mathbb{C}[X]$.

Now it is easy to see that R is seminormal. From the Cartesian square

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
R/f \arrow[r] & S/fS \simeq S_1
\end{tikzcd}$$

we have an exact Mayer-Vietoris sequence (see [Ba, pp. 481]) as given below:

$$0 \longrightarrow R^* \longrightarrow S^* \oplus (R/f)^* \longrightarrow S_1^* \longrightarrow \text{Pic}(R) \longrightarrow \text{Pic}(S) \oplus \text{Pic}(R/f) \longrightarrow \text{Pic}(S_1).$$

Consequently we have

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathbb{C}^* \oplus \mathbb{C}^* \longrightarrow \mathbb{C}^* \longrightarrow \text{Pic}(R) \longrightarrow 0,$$

from which it follows that $\mathrm{Pic}(R) = 0$.

Now using the above Cartesian square, we have another exact sequence

$$SK_1(R) \longrightarrow SK_1(S) \oplus SK_1(R/f) \longrightarrow SK_1(S_1) \longrightarrow SK_0(R) \longrightarrow SK_0(S) \oplus SK_0(R/f) \longrightarrow SK_0(S_1).$$

It is easy to see that $SK_1(R/f)$, $SK_0(S)$ and $SK_0(R/f)$ are all trivial. Also by Suslin's stability theorem [Su 1] (or see [L, pp. 220]), we have $SK_1(S) = 0$. Hence we have

$$SK_1(S_1) \simeq SK_0(R).$$

As $\mathrm{Pic}(R) = 0$, we have $\widetilde{K_0}(R) \simeq SK_0(R)$.

Hence $\widetilde{K_0}(R) = SK_1(S_1) \neq 0$. Therefore there exists a projective R-module P of rank two (with trivial determinant) which is not stably free. Fix an isomorphism $\chi : R \simeq \wedge^2(P)$ and consider the Euler class $e(P, \chi) \in E(R)$. As P has trivial determinant, $e(P, \chi) = 0$ would imply that P is free. Therefore, $E(R)$ is not the trivial group whereas $E(S)$ is trivial, showing that the map from $E(R)$ to $E(S)$ is not injective. □

5. Subintegral extension of 2-dimensional rings

The following question, mentioned in [I, Remark (b), pp 331], is still open.

Question 5.1. Let $R \hookrightarrow S$ be a subintegral extension. Let P and Q be two projective R-module with $\det(P) \simeq \det(Q)$ and $P \otimes S \simeq Q \otimes S$. Is $P \simeq Q$?

In [I] it is suggested that perhaps the compatibility of the two isomorphisms is required as an additional hypothesis. Following that suggestion we give an affirmative answer in the case when $\dim(R) = 2$.

We need the following crucial lemma from [B 2].

Lemma 5.2. *Let A be a ring and P and Q be two projective A-module of rank 2 such that $det(P) \simeq det(Q)$. Let $\chi : det(P) \xrightarrow{\sim} det(Q)$ be an isomorphism. Let $J \subset A$ be an ideal of height 2. Let $\alpha : P \twoheadrightarrow J$ and $\beta : Q \twoheadrightarrow J$ be two surjections. Let bar denote the reduction modulo J and $\overline{\alpha} : \overline{P} \twoheadrightarrow J/J^2$ and $\overline{\beta} : \overline{Q} \twoheadrightarrow J/J^2$ be the surjections induced from α and β, respectively. Suppose that there exists an isomorphism $\delta : \overline{P} \xrightarrow{\sim} \overline{Q}$ such that : (i) $\overline{\beta}\delta = \overline{\alpha}$ and (ii) $\wedge^2\delta = \overline{\chi}$. Then there exists an isomorphism $\sigma : P \xrightarrow{\sim} Q$ such that $\beta\sigma = \alpha$, σ is a lift of δ and $\wedge^2\sigma = \chi$.*

We now prove the main result of this section. Note that in view of (2.19), we do not need to assume that $\mathbb{Q} \subset R$.

Theorem 5.3. *Let R be a ring of dimension 2 and $R \hookrightarrow S$ be a subintegral extension. Let P and Q be two projective R-modules of rank 2 such that $det(P) \simeq det(Q)$ and $P \otimes S \simeq Q \otimes S$. Let $\chi : det(P) \xrightarrow{\sim} det(Q)$ and $\theta : P \otimes S \xrightarrow{\sim} Q \otimes S$ be isomorphisms. Assume that $\chi \otimes S = \wedge^2\theta$. Then $P \simeq Q$.*

Proof. Now $\det(P) = \wedge^2(P)$ and $\det(Q) = \wedge^2(Q)$ are projective R-modules of rank one. Since they are isomorphic, there is a projective R-module L of rank one which is isomorphic to both. We fix $\chi_1 : L \xrightarrow{\sim} \wedge^2 P$. Let $\chi_2 := (\chi)\chi_1 : L \xrightarrow{\sim} \wedge^2 P \xrightarrow{\sim} \wedge^2 Q$.

We now point out a general fact. Let A be a ring of dimension n and P be a projective A-module of rank n with an isomorphism $\chi : L \xrightarrow{\sim} \wedge^n(P)$. Recall from the definition of the Euler class of the pair (P, χ) that $e(P, \chi)$ is an invariant of the isomorphism class of (P, χ).

From the above paragraph we conclude that the Euler classes $e(P \otimes S, \chi_1 \otimes S)$ and $e(Q \otimes S, \chi_2 \otimes S)$ are equal in the Euler class group $E(S, L \otimes S)$.

The two elements $e(P \otimes S, \chi_1 \otimes S)$ and $(Q \otimes S, \chi_2 \otimes S)$ of $E(S, L \otimes S)$ are the images of the elements $e(P, \chi_1)$ and $e(P, \chi_2)$, respectively, under the natural map $\Phi : E(R, L) \longrightarrow E(S, L \otimes S)$. The map Φ is injective. Therefore $e(P, \chi_1) = e(Q, \chi_2)$ in $E(R, L)$. Let $e(P, \chi_1) = e(Q, \chi_2) = (I, \omega_I)$ in $E(R, L)$.

Now using [B-RS 3, 4.3], there exist two surjections $f : P \twoheadrightarrow I$ and $g : Q \twoheadrightarrow I$ such that (I, ω_I) is obtained from (f, χ_1) and (g, χ_2).

Let $\mu : (R/I)^2 \xrightarrow{\sim} P/IP$ and $\tau : (R/I)^2 \xrightarrow{\sim} Q/IQ$ be two isomorphisms such that $\wedge^2 \mu = \overline{\chi}_1$ and $\wedge^2 \tau = \overline{\chi}_2$. From the definition of the Euler class of a projective module, it follows that $\omega_I = \overline{f}\mu = \overline{g}\tau$

Now consider the isomorphism $\delta = \tau\mu^{-1} : P/IP \xrightarrow{\sim} Q/IQ$. Then we have $\overline{g}\delta = \overline{f}$ and $\wedge^2(\delta) = (\wedge^2\tau)(\wedge^2\mu^{-1}) = \overline{\chi_2\chi_1}^{-1} = \overline{\chi}$.

Therefore we have two surjections $f : P \twoheadrightarrow I$ and $g : Q \twoheadrightarrow I$ and an isomorphism $\delta : P/IP \xrightarrow{\sim} Q/IQ$ such that $\wedge^2\delta = \overline{\chi}$ and $\overline{g}\delta = \overline{f}$. Therefore by the above lemma there exists an isomorphism $\phi : P \xrightarrow{\sim} Q$ such that: (i) $\beta\phi = \alpha$, (ii) ϕ is a lift of δ, and (iii) $\wedge^2\phi = \chi$. Hence the theorem is proved. $\square$

Corollary 5.4. *Let $R \hookrightarrow S$ be a subintegral extension with dim$(R) = 2$. Let L be a rank 1 projective R-module such that $(L \oplus R) \otimes S$ is cancellative. Then $L \oplus R$ is also cancellative.*

Proof. We give two proofs of this corollary.

Proof 1. Let P be a projective R-module of rank 2 such that $L \oplus R^2 \simeq P \oplus R$. Then $L \simeq \wedge^2(P)$. We fix an isomorphism $\chi : L \xrightarrow{\sim} \wedge^2(P)$.

Let us denote $L \otimes_R S$ by $\widetilde{L}$ and $P \otimes_R S$ as $\widetilde{P}$. As $\widetilde{L}$ is cancellative, there is an isomorphism $\phi : \widetilde{L} \oplus S \xrightarrow{\sim} \widetilde{P}$. Then ϕ induces $\wedge^2(\phi) : \widetilde{L} \xrightarrow{\sim} \wedge^2(\widetilde{P})$. The isomorphism $\chi \otimes S : \widetilde{L} \xrightarrow{\sim} \wedge^2(\widetilde{P})$ (induced by χ) and the isomorphism $\wedge^2(\phi)$ differ by a unit $u \in S$. Define $\tau : \widetilde{L} \oplus S \longrightarrow \widetilde{L} \oplus S$ by sending (l, s) to (l, us). Then τ is an isomorphism and moreover, $\wedge^2(\tau) : \widetilde{L} \xrightarrow{\sim} \widetilde{L}$ is just scalar multiplication by u. Then the composition $\theta = \phi\tau : \widetilde{L} \oplus S \xrightarrow{\sim} \widetilde{P}$ has the property that $\wedge^2(\theta) = \chi \otimes S$. Now we can apply the above theorem to conclude that $L \oplus R \xrightarrow{\sim} P$. Thus, $L \oplus R$ is cancellative.

Proof 2. In this proof we do not use the above theorem. We shall denote $L \otimes_R S$ by $\widetilde{L}$. We use the following observation of Bhatwadekar in [B 2, p. 348] : *for a ring A and a projective A-module L of rank one, $L \oplus A$ is cancellative if and only if $E(A, L) \simeq E_0(A, L)$.*

Now assume that $\widetilde{L} \oplus S$ is cancellative. Then $E(S, \widetilde{L}) \simeq E_0(S, \widetilde{L})$. Since $R \hookrightarrow S$ is subintegral, using (3.12) and (3.27) it follows that $E(R, L) \simeq E_0(R, L)$. Therefore $L \oplus R$ is cancellative. $\square$.

For the next result we need the following proposition from [B 2].

Proposition 5.5. *Let A be ring of dimension 2 and let P be a rank 2 projective A-module. If $\wedge^2 P \oplus A$ is cancellative, then P is cancellative.*

Corollary 5.6. *Let $R \hookrightarrow S$ be a subintegral extension with $dim(R) = 2$. Suppose that all projective S-modules of rank 2 are cancellative. Then all projective R-modules of rank 2 are cancellative.*

Proof. By the above proposition it is sufficient to consider rank 2 projective R-modules of the form $L \oplus R$. Rest of the proof follows from the above corollary. $\square$.

We can easily extend (5.3) to $R[T]$ for projective modules with trivial determinants, in the following way.

Theorem 5.7. *Let R be a ring (containing $\mathbb{Q}$) of dimension 2 and $R \hookrightarrow S$ be a subintegral extension. Let P and Q be two projective $R[T]$-modules of rank 2 with trivial determinants such that $P \otimes S[T] \simeq Q \otimes S[T]$. Let $\chi : det(P) \xrightarrow{\sim} det(Q)$ and $\theta : P \otimes S[T] \xrightarrow{\sim} Q \otimes S[T]$ be isomorphisms. Assume that $\chi \otimes S = \wedge^2 \theta$. Then $P \simeq Q$.*

Proof. We just follow the proof of (5.3). We repeat Step 1 word by word, only replacing L by $R[T]$. In Step 2 we only need to use [D 1, 7.6] in place of [B-RS 3, 4.3]. $\square$

Remark 5.8. A nontrivial result that is hidden in the proof of (5.7) is the symplectic cancellation theorem of Bhatwadekar [B 2, 4.8], which is used to prove [D 1, 7.6].

6. A PROOF BY BHATWADEKAR

Let A be a ring and n be an integer such that $2 \leq \dim(A) \leq 2n - 4$. Van der Kallen [VK2] proved that the orbit set $\mathrm{Um}_n(A)/\mathcal{E}_n(A)$ carries a group structure. Now let $A \hookrightarrow B$ be a subintegral extension. Let $u, v \in \mathrm{Um}_n(A)$. In [G], Gubeladze proves that

$$u \underset{\mathcal{E}_n(A)}{\sim} v \iff u \underset{\mathcal{E}_n(B)}{\sim} v.$$

Consequently, one obtains that the canonical group homomorphism $\mathrm{Um}_n(A)/\mathcal{E}_n(A) \to \mathrm{Um}_n(B)/\mathcal{E}_n(B)$ is injective. We learnt from S. M. Bhatwadekar, through private communication, the proof of the following result. Bhatwadekar's approach has a different appeal and we believe the reader will find it interesting. We thank Bhatwadekar most sincerely for allowing us to include it here.

Theorem 6.1. *Let $R \hookrightarrow S$ be a subintegral extension of rings of dimension $n \geq 2$. Then the induced map $\phi : Um_{n+1}(R)/\mathcal{E}_{n+1}(R) \longrightarrow Um_{n+1}(S)/\mathcal{E}_{n+1}(S)$ is an isomorphism.*

Proof. (**Bhatwadekar**) Without loss of generality we may assume that R, S are both reduced. Further, we may assume that $R \hookrightarrow S$ is an elementarily subintegral extension and, say, $S = R[b]$ with $b^2, b^3 \in R$, $b \notin R$. Let C be the conductor ideal of R in S.Then $S/C \simeq R/C \oplus (R/C)\cdot \overline{b}$ and $\overline{b}^2 = 0$, where $\overline{b}$ is the residue class of b modulo C. Further, we have $\text{ht}(C) \geq 1$.

Let $(a_0, \cdots, a_n) \in \text{Um}_{n+1}(R)$. Since $\dim(R/C) \leq n-1$, the row, when reduced modulo C, is elementarily completable. Therefore we can assume that $(a_0, \cdots, a_n) \equiv (1, 0, \cdots, 0)$ modulo C.

We first show that the map ϕ is injective. Let $(a_0, \cdots, a_n) \in \text{Um}_{n+1}(R)$ be such that $(a_0, \cdots, a_n) \underset{\mathcal{E}_{n+1}(S)}{\sim} (1, 0, \cdots, 0)$. Therefore there exists $\sigma \in \mathcal{E}_{n+1}(S)$ whose first row is $(a_0, \cdots, a_n)$. Let $\overline{\sigma}$ be the image of σ in $\mathcal{E}_{n+1}(S/C)$. By the observation in the preceding paragraph, the first row of $\overline{\sigma}$ is $(1, 0, \cdots, 0)$. Applying some elementary transformations, we may assume that $\overline{\sigma}$ has the following form:

$$\overline{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}$$

where $\theta \in SL_n(S/C) \cap \mathcal{E}_{n+1}(S/C)$. We first show that there exist $\delta \in SL_n(R/C)$ and $\varepsilon \in \mathcal{E}_n(S/C)$ such that $\varepsilon\theta = \delta$.

Consider the conductor diagram:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/C & \longrightarrow & S/C \simeq R/C \oplus (R/C)\overline{b} \end{array}$$

Clearly θ induces a matrix $\delta \in SL_n(R/C)$. Now consider $\varepsilon = \delta\theta^{-1} \in SL_n(S/C)$. Observe that ε is identity modulo $\text{nil}(S/C)$ and therefore $\varepsilon \in \mathcal{E}_n(S/C)$.

Let $\tau \in \mathcal{E}_n(S)$ be a lift of ε. Consider the matrix

$$\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \sigma \in \mathcal{E}_{n+1}(S)$$

Then

$$\overline{\lambda} = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{\tau} \end{pmatrix} \overline{\sigma} = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \varepsilon \end{pmatrix} \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \theta \end{pmatrix} = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \delta \end{pmatrix}$$

Since the image of $\overline{\sigma}$ is in $\mathcal{E}_{n+1}(R/C)$, it follows that $\delta \in SL_n(R/C) \cap \mathcal{E}_{n+1}(R/C)$. Therefore, $\overline{\lambda} \in \mathcal{E}_{n+1}(R/C)$. Hence we have $\lambda \in SL_{n+1}(R)$. Also $(a_0, \cdots, a_n)$ is the first row of λ. Let $\alpha \in \mathcal{E}_{n+1}(R)$ be a lift of $\overline{\lambda}$. Now consider the matrix $\beta = \lambda\alpha^{-1}$, which is elementary in S and also $\overline{\beta} = I_{n+1}$ in R/C.

Recall that by [St, 2.2] the relative group $\mathcal{E}_{n+1}(S, C)$ is isomorphic to the kernel of the natural map $\mathcal{E}_{n+1}(S) \twoheadrightarrow \mathcal{E}_{n+1}(S/C)$. Therefore, $\beta \in \mathcal{E}_{n+1}(S, C)$. The first row of β is $(a_0, a_1, \cdots, a_n)\alpha^{-1}$. Let us call it $(b_0, b_1, \cdots, b_n)$. Then $(b_0, \cdots, b_n) \equiv (1, 0, \cdots, 0)$ modulo C. In other words, $(b_0, \cdots, b_n) \in \mathrm{Um}_{n+1}(R, C)$ (Recall that the set $\mathrm{Um}_{n+1}(R, C)$ consists of unimodular rows $(v_0, \cdots, v_n)$ of R such that $v_0 \equiv 1$ modulo C, and $v_i \in C$ for $i = 1, \cdots, n$). Now it is enough to prove that $(b_0, \cdots, b_n) \underset{\mathcal{E}_{n+1}(R)}{\sim} (1, 0, \cdots, 0)$.

We know that $\mathrm{Um}_{n+1}(R, C)/\mathcal{E}_{n+1}(R, C)$ and $\mathrm{Um}_{n+1}(S, C)/\mathcal{E}_{n+1}(S, C)$ have group structures due to Wander Kallen. By [VK1, 3.21, pp 382], the natural maps

$$\mathrm{Um}_{n+1}(\mathbb{Z} \oplus C, C)/\mathcal{E}_{n+1}(\mathbb{Z} \oplus C, C) \longrightarrow \mathrm{Um}_{n+1}(R, C)/\mathcal{E}_{n+1}(R, C)$$

$$\mathrm{Um}_{n+1}(\mathbb{Z} \oplus C, C)/\mathcal{E}_{n+1}(\mathbb{Z} \oplus C, C) \longrightarrow \mathrm{Um}_{n+1}(S, C)/\mathcal{E}_{n+1}(S, C)$$

are isomorphisms and hence $\mathrm{Um}_{n+1}(R, C)/\mathcal{E}_{n+1}(R, C) \longrightarrow \mathrm{Um}_{n+1}(S, C)/\mathcal{E}_{n+1}(S, C)$ is an isomorphism. Recall that we have $(b_0, \cdots, b_n) \underset{\mathcal{E}_{n+1}(S,C)}{\sim} (1, 0, \cdots, 0)$. Therefore $(b_0, \cdots, b_n) \underset{\mathcal{E}_{n+1}(R,C)}{\sim} (1, 0, \cdots, 0)$, proving that ϕ is injective.

Now we show that ϕ is surjective. Let $(a_0, \cdots, a_n) \in \mathrm{Um}_{n+1}(S)$. Then $(\overline{a_0}, \overline{a_1}, \cdots, \overline{a_n}) \in \mathrm{Um}_{n+1}(S/C)$, where bar denotes reduction modulo C. Since $\dim(S/C) \leq n - 1$, the row $(\overline{a}_0, \cdots, \overline{a}_n)$ is elementarily completable. Say $(\overline{a}_1, \cdots, \overline{a}_n)\overline{\theta} = (\overline{1}, \cdots, \overline{0})$. Let $\theta \in \mathcal{E}_{n+1}(S)$ be a lift of θ. Write $(a_0, \cdots, a_n)\theta = (b_0, \cdots, b_n)$. Then $(\overline{b}_0, \cdots, \overline{b}_n) = (\overline{1}, \overline{0}, ..., \overline{0})$. Therefore we have $b_0 - 1 \in C \subset R$ and $b_1, \cdots, b_n \in C \subset R$. Hence $b_0, \cdots, b_n \in R$. Let $\mathfrak{m}$ be any maximal ideal of R. If $C \subset \mathfrak{m}$, then $b_0 \notin \mathfrak{m}$. If $C \not\subset \mathfrak{m}$, then $b_i \notin \mathfrak{m}$ for some $i = 1, \cdots, n$. In any case, the ideal in R generated by $b_0, \cdots, b_n$ is not contained in any maximal ideal of R. Therefore, $(b_0, \cdots, b_n) \in \mathrm{Um}_{n+1}(R)$. $\square$

An alternative proof of (3.25) can now be given.

Proof of (3.25): Let $R \hookrightarrow S$ be a subintegral extension of $\mathbb{Q}$-algebras with $\dim(R) = n = \dim(S)$, where n is even. By [B-RS 3, 7.6] we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\frac{Um_{n+1}(R)}{\mathcal{E}_{n+1}(R)} \arrow[r] \arrow[d, "\phi"] & E(R) \arrow[r] \arrow[d, "\Phi"] & E_0(R) \arrow[r] \arrow[d, "\Phi_0"] & 0 \\
\frac{Um_{n+1}(S)}{\mathcal{E}_{n+1}(S)} \arrow[r] & E(S) \arrow[r] & E_0(S) \arrow[r] & 0
\end{tikzcd}$$

As Φ and ϕ are both isomorphisms, it follows that so is Φ_0. $\square$

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