THE EULER CLASS GROUP OF A POLYNOMIAL ALGEBRA WITH COEFFICIENTS IN A LINE BUNDLE

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ABSTRACT. Let A be a commutative Noetherian ring and P be a projective A-module of rank $= (\dim(A) - 1)$. An intriguing open question is to find the precise obstruction for P to split as: $P \simeq Q \oplus A$ for some A-module Q. In this paper we settle this question when $A = R[T]$ for some ring R containing the field of rationals and P is a projective A-module of rank $= \dim(R)$.

1. INTRODUCTION

Let A be a Noetherian ring of (Krull) dimension n. A classical result of Serre [Se] asserts that if rank$(P) \geq n+1$, then $P \simeq Q \oplus A$ for some A-module Q. There are well-known examples of rings A and indecomposable projective A-modules of rank $\leq \dim(A)$ to show that Serre's result is best possible. Most of the research in projective modules in last thirty years is centred around the following question.

Question 1.1. Let A be a Noetherian ring of dimension n and P be a projective A-module of rank $r \leq n$. What is the precise obstruction for P to split off a free summand of rank one?

To tackle the above question one would like to find a suitable "obstruction group" $G^r(A)$ so that given a projective A-module P of rank r, an element $x_r(P) \in G^r(A)$ can be associated such that $x_r(P)$ is trivial in $G^r(A)$ if and only if $P \simeq Q \oplus A$. This has been achieved in the case $r = n$ through the following path-breaking works.

(1) [MK-M, Mu] Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension n over an algebraically closed field k. Then the Chow group $CH^n(X)$ is the obstruction group and $c_n(P)$ (the top Chern class of P) is the obstruction element. It is well-known that this result is no longer valid for arbitrary base field k.

(2) [Bh-RS 1, Bh-RS 3] Let A be a Noetherian $\mathbb{Q}$-algebra of dimension n. The n-th Euler class group $E^n(A, L)$ of A with coefficients in a line bundle L (defined in [Bh-RS 3]) takes the role of $G^n(A)$. Given a projective module P of rank

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n, its Euler class $e(P,\chi)$ takes the role of $x_n(P)$, where $\chi : \wedge^n(P) \xrightarrow{\sim} L$ is an isomorphism.

(3) [B-M, F, F-Sr, Mo] Let X be a smooth affine scheme of dimension n and L be a line bundle over X. Let $\mathcal{E}$ be a vector bundle of rank n with determinant L. Then, one can take the Chow-Witt group $\widetilde{CH}^n(X,L)$ (defined in [B-M]) as the obstruction group. The Euler class associated to $\mathcal{E}$ in this group works as the obstruction element.

It is not known if the groups in (2) and (3) are isomorphic for a smooth affine scheme.

As described above, Question 1.1 has a satisfactory solution in the case: rank$(P) =$ dim(A). However, not much progress has been made for rank$(P) <$ dim(A) (see [Bh-RS 5] for some results in this direction). The first case that one would like to investigate is obviously when rank$(P) =$ dim$(A) - 1$. In this context, it is most natural to understand first what happens if A is a polynomial algebra, i.e., $A = R[T]$, where R is a Noetherian $\mathbb{Q}$-algebra of dimension n, and P is a projective $R[T]$-module of rank n. In this setup, we settle Question 1.1 in this paper. A partial solution in the same setup has been obtained by the first author in [D 1]. We give the details below.

Let R be a commutative Noetherian ring of dimension $n \geq 2$ containing $\mathbb{Q}$. Following the works of Bhatwadekar and Raja Sridharan [Bh-RS 1, Bh-RS 3] on the Euler class groups, the notion of the n-th Euler class group $E^n(R[T])$ has been defined and explored in detail in [D 1, D 2]. This group serves as an obstruction group to detect whether a given projective $R[T]$-module P of rank n, *with trivial determinant*, splits as $P \simeq Q \oplus R[T]$. To achieve this, given such a P and a trivialization $\chi : R[T] \xrightarrow{\sim} \wedge^n(P)$, an element of $E^n(R[T])$ was associated to the pair (P,χ), which is called the Euler class of (P,χ). It was then proved [D 1, 4.11] that $P \simeq Q \oplus R[T]$ if and only if this Euler class vanishes in $E^n(R[T])$.

Evidently the theory was limited, as it could only capture projective $R[T]$-modules with trivial determinant. In this paper we eliminate that restriction. We extend the theory to $E^n(R[T],L)$ (the n-th Euler class group of $R[T]$ with respect to a line bundle L over $R[T]$) and define the Euler class of a pair (P,χ), where P is a projective $R[T]$-module of rank n and $\chi : L \xrightarrow{\sim} \wedge^n(P)$ an isomorphism. We prove that (Theorem 6.12 below) this Euler class vanishes in $E^n(R[T],L)$ if and only if P splits off a free summand of rank one.

We carry this out in two steps. First, in Section 4 we tackle the case when the line bundle L is extended from R. An expert who is also familiar with [D 1] will realize that this part is not much difficult if one keeps [D 1] as a guide. We have been able to extend almost all the relevant results from [D 1] here. These results are then crucially used to develop the theory further, as discussed below.

But before we describe our next step, let us digress a bit. It is not hard to believe that for all practical purposes we may assume that R is reduced. Now let R_1 be the seminormalization of R. Then R_1 is seminormal and as a consequence, $\text{Pic}(R_1) \simeq \text{Pic}(R_1[T])$. In other words, line bundles over $R_1[T]$ are extended from R_1. We successfully exploit this phenomenon to define and study the n-th Euler class group $E^n(R[T], L)$ for $n \geq 4$, when L is not necessarily extended from R.

For an arbitrary L, the idea is to find a suitable finite subintegral extension S of R (i.e., $R \hookrightarrow S \hookrightarrow R_1$), such that the projective $S[T]$-module $L \otimes S[T]$ is extended from S. Note that, $E^n(S[T], L \otimes S[T])$ is now well-understood due to the results in Section 4. In Section 5 we introduce a machinery, which is modeled on a series of lemmas from [Bh 1], to descend from $S[T]$ to $R[T]$. Then in Section 6 we develop the theory of the Euler class group $E^n(R[T], L)$ by going forth and back by using these 'descent lemmas'. Finally, it turns out that $E^n(R[T], L) \simeq E^n(S[T], L \otimes S[T]) \simeq E^n(R_1[T], L \otimes R_1[T])$.

Unfortunately the method we just described above does not work so well in the case $\dim(R) = 3$ due to the lack of a suitable "subtraction principle". We treat this case separately in Section 7 by defining a "restricted" Euler class group which serves most of our purposes. For instance, here also we prove that the Euler class of a projective $R[T]$-module P of rank 3 is the precise obstruction for P to split off a free summand of rank one. In this section we also treat the case $\dim(R) = 2$ and extend most of the results from [D 1] for a two dimensional ring R.

We should probably warn our reader that the theory of $E^n(R[T])$ or $E^n(R[T], L)$ is not obtained by just replacing R by $R[T]$ in the works of Bhatwadekar and Raja Sridharan. We note that $\dim(R[T]) = n + 1$. By a result of Mandal [M1], the top Euler class group (the one considered in [Bh-RS 3]), $E^{n+1}(R[T])$ is trivial. Using some additional arguments it can be proved that $E^{n+1}(R[T], L)$ is trivial too (see (6.1) below). In this paper we are dealing only with the n-th Euler class groups and therefore we shall drop the superscript and denote a group like $E^n(R[T], L)$ by $E(R[T], L)$.

2. Preliminaries

All the rings considered in this paper are assumed to be commutative, Noetherian. By the dimension of a ring we mean its Krull dimension. Modules are assumed to be finitely generated. Projective modules are assumed to have constant rank.

Definition 2.1. Let R be a ring and P be a projective R-module. An element $p \in P$ is called *unimodular* if there is a surjective R-linear map $\phi : P \twoheadrightarrow R$ such that $\phi(p) = 1$. Note that P has a unimodular element if and only if $P \simeq Q \oplus R$ for some R-module Q. The set of all unimodular elements of P will be denoted by $\text{Um}(P)$.

The following "splitting theorem" is due to Serre [Se].

Theorem 2.2. *Let R be a ring and P be a projective R-module. If rank$(P) \geq$ dim$(R)+1$, then $P \simeq Q \oplus R$ for some R-module Q.*

Let P be a projective R-module. The group of automorphisms of P of determinant one will be denoted by $SL(P)$. We now recall the definition of a subgroup of $SL(P)$. Given $\varphi \in P^*(= \mathrm{Hom}_R(P,R))$ and $p \in P$, let φ_p denote the composite $P \xrightarrow{\varphi} R \xrightarrow{p} P$. If $\varphi(p) = 0$, then $\varphi_p^2 = 0$ and $1+\varphi_p$ is an automorphism of P.

Definition 2.3. An automorphism of P is called a *transvection* if it is of the form $1+\varphi_p$ where $\varphi(p) = 0$ and either φ is unimodular in P^* or p is unimodular in P. The subgroup of $SL(P)$ generated by all transvections will be denoted by $\mathcal{E}(P)$.

The following classical result is due to Bass [Ba].

Theorem 2.4. *Let A be a ring and let P be a projective A-module of rank $>$ dim$A$. Then the group $\mathcal{E}(P \oplus A)$ of transvections of $P \oplus A$ acts transitively on Um$(P \oplus A)$.*

The following result is due to Lindel [Li 2, 2.6].

Theorem 2.5. *Let A be a commutative Noetherian ring with* $\dim A = d$ *and* $R = A[T_1, \cdots, T_n]$. *Let P be a projective R-module of rank $\geq \max(2, d+1)$. Then $\mathcal{E}(P \oplus R)$ acts transitively on the set of unimodular elements of $P \oplus R$.*

We shall need the following proposition on lifting of a transvection.

Theorem 2.6. [Bh-R 1, 4.1] *Let A be a ring, $J \subset A$ be an ideal and P be a projective A-module of rank n. Then any transvection $\widetilde{\theta} \in \mathcal{E}(P/JP)$ can be lifted to a (unipotent) automorphism θ of P. If in addition, the map Um$(P) \to$ Um(P/JP) is surjective, then the map $\mathcal{E}(P) \to \mathcal{E}(P/JP)$ is surjective.*

The next two propositions from [D-Z] are crucial to this paper. To make this paper self-contained, we reproduce the proofs.

Proposition 2.7. [D-Z, 2.7] *Let S be a ring and J, C be ideals of S such that $J + C = S$. Let P be a projective S-module and σ be a transvection of P/CP. Then σ can be lifted to $\tau \in$ Aut(P) with the property that τ is identity modulo J.*

Proof. Let $\sigma = 1+\psi_q$, where $\psi \in (P/CP)^*$ and $q \in P/CP$ such that $\psi(q) = 0$. Suppose $p \in P$ and $\theta \in P^*$ be lifts of q and ψ. Then we have $\theta(p) = c$, for some $c \in C$.

We first consider the case when q is a unimodular element of P/CP. Then there exists $\varphi \in P^*$ such that $\varphi(p) = 1+d$, for some $d \in C$.

Set $\phi' = (1+d)\theta - c\varphi$. Then $\phi'(p) = 0$ and ϕ' is a lift of ψ. Therefore $1+\phi'_p \in \mathrm{Aut}\,(P)$ and it lifts σ. Since $J + C = S$, therefore there exist $a \in J$ and $b \in C$ such that $a + b = 1$.

Finally we consider $\tau = 1 + a\phi'_p$. Then again $\tau \in \text{Aut}\,(P)$, $\tau = \text{Id}$ modulo J and τ is a lift of σ.

Next we consider the case when $\psi \in \text{Um}((P/CP)^*)$. Then there exists $p' \in P$ such that $\theta(p') = 1 + e$, for some $e \in C$. Consider the element $q' = (1+e)p + cp'$, then $\theta(q') = 0$. Therefore, $\tau = 1 + a\theta_{q'}$ will work. $\square$

Proposition 2.8. [D-Z, 2.6] *Let R be a ring and P be a projective R-module such that P has a unimodular element. Let $\alpha, \beta \in Um(P^*)$ be such that $\alpha \equiv \beta$ modulo the nil radical $\mathfrak{n}$ of R. Then there is $\theta \in \mathcal{E}(P)$ such that $\beta = \alpha\theta$.*

Proof. Applying [MK-M-R, Remark 2.3] it follows that there is $\Theta \in \mathcal{E}(P^*)$ such that $\Theta(\alpha) = \beta$. Therefore, Θ is a finite product of transvections of the projective module P^*. For simplicity, we prove this proposition by assuming that Θ itself is a transvection. The general case can be worked out in a similar manner.

Let $\Theta = 1 + \psi_\phi$, where $\psi \in P^{**}$ and $\phi \in P^*$ such that $\psi(\phi) = 0$. Since P is a projective module, P^{**} can be identified with P. Therefore, we may assume $\psi = p$ for some $p \in P$. With this identification we have $\psi(\phi) = \phi(p) = 0$. Now from the definition of a transvection, we have, either $\psi \in Um(P^{**})$ or $\phi \in Um(P^*)$. If $\psi \in Um(P^{**})$, then note that $p \in \text{Um}(P)$. Therefore $1 + \phi_p$ is a transvection of P (as $\phi(p) = 0$).

Now we have $\Theta(\alpha) = (1 + \psi_\phi)(\alpha) = \beta$. Therefore, for any $q \in P$, we have $(1 + \psi_\phi)(\alpha)(q) = \beta(q)$. But $(1 + \psi_\phi)(\alpha)(q) = \alpha(q) + (\phi\psi)(\alpha)(q) = \alpha(q) + (\phi\alpha(p))(q) = \alpha(q) + \alpha(p)\phi(q)$.

On the other hand, $\alpha(1 + \phi_p)(q) = \alpha(q + p\phi(q)) = \alpha(q) + \alpha(p)\phi(q)$ and hence $\alpha(1 + \phi_p)(q) = \beta(q)$ for all $q \in P$. Therefore, if we write $\theta = 1 + \phi_p$, then θ is a transvection of P and $\alpha\theta = \beta$. $\square$

We collect a bunch of lemmas from [Bh-RS 3].

Lemma 2.9. [Bh-RS 3, 2.11] *Let R be a Noetherian ring and $J \subset R$ be an ideal of R. Let $K \subset J$ and $L \subset J^2$ be two ideals of R such that $K + L = J$. Then $J = K + (e)$ for some $e \in L$ with $e(1-e) \in K$ and $K = J \cap J'$ where $J' + L = R$.*

Lemma 2.10. [Bh-RS 3, 2.13] *Let A be a ring and P be a projective A-module of rank n. Let $(\alpha, a) \in (P^* \oplus A)$. Then there exists an element $\beta \in P^*$ such that* $\text{ht}\,(I_a) \geq n$, *where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq n$ then* $\text{ht}\, I \geq n$. *Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and I is a proper ideal of A, then* $\text{ht}\, I = n$.

Lemma 2.11. [Bh-RS 3, 2.14][Bh-RS 4, 2.4] (Moving Lemma) *Let R be a Noetherian ring of dimension d and let P be a projective R-module of rank n, where $2n \geq d+2$. Let $J \subset R$ be an ideal of height n and let $\overline{\alpha} : P/JP \twoheadrightarrow J/J^2$ be a surjection. Then there exists an ideal $J' \subset R$ and a surjection $\beta : P \twoheadrightarrow J \cap J'$ such that:*

(1) $J + J' = R$.
(2) $\beta \otimes R/J = \overline{\alpha}$.
(3) $ht(J') \geq n$.
(4) *Given finitely many ideals $J_1, \cdots, J_r$ of R, each of height $\geq d-n+1$, the ideal J' can be chosen with the additional property that it is comaximal with J_i for $i = 1, \cdots, r$.*

The following lemma is an easy application of (2.9) and (2.10). A proof can be found in [K].

Lemma 2.12. *Let A be a commutative Noetherian ring of dimension d and I be an ideal of A. Let P be a projective A module with rank$(P) = n \geq d+1$. Assume that there exists a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. Then α can be lifted to a surjection $\beta : P \twoheadrightarrow I$.*

We improve [Bh-RS 2, 2.5] in the following form to suit our needs. The proof is similar to the one given in [Bh-RS 2].

Proposition 2.13. *Let A be a commutative Noetherian ring of dimension $d \geq 2$. Let I be an ideal of $A[T]$ of height ≥ 2 and P be a projective $A[T]$-module of rank $n \geq d+1$. Suppose that there exists $\phi : P/IP \twoheadrightarrow I/I^2$. Then ϕ can be lifted to a surjection $\psi : P \twoheadrightarrow I$.*

Proof. Since rank$(P) = n \geq \dim(A) + 1$, by a result of Plumstead [P, Corollary 2 of section 3], P has a free summand of rank one. Let $P = Q \oplus A[T]$, where Q is a projective $A[T]$ module. Let $J = I \cap A$. Since ht$(I) \geq 2$, we have $ht(J) \geq 1$. Therefore we can choose $b \in J^2$ such that ht$(b) = 1$. Now

$$n \geqq \dim(A) + 1 = \dim(A[T]) - 1 + 1 \geq \dim(A[T]/bA[T]) + 1$$

Hence by the previous lemma we have $\Phi : P/bP \twoheadrightarrow I/(b)$, which is a lift of $\phi \otimes A[T]/bA[T]$. Let $\gamma \in \text{Hom}_{A[T]}(P, I)$ be a lift of Φ (not necessarily surjective). Then, as $I/bA[T] = \Phi(P/bP)$, we have $\gamma(P) + bA[T] = I$. Now by applying (2.10) to the pair $(\gamma, b) \in \text{Hom}_{A[T]}(P, A[T]) \oplus A[T]$, we see that there exists $\beta \in \text{Hom}_{A[T]}(P, A[T])$ such that ht$(K_b) \geq n$, where $K = (\gamma + b\beta)(P)$.

Note that $K + bA[T] = I$ and $b \in I^2$. By (2.9), there exists an ideal I' of $A[T]$ such that $K = I \cap I'$ and $I' + bA[T] = A[T]$. Now ht$(I') =$ ht$(I'_b) =$ ht$(K_b) \geq n$. Setting $\mu = \gamma + b\beta$ we further observe:

(1) $\mu : P \twoheadrightarrow I \cap I'$,
(2) $\mu \otimes A[T]/I = \phi$,

If $I' = A[T]$, then we are done because then $\mu : P \twoheadrightarrow I \cap I' = I$ and μ is a lift of ϕ (recall that $b \in I^2$). If I' is a proper ideal then clearly I' contains a monic polynomial and by [La, Lemma 1.1, p. 79] we have $I' \cap A + bA = A$. Therefore I' contains an element of the form $1 + ba$ for some $a \in A$. Hence $I'_{1+bA} = A[T]_{1+bA}$, and therefore we have a surjection $\mu_{1+bA} : P_{1+bA} \twoheadrightarrow I_{1+bA}$.

Now $P = Q \oplus A[T]$ Let $\theta_b : Q_b \oplus A_b[T] \twoheadrightarrow A_b[T]$ be the projection onto the second factor. Now consider the following surjections:

$$\mu_{b(1+bA)} : P_{b(1+bA)} \twoheadrightarrow A_{b(1+bA)}[T]$$

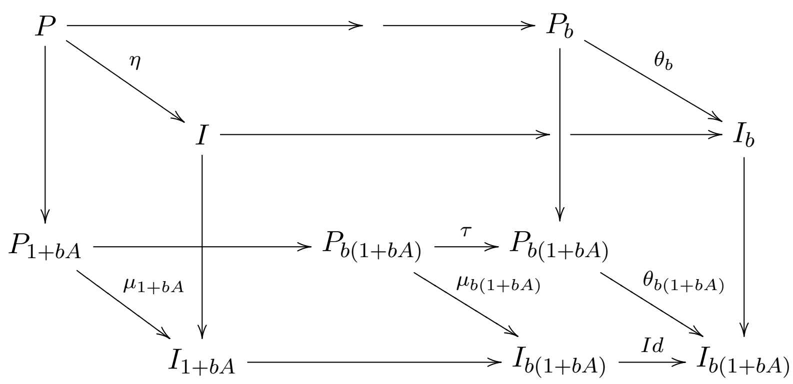
$$\theta_{b(1+bA)} : P_{b(1+bA)} \twoheadrightarrow A_{b(1+bA)}[T]$$

So we have two unimodular elements of $P^*_{b(1+bA)}$.

Observe that rank$(Q_{b(1+bA)}) = n - 1 \geq$ max$(2, d)$ and $d - 1 =$ dim$(A_{b(1+bA)})$. Therefore, by (2.5) we have a transvection $\tau \in \mathcal{E}(P_{b(1+bA)})$ such that

$$\tau\mu_{b(1+bA)} = \theta_{b(1+bA)}$$

Now consider the following fiber product diagram:



By a standard patching argument we have a surjection $\eta : P \twoheadrightarrow I$. It is easy to see that η is a lift of ϕ. □

Lemma 2.14. *Let A be a commutative Noetherian ring. Let I and K be two ideals of $A[T]$ such that $K \subset I^2$. Let P be a projective $A[T]$-module and $\mathfrak{n}$ be the nil radical of A. Let bar denote reduction modulo $\mathfrak{n}$. Suppose that $\alpha : P \twoheadrightarrow I/K$ be a surjection such that the induced map $\overline{\alpha} : P \twoheadrightarrow \overline{I}/\overline{K}$ can be lifted to a surjection $\beta : P \twoheadrightarrow \overline{I}$. Then α can also be lifted to a surjection $\phi : P \twoheadrightarrow I$.*

Proof. We have $\beta : P \twoheadrightarrow \overline{I}$, which is a lift of $\overline{\alpha}$. Therefore, we have

$$\beta : P \twoheadrightarrow (I + \mathfrak{n}[T])/\mathfrak{n}[T] \xrightarrow{\sim} I/(I \cap \mathfrak{n}[T]).$$

We note that $I/(K \cap \mathfrak{n}[T])$ is the fiber product of I/K and $I \cap \mathfrak{n}[T]$ over $I/(K, I \cap \mathfrak{n}[T])$:

$$\begin{array}{ccc} I/(K \cap \mathfrak{n}[T]) & \longrightarrow & I/K \\ \downarrow & & \downarrow \\ I/(I \cap \mathfrak{n}[T]) & \longrightarrow & I/(K, I \cap \mathfrak{n}[T]) \end{array}$$

Therefore α and β will patch to yield a surjection $\gamma : P \twoheadrightarrow I/(K \cap \mathfrak{n}[T])$. Let $\phi : P \longrightarrow I$ be a lift of γ. We prove that ϕ is surjective. We have $\phi(P) + (K \cap \mathfrak{n}[T]) = I$. Since $\mathfrak{n}$ is the nil radical of A, it follows that $V(I) = V(\phi(P))$ and hence $I = \phi(P)$. Since ϕ lifts α, we are done. $\square$

Here are some generalities on subintegral extensions that are relevant to this paper.

Definition 2.15. An extension $R \hookrightarrow S$ of rings is called *subintegral* if : (1) it is integral, (2) the induced map Spec$(S) \longrightarrow$ Spec(R) is bijective, and (3) for each $\mathfrak{P} \in$ Spec(S) the induced field extension $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \hookrightarrow S_{\mathfrak{P}}/\mathfrak{P}S_{\mathfrak{P}}$ is trivial, where $\mathfrak{p} = \mathfrak{P} \cap R$.

Definition 2.16. An extension of the form $R \hookrightarrow R[b]$ with $b^2, b^3 \in R$ is subintegral. It is called *elementarily subintegral*.

Remark 2.17. We record the following fundamental facts about subintegral extensions.

(1) $R \hookrightarrow S$ is subintegral if and only if S is the filtered union of subrings which can be obtained from R by a finite number of elementarily subintegral extensions.
(2) Let $R \hookrightarrow S$ be an extension of rings and $R \hookrightarrow R'$ be a faithfully flat extension. Write $S' = S \otimes_R R'$. Then $R \hookrightarrow S$ is subintegral if and only if $R' \hookrightarrow S'$ is subintegral. (See [Sw, Page 215] for details). In particular, $R \hookrightarrow S$ is subintegral if and only if $R[T] \hookrightarrow S[T]$ is subintegral.
(3) Let $R \hookrightarrow S$ be subintegral and $I \subset R$ be an ideal. Then we have ht$(I) =$ ht(IS). For a proof, see [D-Z, 3.6].

3. Some subtraction principles

One of the most important tools for the type of questions we are tackling in this paper are the so called "subtraction principles", and we shall need a couple of variants of it. This short section is devoted to collecting them for future use. We first state an available one and then prove some versions suited to fit our needs.

Proposition 3.1. [Bh-K, 3.7] *Let A be a ring of dimension d and J, J' be two comaximal ideals of A of height n where $2n \geq d + 3$. Let $P = Q \oplus A$ be a projective A-module of rank n. Let $\alpha : P \twoheadrightarrow J \cap J'$ and $\beta : P \twoheadrightarrow J'$ be two surjections such that $\alpha \otimes A/J' = \beta \otimes A/J'$. Then there exists a surjection $\theta : P \twoheadrightarrow J$ such that $\theta \otimes A/J = \alpha \otimes A/J$.*

Modifying the proof of [Bh-RS 3, 3.3] we obtain the following subtraction principle.

Proposition 3.2. *Let $n \geq 4$ and A be a Noetherian ring of dimension $n+1$. Let P and L be projective A-modules of rank n and 1, respectively, such that $P \oplus A \simeq L \oplus A^n$. Write $Q = L \oplus A^{n-2}$. Let $\chi : \wedge^n(P) \xrightarrow{\sim} L$ be an isomorphism. Let $J \subseteq A$ be an ideal of height $\geq n$ and J' be an ideal of height n such that $J + J' = A$. Let $\alpha : P \twoheadrightarrow J \cap J'$ and $\beta : Q \oplus A \twoheadrightarrow J'$ be surjections. Let $\overline{\beta} = \beta \otimes A/J' : P/J'P \twoheadrightarrow J'/J'^2$ and $\overline{\alpha} = \alpha \otimes A/J' : P/J'P \twoheadrightarrow J'/J'^2$ be the induced surjections. Suppose that there exists an isomorphism $\delta : P/J'P \xrightarrow{\sim} (Q \oplus A)/J'(Q \oplus A)$ such that: (i) $\overline{\beta}\delta = \overline{\alpha}$ and (ii) $\wedge^n(\delta) = \overline{\chi}$. Then there exists a surjection $\theta : P \twoheadrightarrow J$ such that $\theta \otimes A/J = \alpha \otimes A/J$.*

Proof. We write down the proof in two steps as it will turn out to be convenient later.
Step 1. We note that to prove this result we can change β by composing with an element of $SL(Q \oplus A)$. Let $\beta = (\nu, a)$, where $\nu \in Q^*$. Let tilde denote reduction modulo J^2. As $J + J' = A$, it follows that $(\widetilde{\nu}, \widetilde{a}) \in \text{Um}((\widetilde{Q} \oplus \widetilde{A})^*))$. Let ν' be any element of $\text{Um}(\widetilde{Q}^*)$. Since $\dim(A/J^2) \leq 1 < \text{rank}(Q)$, by (2.4) there exists $\widetilde{\sigma} \in \mathcal{E}((\widetilde{Q} \oplus \widetilde{A})^*))$ such that $(\widetilde{\nu}, \widetilde{a})\widetilde{\sigma} = (\nu', 0)$. By (2.6), $\widetilde{\sigma}$ can be lifted to an automorphism $\sigma^* \in SL((Q \oplus A)^*)$. But σ^* induces an automorphism $\sigma \in SL(Q \oplus A)$. Therefore, by replacing β with $\beta\sigma$, we may assume that $\beta = (\nu, a)$ has the property that $a \in J^2$ and $\nu(Q) + J^2 = A$. We can further apply (2.10) to obtain $\tau \in Q^*$ such that if $I = (\nu + a\tau)(Q)$, then $\text{ht}(I_a) \geq n - 1$. Note that $(I, a) = J'$. As $\text{ht}(J') = n$, it follows that $\text{ht}(I) = n-1$ and thus $\dim(A/I) \leq 2$. As (ν, a) and $(\nu + a\tau, a)$ are connected by a transvection, by replacing (ν, a) by $(\nu + a\tau, a)$, we can assume that:

(1) $\nu(Q) + J^2 = A$.
(2) $\dim(A(\nu(Q)) \leq 2$.

Using (1) we may further assume that $a = 1$ modulo J^2.

Step 2. Consider the following ideals in $A[Y]$: $K_1 = (\nu(Q), Y + a)$, $K_2 = J[Y]$, and $K_3 = K_1 \cap K_2$.

We claim that there is a surjection $\eta(Y) : P[Y] \twoheadrightarrow K_3$ such that $\eta(0) = \alpha$. We first check that the theorem follows from the claim, as it is easier! Putting $Y = 1 - a$ we obtain a surjection $\theta : P \twoheadrightarrow J$. Since $a = 1$ modulo J^2, we have $\theta \otimes A/J = \eta(1-a) \otimes A/J = \eta(0) \otimes A/J = \alpha \otimes A/J$, which proves the theorem.

Now for the claim, note that $A[Y]/K_1 \simeq A/(\nu(Q))$ and we have $\dim(A[Y]/K_1) \leq 2$. As P and $Q \oplus A$ are stably isomorphic, it is easy to see that there is an isomorphism, say, $\kappa(Y) : P[Y]/K_1P[Y] \xrightarrow{\sim} Q[Y]/K_1Q[Y] \oplus A[Y]/K_1$. We choose $\kappa(Y)$ so that $\wedge^n\kappa(Y) = \chi \otimes A[Y]/K_1$. Since $\wedge^n(\delta) = \chi \otimes A/J'$, it follows that $\kappa(0)$ and δ differ by an element of $SL(Q/J'Q \oplus A/J')$. We can apply (2.6) and alter $\kappa(Y)$ by an element of $SL(Q[Y]/K_1Q[Y] \oplus A[Y]/K_1)$ and assume that $\kappa(0) = \delta$. Now, tensoring the surjection $(\nu \otimes A[Y], Y + a) : Q[Y] \oplus A[Y] \twoheadrightarrow K_1$ with $A[Y]/K_1$, we obtain a surjection $\epsilon(Y) : Q[Y]/K_1Q[Y] \oplus A[Y]/K_1 \twoheadrightarrow K_1/K_1^2$. Therefore, we have a surjection $\pi(Y) = \epsilon(Y)\kappa(Y) : P[Y]/K_1P[Y] \twoheadrightarrow K_1/K_1^2$. Since $\overline{\beta}\delta = \overline{\alpha}$, $\epsilon(0) = \overline{\beta}$, and $\kappa(0) = \delta$, we have $\pi(0) = \alpha \otimes A/J'$. Therefore, applying [M-RS, 2.3] we obtain a surjection $\eta(Y) : P[Y] \twoheadrightarrow K_3$ such that $\eta(0) = \alpha$. This proves the claim. □

Using a similar method we have another subtraction principle.

Proposition 3.3. *Let R be a ring of dimension $n \geq 3$ and write $A = R[T]$. Assume that ht $\mathcal{J}(R) \geq 2$, where $\mathcal{J}(R)$ is the Jacobson radical of R. Let P and L be projective A-modules of rank n and 1, respectively, together with an isomorphism $\chi : \wedge^n(P) \xrightarrow{\sim} L$. Write $Q = L \oplus A^{n-2}$. Let $J \subseteq A$ be an ideal of height $\geq n$ and J' be an ideal of height n such that $J' + (K^2T) = A$, where $K = \mathcal{J}(R) \cap J$. Let $\alpha : P \twoheadrightarrow J \cap J'$ and $\beta : Q \oplus A \twoheadrightarrow J'$ be surjections such that $\alpha(P) + (K^2T) = J$. Let $\overline{\beta} = \beta \otimes A/J' : P/J'P \twoheadrightarrow J'/J'^2$ and $\overline{\alpha} = \alpha \otimes A/J' : P/J'P \twoheadrightarrow J'/J'^2$ be the induced surjections. Suppose that there exists an isomorphism $\delta : P/J'P \xrightarrow{\sim} (Q \oplus A)/J'(Q \oplus A)$ such that: (i) $\overline{\beta}\delta = \overline{\alpha}$ and (ii) $\wedge^n(\delta) = \overline{\chi}$. Then there exists a surjection $\theta : P \twoheadrightarrow J$ such that $(\theta - \alpha)(P) \subset (K^2T)$.*

Proof. To be consistent with the above proposition, we write the proof in steps.
Step 1. As in the proof of (3.2), write $\beta = (\nu, a)$, where $\nu \in Q^*$. Let tilde denote reduction modulo (K^2T). Then $(\widetilde{\nu}, \widetilde{a}) \in \text{Um}((\widetilde{Q} \oplus \widetilde{A})^*))$. Let ν' be any element of $\text{Um}(\widetilde{Q}^*)$. Write $D = R[T]/(K^2T)$. Note that KD is contained in the Jacobson radical of D and $D/KD \simeq (R/K)[T]$. With an argument combining (2.5) and (2.6), it is easy to check that, changing β if necessary, we can assume that $\beta = (\nu, a)$ has the property that $a \in (K^2T)$ and $\nu(Q) + (K^2T) = A$. Now apply (2.10) to finally ensure that $\text{ht}(\nu(Q)) = n-1$, and therefore by [Bh-RS 1, 3.1] $\dim(A/(\nu(Q)) \leq 1$. Note that we may further assume that $a = 1$ modulo (K^2T).

Step 2. This step is exactly the same as its counterpart in (3.2). Note that here we have the advantage that $\dim(A/(\nu(Q)) \leq 1$ and therefore we do not need P and $Q \oplus A$ to be stably isomorphic. □

4. The Euler class group $E(R[T], L[T])$

By a ring we shall mean a commutative Noetherian ring containing $\mathbb{Q}$.

Let R be a ring of dimension $n \geq 3$. The aim of this section is to extend the theory of the Euler class group $E(R[T])$ of $R[T]$, as developed in [D 1, D 2], to $E(R[T], L[T])$, where L is a projective R-module of rank one. Obviously, when L is free, $E(R[T], L[T])$ should coincide with $E(R[T])$. Experts in this area, who are familiar with [D 1, D 2], would agree that this extension is not really difficult. However, we need to carry out the details not only for the convenience of a general reader but also for keeping a ready reference for Section 6 because Section 6 depends heavily on this section with Section 5 providing the technical bridge.

The following theorem was stated without proof in [D 1, 3.11]. One can actually mimic the proof of [D 1, 3.10] with necessary modifications to prove this result directly. For the sake of completeness we give here a quick proof, using [D 1, 3.10]. As [D 1, 3.10] played a pivotal role in studying the Euler class group $E(R[T])$, this theorem will do the same for $E(R[T], L[T])$. Recall that $R(T)$ is the ring obtained from $R[T]$ by inverting all the monic polynomials and that $\dim(R(T)) = \dim(R)$.

Theorem 4.1. *Let R be a ring of dimension $n \geq 3$ and P be a projective R-module of rank n. Let $I \subset R[T]$ be an ideal of height n such that there is a surjection*

$$\psi : P[T] \twoheadrightarrow I/(I^2T).$$

Assume that $\psi \otimes R(T) : (P[T] \otimes R(T)) \twoheadrightarrow IR(T)/I^2R(T)$ can be lifted to a surjection

$$\widetilde{\theta} : (P[T] \otimes R(T)) \twoheadrightarrow IR(T).$$

Then ψ also has a lift to a surjective map $\theta : P[T] \twoheadrightarrow I$.

Proof. We first note that if I contains a monic polynomial, then the conditions of the theorem are trivially satisfied. In this case, the theorem has been proved by Mandal [M 2, 2.1]. Therefore, in what follows, we may assume that I does not contain a monic polynomial.

Let $J = I \cap R$. Applying [D 1, 3.9], we get a lift $\phi \in \mathrm{Hom}_{R[T]}(P[T], I)$ of ψ, such that the ideal $\phi(P[T]) = I''$ satisfies the following properties:

(i) $I'' + (J^2T) = I$.
(ii) $I'' = I \cap I'$, where $\mathrm{ht}(I') \geq n$.
(iii) $I' + (J^2T) = R[T]$.

Let $J' = I' \cap R$. It can be deduced that $\dim(R/(J + J')) = 0$. This was proved in [D 1, 3.10].

Write $B = R_{1+J}$. Tensoring the surjection $\phi \otimes B : P_{1+J}[T] \twoheadrightarrow (I \cap I')B[T]$ with $B[T]/I'B[T]$ we obtain a surjection

$$\phi_1 : P_{1+J}[T] \twoheadrightarrow I'B[T]/{I'}^2B[T].$$

Now we note two things. First, as $I' + (J^2T) = A[T]$, it follows that $I'(0)B = B$. Secondly, since JB is contained in the Jacobson radical of B and $\dim(B/JB) \leq 1$, it is easy to see using (2.2) that P_{1+J} has a free summand of rank one and hence there is a surjective map $\alpha : P_{1+J} \twoheadrightarrow I'(0)B(= B)$. Combining these two, it is not hard to see that there is a surjection

$$\overline{\beta} : P_{1+J}[T] \twoheadrightarrow I'B[T]/({I'}^2T)B[T],$$

which is a lift of ϕ_1 (see [Bh-RS 1, 3.9] for a proof).

Consider the ring $C = B_{1+J'} = R_{1+J+J'}$. As $\dim(R/(J+J')) = 0$, it follows that C is semilocal, and consequently $P_{1+J+J'}$ is a free C-module. Applying the subtraction principle (3.1) over $C(T)$, we see that there is a surjection $\gamma : P \otimes C(T) \twoheadrightarrow I'C(T)$ which lifts $\overline{\beta} \otimes C(T)$. Since $P_{1+J+J'}$ is a free C-module, it follows from [D 1, 3.10], that $\overline{\beta} \otimes C[T]$ has a lift to a surjective map $\widetilde{\beta} : P_{1+J+J'}[T] \twoheadrightarrow I'C[T]$. It now follows from [D 1, 3.8], that $\overline{\beta}$ has a lift to a surjection $\beta : P_{1+J}[T] \twoheadrightarrow I'B[T]$, i.e., $(\beta - \overline{\beta})(P_{1+J}[T]) \subset (I'^2T)B[T]$.

Now we can apply [Bh-K, 4.7], and obtain a surjection $\eta : P_{1+J}[T] \twoheadrightarrow IB[T]$, such that $(\eta - \phi)(P_{1+J}[T]) \subset (I^2T)B[T]$. Applying [D 1, 3.8] again, we are done. $\square$

In this section we shall frequently apply the above theorem taking $L[T] \oplus R[T]^{n-1}$ in place of $P[T]$, where L is a projective R-module of rank one. Let us illustrate one such application in the form of following proposition which will be used later.

Notation. Let L be a projective R-module of rank one. Throughout this section we shall denote $L \oplus R^{n-1}$ by $\mathcal{L}$ and $L[T] \oplus R[T]^{n-1}$ by $\mathcal{L}[T]$.

Proposition 4.2. *Let R be a ring and $I \subset R[T]$ be an ideal of height n. Let α and β be two surjections from $\mathcal{L}[T]/I\mathcal{L}[T]$ to I/I^2 such that there exists $\sigma \in SL(\mathcal{L}[T]/I\mathcal{L}[T])$ with the property that $\alpha\sigma = \beta$. Suppose that α can be lifted to a surjection $\theta : \mathcal{L}[T] \twoheadrightarrow I$. Then β can also be lifted to a surjection $\phi : \mathcal{L}[T] \twoheadrightarrow I$.*

Proof. Since $\mathbb{Q} \subset R$, there exists $\lambda \in \mathbb{Q}$ such that $I(\lambda) = R$ or $I(\lambda)$ is an ideal of R of height n. Without loss of generality we may assume that $\lambda = 0$.

If $I(0) = R$, then by [Bh-RS 1, 3.9], we can lift β to a surjection $\widetilde{\beta} : \mathcal{L}[T] \twoheadrightarrow I/(I^2T)$. We now show that the same can be done if $\text{ht}(I(0)) = n$. Let $\alpha(0) : \mathcal{L}/I(0)\mathcal{L} \twoheadrightarrow I(0)/I(0)^2$, $\beta(0) : \mathcal{L}/I(0)\mathcal{L} \twoheadrightarrow I(0)/I(0)^2$ be surjections induced by α, β, respectively. Therefore $\alpha(0)\sigma(0) = \beta(0)$. As $\dim(R/I(0)) = 0$, we have $\sigma(0) \in \mathcal{E}(\mathcal{L}/I(0)\mathcal{L})$. As $\mathcal{E}(\mathcal{L}) \longrightarrow \mathcal{E}(\mathcal{L}/I(0)\mathcal{L})$ is surjective, there exists $\tau \in \mathcal{E}(\mathcal{L})$, which is a lift of $\sigma(0)$. As $\theta(0)$ lifts $\alpha(0)$, the composition $\theta(0)\tau$ lifts β. Again by [Bh-RS 1, 3.9], we can lift β to a surjection $\widetilde{\beta} : \mathcal{L}[T] \twoheadrightarrow I/(I^2T)$.

Now consider the ring $R(T)$ and the induced surjections $\alpha \otimes R(T)$ and $\widetilde{\beta} \otimes R(T)$. Again since $\dim(R(T)/IR(T)) = 0$, we have $SL(\mathcal{L} \otimes R(T)/I\mathcal{L} \otimes R(T)) = \mathcal{E}(\mathcal{L} \otimes R(T)/I\mathcal{L} \otimes R(T))$ and as above , $\widetilde{\beta} \otimes R(T)$ can be lifted to a surjection from $\mathcal{L} \twoheadrightarrow IR(T)$. Now we can apply (4.1) and conclude that β can be lifted to a surjection $\phi : \mathcal{L} \twoheadrightarrow I$. $\square$

Remark 4.3. The above proposition was proved in [D 1, 4.4] in the case when $\mathcal{L}$ is free. We may justifiably call the technique involved in the proof as a *"monic inversion technique"*. This was ubiquitous in [D 1]. As we are proving results analogous to [D 1] in this section, therefore, whenever we present a result which can be proved by this *monic inversion technique*, either we give a quick sketch or we leave the proof to the reader.

The following addition and subtraction principles, like their counterparts in [D 1], can be proved using the *monic inversion technique* illustrated above.

Proposition 4.4. (Addition principle) *Let R be a ring of dimension $n \geq 3$ and let $I_1, I_2 \subset R[T]$ be two comaximal ideals, each of height n. Assume that there exist surjections $\theta_1 : \mathcal{L}[T] \twoheadrightarrow I_1$ and $\theta_2 : \mathcal{L}[T] \twoheadrightarrow I_2$. Then there exists surjection $\theta : \mathcal{L}[T] \twoheadrightarrow I_1 \cap I_2$ such that $\theta \otimes R[T]/I_i = \theta_i \otimes R[T]/I_i$, $i = 1, 2$.*

Proposition 4.5. (Subtraction principle) *Let R be a ring of dimension $n \geq 3$ and let $I_1, I_2 \subset R[T]$ be two comaximal ideals, each of height n. Assume that there exist surjections $\theta_1 : \mathcal{L}[T] \twoheadrightarrow I_1$ and $\theta : \mathcal{L}[T] \twoheadrightarrow I_1 \cap I_2$ such that $\theta \otimes R[T]/I_1 = \theta_1 \otimes R[T]/I_1$. Then there exists surjection $\theta_2 : \mathcal{L}[T] \twoheadrightarrow I_2$ such that $\theta \otimes R[T]/I_2 = \theta_2 \otimes R[T]/I_2$.*

Let R be a commutative Noetherian ring of dimension $n \geq 3$ containing $\mathbb{Q}$. Let L be a projective R-module of rank one. We now go on to define the (n-th) Euler class group $E^n(R[T], L[T])$. For brevity we denote this group by $E(R[T], L[T])$. As above, we shall denote $L \oplus R^{n-1}$ by $\mathcal{L}$ and $L[T] \oplus R[T]^{n-1}$ as $\mathcal{L}[T]$.

We first define some terms. Let $I \subset R[T]$ be an ideal of height n such that there exists a surjection $\mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$. Two surjections $\alpha, \beta : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$ are said to be *related* if there exists $\sigma \in SL(\mathcal{L}[T]/I\mathcal{L}[T])$ such that $\alpha\sigma = \beta$. It easily follows that this defines an equivalence relation on the set of surjections from $\mathcal{L}[T]/I\mathcal{L}[T]$ to I/I^2. Let $[\alpha]$ denote the equivalence class of α. We call such an equivalence class $[\alpha]$ a *local $L[T]$-orientation* of I.

We call a local $L[T]$-orientation $[\alpha]$ of I a *global $L[T]$-orientation* if the surjection $\alpha : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : \mathcal{L}[T] \twoheadrightarrow I$. Note that by (4.2), if α can be lifted to a surjection $\theta : \mathcal{L}[T] \twoheadrightarrow I$, then β can also be lifted to a surjection $\eta : \mathcal{L}[T] \twoheadrightarrow I$. Therefore, by a slight abuse of notations, we denote $[\alpha]$ by α.

Let G be the free abelian group on the set of pairs (I, ω_I) where $I \subset R[T]$ is an ideal of height n with the property that $\text{Spec}(R[T]/I)$ is connected and I/I^2 is a surjective image of $\mathcal{L}[T]/I\mathcal{L}[T]$ and $\omega_I : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$ is a local $L[T]$-orientation of I.

Let I be any ideal of $R[T]$ of height n such that I/I^2 is surjective image of $\mathcal{L}[T]/I\mathcal{L}[T]$. Then there is a unique decomposition (see [D 1] for details), $I = I_1 \cap \cdots \cap I_k$, where $\text{Spec}(R[T]/I_i)$ is connected and $\text{ht}\, I_i = n$ for each i, and $I_i + I_j = R[T]$ for $i \neq j$. Now if ω_I is a local $L[T]$-orientation of I then it naturally gives rise to $\omega_{I_i} : \mathcal{L}[T]/I_i\mathcal{L}[T] \twoheadrightarrow I_i/{I_i}^2$ for $1 \leq i \leq k$. By (I, ω_I) we mean the element $\sum (I_i, \omega_{I_i}) \in G$.

Let H be the subgroup of G generated by the set of pairs (I, ω_I) in G such that ω_I is a global orientation.

Definition 4.6. We define the (n-th) Euler class group of $R[T]$ with respect to $L[T]$ as $E(R[T], L[T]) := G/H$.

Let P be a projective $R[T]$-module of rank n having determinant $L[T]$, where L is a projective R-module of rank 1. Let $\chi : L[T] \xrightarrow{\sim} \wedge^n P$ be an isomorphism. To the pair (P, χ), we associate an element $e(P, \chi)$ of $E(R[T], L[T])$ as follows: Let $\lambda_0 : P \twoheadrightarrow I_0$ be a surjection, where I_0 is an ideal of $R[T]$ of height n. Let bar denote reduction modulo I_0. We obtain an induced surjection $\overline{\lambda}_0 : P/I_0P \twoheadrightarrow I_0/I_0^2$. Note that, since P has determinant $L[T]$ and $\dim(R[T]/I_0) \leq 1$, by Serre's splitting theorem (2.2) we have $P/I_0P \simeq L[T]/I_0L[T] \oplus (R[T]/I_0)^{n-1}$ ($= \mathcal{L}[T]/I_0\mathcal{L}[T]$ in our notation). We choose an isomorphism $\overline{\gamma} : \mathcal{L}[T]/I_0\mathcal{L}[T] \xrightarrow{\sim} P/I_0P$, such that $\wedge^n \overline{\gamma} = \overline{\chi}$. Let ω_{I_0} be the composite surjection

$$\mathcal{L}[T]/I_0\mathcal{L}[T] \overset{\overline{\gamma}}{\xrightarrow{\sim}} P/I_0P \overset{\overline{\lambda_0}}{\twoheadrightarrow} I_0/I_0^2.$$

Let $e(P, \chi)$ be the image in $E(R[T], L[T])$ of the element (I_0, ω_{I_0}). We say that (I_0, ω_{I_0}) is obtained from the pair (λ_0, χ).

As yet another application of the *monic inversion technique*, we have the following

Lemma 4.7. *The assignment sending the pair (P, χ) to the element $e(P, \chi)$, as described above, is well defined.*

Proof. (Sketch) Let $\lambda_i : P \twoheadrightarrow I_i$ ($i = 0, 1$) be two surjections so that (λ_i, χ) induce (I_i, ω_{I_i}). Apply the moving lemma 2.11 to find an ideal $K \subset R[T]$ and a local $L[T]$-orientation ω_K of K such that $\text{ht}(K) \geq n$, $K + I_i = R[T]$ for $i = 0, 1$ and $(I_0, \omega_{I_0}) + (K, \omega_K) = 0$ in $E(R[T], L[T])$. Now let $I = I_1 \cap K$ and ω_I be the local $L[T]$-orientation of I induced by ω_{I_1} and ω_K. Now use the facts that the Euler class of a projective R-module (resp., $R(T)$-module) is well-defined, and the monic inversion technique as in (4.1) to show that ω_I is a global orientation. This will prove $0 = (I, \omega_I) = (I_1, \omega_{I_1}) + (K, \omega_K)$ in $E(R[T], L[T])$ and therefore, $(I_0, \omega_{I_0}) = (I_1, \omega_{I_1})$. $\square$

Definition 4.8. We define the Euler class of (P, χ) to be $e(P, \chi)$.

Remark 4.9. It is easy to see from the definition of $E(R, L)$ in [Bh-RS 3] and the definition of $E(R[T], L[T])$ given above, that there is a canonical group homomorphism $\Phi : E(R, L) \to E(R[T], L[T])$. Following the method of proof of [D 2, 3.3] with obvious modifications, one can check that there is a surjective group homomorphism $\Psi : E(R[T], L[T]) \twoheadrightarrow E(R, L)$ with the property that if $(I, \omega_I) \in E(R[T], L[T])$ is such that the ideal $I(0)$ is an ideal of R of height n, then $\Psi((I, \omega_I)) = (I_0, \omega_{I(0)})$, where $\omega_{I(0)}$ is the local L-orientation of $I(0)$ induced by ω_I (if $I(0) = R$, then $\Psi((I, \omega_I)) = 0$). Moreover, $\Psi\Phi = \text{id}_{E(R,L)}$ and therefore Φ is injective. On the other hand, as the extension $R[T] \hookrightarrow R(T)$ is flat, we have a canonical group homomorphism $\varphi : E(R[T], L[T]) \to E(R(T), L[T] \otimes R(T))$.

With the above remark in hand, one can prove the following theorem. The method of proof again involves a straightforward *monic inversion technique*.

Theorem 4.10. *Let R be a ring of dimension $n \geq 3$. Let $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements and let $\omega_I : \mathcal{L}[T]/I\mathcal{L}[T] \twoheadrightarrow I/I^2$ be a local $L[T]$-orientation of I. Suppose that the image of (I, ω_I) is zero in $E(A[T], L[T])$. Then ω_I can be lifted to a surjection $\theta : \mathcal{L}[T] \twoheadrightarrow I$ (i.e., ω_I is a global orientation).*

Proof. We leave the proof to the reader. □

The following theorem extends [D 1, 4.8] and a theorem of Mandal [M 2, 2.1]. The method of proof is similar to [D 1, 4.8]. As the proof is rather involved, we give an outline.

Theorem 4.11. *Let R be a ring of $dim R = n \geq 3$ and $J \subseteq R[T]$ be an ideal of height n. Let P be a projective $R[T]$-module of rank n whose determinant is $L[T]$. Assume that we are given a surjection $\psi : P \twoheadrightarrow J/(J^2T)$. Assume further that $\psi \otimes R(T)$ can be lifted to a surjection $\psi' : P \otimes R(T) \twoheadrightarrow JR(T)$. Then, there exists a surjection $\Psi : P \twoheadrightarrow J$ such that Ψ is a lift of ψ .*

Proof. We fix an isomorphism $\chi : L[T] \xrightarrow{\sim} \wedge^n P$. Let $\mathcal{J}(R, P)$ denote the Quillen ideal of P in R and write $K = \mathcal{J}(R, P) \cap J$. Since the determinant of P is extended from R, we have, $\text{ht}(\mathcal{J}(R, P)) \geq 2$. Therefore, $\text{ht}\, K \geq 2$. We can apply [D 1, 3.9] and obtain a lift $\alpha \in \text{Hom}_{R[T]}(P, J)$ of ψ and an ideal $J' \subset R[T]$ of height n such that (1) $J' + (K^2T) = R[T]$, (2) $\alpha : P \twoheadrightarrow J \cap J'$ is a surjection and (3) $\alpha(P) + (K^2T) = J$.

It follows that $e(P, \chi) = (J \cap J', \omega_{J\cap J'})$ in $E(R[T], L[T])$ where the local orientation $\omega_{J\cap J'}$ is obtained by composing $\alpha \otimes R[T]/(J \cap J')$ with a suitable isomorphism $\overline{\lambda} : (R[T]/J \cap J')^n \simeq P/(J \cap J')P$, as described in the definition of an Euler class.

Therefore, $e(P, \chi) = (J, \omega_J) + (J', \omega_{J'})$. We note that since $J'(0) = R$, by [Bh-RS 1, 3.9] we can lift $\omega_{J'}$ to a surjection from $\mathcal{L}[T] \twoheadrightarrow J'/(J'^2T)$. Moreover, considering the equation $e(P \otimes R(T), \chi \otimes R(T)) = (JR(T), \omega_J \otimes R(T)) + (J'R(T), \omega_{J'} \otimes R(T))$ in $E(R(T), L[T] \otimes R(T))$ and using the condition of the theorem it is easy to deduce that $(J'R(T), \omega_{J'} \otimes R(T)) = 0$ in $E(R(T), L[T] \otimes R(T))$. (Actually, the condition of the theorem tells that $e(P \otimes R(T), \chi \otimes R(T)) = (JR(T), \omega_J \otimes R(T))$). As $\omega_{J'}$ is induced by a surjection $\mathcal{L}[T] \twoheadrightarrow J'/(J'^2T)$, it follows from (4.1) that there is a surjection $\beta : \mathcal{L}[T] \twoheadrightarrow J'$ which lifts $\omega_{J'}$.

Let us write $B = R_{1+K}$. By [D 1, 3.8] it is enough to prove that there is a surjection $\theta : \mathcal{L}[T] \otimes B[T] \twoheadrightarrow J$ such that $(\theta - \alpha)(\mathcal{L}[T]) \subset (K^2T)$. We can apply (3.3) to obtain such a θ. □

Remark 4.12. Let the notations be as in the above theorem. Note that if J contains a monic polynomial, the conditions of the theorem are trivially satisfied. The conclusion of the theorem asserts that if J contains a monic polynomial, then any surjection ψ :

$P \twoheadrightarrow J/(J^2T)$ can be lifted to a surjection $\Psi : P \twoheadrightarrow J$. This improves [M 2, 2.1], where P was assumed to be extended from R.

To derive some corollaries of the above two theorems, we need the following lemma.

Lemma 4.13. [D 1, 4.9] *Let A be a ring, $I \subset A[T]$ be an ideal and P be a projective $A[T]$-module. Suppose that we are given surjections $\alpha : P \twoheadrightarrow I/I^2$ and $\beta : P \twoheadrightarrow I(0) = I/I \cap (T)$ such that $\alpha \otimes_{A[T]/I} A/I(0) = \beta \otimes_A A/I(0)$. Then there is a surjection $\theta : P \twoheadrightarrow I/(I^2T)$ such that θ lifts both α and β.*

We have the following set of corollaries. These can be derived easily from (4.10), (4.11). For assistance, the reader may consult [D 1].

Corollary 4.14. *Let R be of dimension $n \geq 3$. Let P be a projective $R[T]$-module of rank n with determinant isomorphic to $L[T]$. Let $\chi : \wedge^n P \overset{\sim}{\to} L[T]$. Let $e(P, \chi) = (I, \omega_I)$ in $E(R[T], L[T])$. Then, there exists a surjection $\alpha : P \twoheadrightarrow I$ such that (I, ω_I) is obtained from (α, χ).*

Corollary 4.15. *Let R be a ring. Let P be a projective $R[T]$-module of rank n with determinant isomorphic to $L[T]$. Let $\chi : \wedge^n P \overset{\sim}{\to} L[T]$. Then, $e(P, \chi) = 0$ if and only if P has a unimodular element. In particular, if P has a unimodular element then P maps onto any ideal of $R[T]$ of height n which is surjective image of $L[T] \oplus R[T]^{n-1}$.*

Corollary 4.16. *Let R be a ring and $I \subset R[T]$ be an ideal of height n. Let P be a projective $R[T]$-module of rank n with trivial determinant and $\alpha : P \twoheadrightarrow I$ be a surjection. Suppose that P has a unimodular element. Then I is surjective image of $L[T] \oplus R[T]^{n-1}$.*

We can also prove the following local-global principle for the Euler class groups.

Theorem 4.17. *Let R, L be as above. The following sequence of groups is exact*

$$0 \longrightarrow E(R, L) \longrightarrow E(R[T], L[T]) \longrightarrow \prod_{\mathfrak{m}} E(R_{\mathfrak{m}}[T], L_{\mathfrak{m}}[T]).$$

Proof. Let $(I_1, \omega_{I_1}) \in E(R[T], L[T])$ be such that its image in $E(R_{\mathfrak{m}}[T], L_{\mathfrak{m}}[T])$ is zero for each maximal ideal $\mathfrak{m}$ of R. We show that (I_1, ω_{I_1}) has a preimage in $E(R, L)$.

As $\mathbb{Q} \subset R$, we can assume that either $I_1(0)$ is an ideal of height n or $I_1(0) = R$.

Case 1. Assume that $I_1(0)$ is proper. Apply the moving lemma (2.11), and obtain an ideal $K \subset R$ of height n which is comaximal with $I_1 \cap R$ and a local $L[T]$-orientation ω_K of K such that $(I_1(0), \omega_{I_1(0)}) + (K, \omega_K) = 0$ in $E(R, L)$.

Let $I = I_1 \cap K[T]$. As I_1 and $K[T]$ are comaximal, ω_{I_1} and ω_K will induce a local $L[T]$-orientation ω_I of I and we have:

$$(I, \omega_I) = (I_1, \omega_{I_1}) + (K[T], \omega_K \otimes R[T]) \text{ in } E(R[T], L[T]).$$

Note that proving $(I,\omega_I) = 0$ will suffice. Observe from the above equation that $(I(0),\omega_{I(0)}) = 0$ in $E(R,L)$ and therefore, by [Bh-RS 3, 4.2] $\omega_{I(0)}$ can be lifted to a surjection $\alpha : \mathcal{L} \twoheadrightarrow I(0)$. Therefore, by [Bh-RS 1, 3.9] ω_I can be lifted to a surjection $\psi : \mathcal{L}[T] \twoheadrightarrow I/(I^2T)$. It is now enough to show that ψ can be lifted to a surjection $\theta : \mathcal{L}[T] \twoheadrightarrow I$.

Now we can proceed as in the first half of the proof of (4.1) and reduce the theorem to the case when R is semilocal. But in that case L is free and the proof in this case is given in [D 1, 5.5].

Case 2. If $I_1(0) = R$, then ω_{I_1} can be lifted to a surjection $\psi_1 : \mathcal{L}[T] \twoheadrightarrow I_1/(I_1^2T)$. We can proceed as in Case 1 to reduce the proof to the semilocal case. □

The following is an analogue of a theorem of Roitman [Ro, Proposition 2], proved for the Euler class groups in [D-RS 2].

Theorem 4.18. *Let R, L be as above. Let $S \subset R$ be a multiplicatively closed set. Assume that the canonical map $\Phi : E(R,L) \longrightarrow E(R[T],L[T])$ is given to be surjective. Then the canonical map $\Phi_S : E(S^{-1}R, S^{-1}L) \longrightarrow E(S^{-1}R[T], S^{-1}L[T])$ is also surjective.*

Proof. Write L_S for $L \otimes_R R_S$. By Theorem 4.17, we have the following exact sequence of abelian groups

$$0 \longrightarrow E(R_S, L_S) \longrightarrow E(R_S[T], L_S[T]) \longrightarrow \prod_{\mathfrak{m}} E((R_S)_{\mathfrak{m}}[T], (L_S)_{\mathfrak{m}}[T]),$$

where $\mathfrak{m}$ is a maximal ideal of R_S of height n. To prove the theorem, it is enough to show that $E((R_S)_{\mathfrak{m}}[T], (L_S)_{\mathfrak{m}}[T]) = 0$ for each such $\mathfrak{m}$. Since $\mathfrak{m}$ is a maximal ideal of R which avoids S, we are reduced to showing that under the hypothesis of the theorem, $E(R_{\mathfrak{m}}, L_{\mathfrak{m}}) \longrightarrow E(R_{\mathfrak{m}}[T], L_{\mathfrak{m}}[T])$ is surjective. Since $E(R_{\mathfrak{m}}, L_{\mathfrak{m}}) = 0$, we need only prove that $E(R_{\mathfrak{m}}[T], L_{\mathfrak{m}}[T]) = 0$. But $L_{\mathfrak{m}}$ is free and we are done by [D-RS 2, 4.2]. □

Theorem 4.19. *Let R be a regular ring of dimension $n \geq 3$ which is essentially of finite type over a field k such that R has infinite residue fields. Let L be a projective R-module of rank one. Then $E(R,L)$ is isomorphic to $E(R[T],L[T])$.*

Proof. We only need to prove that the canonical map from $E(R,L)$ to $E(R[T],L[T])$ is surjective. In view of the local-global principle (4.17), it is enough to prove that $E(R_{\mathfrak{m}}[T], L_{\mathfrak{m}}[T]) = 0$ for each maximal ideal $\mathfrak{m}$ of R of height n. But $L_{\mathfrak{m}}$ is free and therefore the result follows from [M-V, Theorem 4], [D 4, 4.9]. □

Theorem 4.20. *Let R be an affine algebra of dimension $n \geq 3$ over an algebraically closed field k of characteristic zero. Then the canonical map $E(R[T],L[T]) \to E(R(T), L[T] \otimes R(T))$ is injective.*

Proof. It can be derived by modifying the proof in [D 1, 5.8] $\square$

Remark 4.21. Let R be a regular domain of dimension d containing an infinite field and n be a positive integer such that $2n \geq d+3$. Then the n-th Euler class group $E^n(R[T])$ has been defined in [D-RS 2] and results analogous to those in [D 1, D 2] have been proved. By a result of Lindel [Li 1], any line bundle on $R[T]$ is extended from R and hence is of the form $L[T]$, where L is a line bundle on R. The theory can easily be extended to define $E^n(R[T], L[T])$ and many results of this section can be proved. For instance, it can be proved that if R is regular and is essentially of finite type over an infinite field, then $E^n(R[T], L[T]) \simeq E^n(R, L)$. A similar result has been proved using different techniques in [M-Y 2]. However we are not going into the details of this setup in this paper.

5. Some descent lemmas and their applications

In this section we prove some technical results which are crucial to the theory and results in Section 6. Motivation for the following lemmas came from [Bh 1, 3.1, 3.2, 3.3]. The basic setup is as follows. We shall try to stick to the notations introduced below throughout this section.

Let $R \hookrightarrow S$ be a finite extension of reduced rings and let C be the conductor ideal of R in S. Let $\mathbb{L}$ be a projective $R[T]$-module of rank one and $I \subset R[T]$ be an ideal such that $IS[T]$ is a proper ideal. Write $\mathfrak{L} = \mathbb{L} \oplus R[T]^{n-1}$. Assume that there is a surjection $\alpha : \mathfrak{L}/I\mathfrak{L} \twoheadrightarrow I/I^2$. Then α naturally induces a surjection from $(\mathfrak{L} \otimes S[T])/IS[T](\mathfrak{L} \otimes S[T])$ to $IS[T]/I^2S[T]$, which we shall denote by α^*. We now explicitly describe how α^* is obtained.

Tensoring α with $S[T]/IS[T]$ over $R[T]/I$ we obtain the surjection

$$\widetilde{\alpha} : \frac{(\mathfrak{L} \otimes_{R[T]} S[T])}{IS[T](\mathfrak{L} \otimes_{R[T]} S[T])} \twoheadrightarrow \frac{(I \otimes_{R[T]} S[T])}{IS[T](I \otimes_{R[T]} S[T])}.$$

Composing $\widetilde{\alpha}$ with the surjective map $\widetilde{f}$ induced by the natural surjection $f : I \otimes_{R[T]} S[T] \twoheadrightarrow IS[T]$, we obtain α^*. Thus α^* is the composition $\widetilde{f}\widetilde{\alpha}$

$$\alpha^* : \frac{(\mathfrak{L} \otimes_{R[T]} S[T])}{IS[T](\mathfrak{L} \otimes_{R[T]} S[T])} \overset{\widetilde{\alpha}}{\twoheadrightarrow} \frac{(I \otimes_{R[T]} S[T])}{IS[T](I \otimes_{R[T]} S[T])} \overset{\widetilde{f}}{\twoheadrightarrow} \frac{IS[T]}{I^2S[T]}.$$

Now suppose it is given that α^* has a lift to a surjection $\beta : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$. In the following three lemmas we investigate, under what additional hypotheses, we may be able to obtain a lift of α to a surjection $\phi : \mathfrak{L} \twoheadrightarrow I$. We fix the above notations for the following lemmas. We shall only mention the additional hypotheses in the statements.

The method of proof of the following lemma is similar to [D-Z, 3.11].

Lemma 5.1. *Let $R, S, C, \mathfrak{L}, I, \alpha$ be as above and assume that : (i) $(R/C)_{red} = (S/C)_{red}$, (ii) $I + C[T] = R[T]$, (iii) $n \geq 2$. Then α can be lifted to a surjective map $\phi : \mathfrak{L} \twoheadrightarrow I$.*

Proof. We give the proof in steps.
Step 1. We first note that since $I + C[T] = R[T]$ and C is the conductor ideal of R in S, the following can be easily verified :

(1) $I \otimes (R/C)[T] \simeq (R/C)[T]$.
(2) $I \otimes (S/C)[T] \simeq (S/C)[T]$.
(3) $R[T]/I \simeq S[T]/IS[T]$.

We have a surjection, $\beta : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$ which is a lift of α^*. Consider

$$\beta_1 := \beta \otimes (S/C)[T] : \mathfrak{L} \otimes (S/C)[T] \twoheadrightarrow IS[T] \otimes (S/C)[T].$$

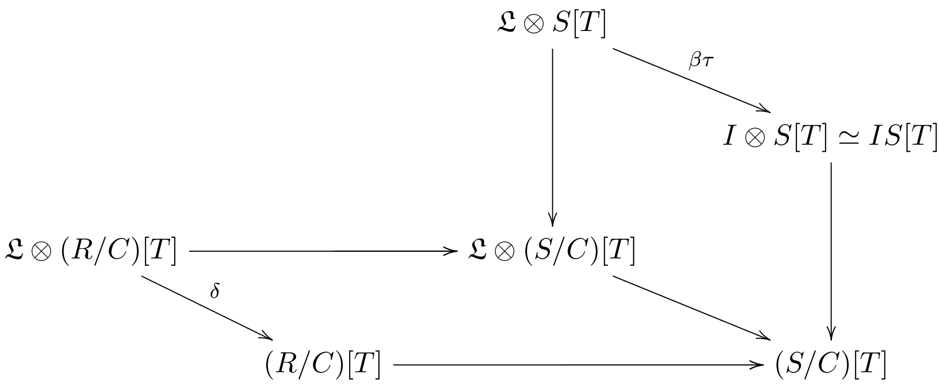
From (2) we have $IS[T] \otimes (S/C)[T] \simeq (S/C)[T]$, implying that the image of $IS[T]$ in $(S/C)[T]$ is $(S/C)[T]$. Therefore, $\beta_1 : \mathfrak{L} \otimes (S/C)[T] \twoheadrightarrow (S/C)[T]$ is a surjection and therefore $\beta_1 \in \text{Um}((\mathfrak{L} \otimes (S/C)[T])^*)$.

Now $\beta_1 \otimes (S/C)_{\text{red}}[T]$ is also a unimodular element of $(\mathfrak{L} \otimes (S/C)_{\text{red}}[T])^*$. Since it is given that $(R/C)_{\text{red}} = (S/C)_{\text{red}}$, it is easy to see that we have a lift of $\beta_1 \otimes (S/C)_{\text{red}}[T]$ to $(R/C)[T]$, say, $\delta : \mathfrak{L} \otimes (R/C)[T] \twoheadrightarrow (R/C)[T]$. In other words, δ is a unimodular element of $(\mathfrak{L} \otimes (R/C)[T])^*$. It is obvious from the way δ is obtained that $\delta \otimes (S/C)[T] = \beta_1$ modulo $\mathfrak{n}((S/C)[T])$, where $\mathfrak{n}((S/C)[T])$ denotes the nil radical of $(S/C)[T]$. So, we have two unimodular elements β_1 and $\delta \otimes (S/C)[T]$ of $(\mathfrak{L} \otimes (S/C)[T])^*$ such that $\delta \otimes (S/C)[T] = \beta_1$ modulo $\mathfrak{n}((S/C)[T])$. Therefore, by (2.8), there exists a transvection σ of $\mathfrak{L} \otimes (S/C)[T]$ such that $\beta_1 \sigma = \delta \otimes (S/C)[T]$. By (2.7), σ can be lifted to an automorphism τ of $\mathfrak{L} \otimes S[T]$ such that τ is identity modulo $IS[T]$.
Step2. Note that $I \otimes (R/C)[T] \simeq (R/C)[T]$ and $I \otimes (S/C)[T] \simeq (S/C)[T]$. Since $I + C[T] = R[T]$, it is easy to check that the natural map $f : I \otimes_{R[T]} S[T] \twoheadrightarrow IS[T]$ is actually an isomorphism.

Consider the following Cartesian diagram :

$$\begin{array}{ccc} I & \longrightarrow & IS[T] \simeq I \otimes S[T] \\ \downarrow & & \downarrow \\ (R/C)[T] & \longrightarrow & (S/C)[T] \end{array}$$

Therefore, we have the following commutative diagram:



Since $\delta \otimes (S/C)[T] = \beta_1\sigma = \beta\tau \otimes (S/C)[T]$, the surjective maps δ and $\beta\tau$ will patch to yield a surjection $\phi : \mathfrak{L} \twoheadrightarrow I$.

Step 3. Finally we need to show that $\phi \otimes R[T]/I = \alpha$. Since $\tau = \text{Id}$ (modulo $IS[T]$), we have $\beta\tau = \beta$ modulo $IS[T]$. Identifying $I \otimes_{R[T]} S[T]$ with $IS[T]$ and using the isomorphism $S[T]/IS[T] \simeq R[T]/I$, we have:

$$\phi \underset{R[T]}{\otimes} (R[T]/I) = \phi \underset{R[T]}{\otimes} (S[T]/IS[T]) = (\phi \underset{R[T]}{\otimes} S[T]) \underset{S[T]}{\otimes} (S[T]/IS[T]) = \beta\tau \underset{S[T]}{\otimes} (S[T]/IS[T])$$

$$= \beta \underset{S[T]}{\otimes} (S[T]/IS[T]) = \alpha \underset{R[T]/I}{\otimes} (S[T]/IS[T]) = \alpha \underset{R[T]/I}{\otimes} (R[T]/I) = \alpha.$$

Thus ϕ lifts α, and the proof of the lemma is complete. □

Lemma 5.2. *Let $R, S, C, \mathfrak{L}, I, \alpha$ be as fixed in the beginning of this section and assume that:*

(1) $(R/C)_{red} = (S/C)_{red}$;
(2) *dim*$(R) =$ *dim*$(S) = n \geq 4$;
(3) *ht*$(C) \geq 1$;
(4) *for any ideal J of $R[T]$, ht$(J) =$ ht$(JS[T])$.*

Then α can be lifted to a surjective map $\phi : \mathfrak{L} \twoheadrightarrow I$.

Proof. We have $\alpha : \mathfrak{L}/I\mathfrak{L} \twoheadrightarrow I/I^2$. Let $J = I^2 \cap C$. Then ht$(J) \geq 1$. Therefore, we can choose an element $b \in J$ such that ht$(b) = 1$. Let bar denote reduction modulo b. Then we have $\overline{\alpha} : \overline{\mathfrak{L}}/\overline{I\mathfrak{L}} \twoheadrightarrow \overline{I}/\overline{I}^2$ and dim$(R/bR) <$ dim(R), where.

Now we can apply (2.13) to get a (surjective) lift $\eta' : \overline{\mathfrak{L}} \twoheadrightarrow \overline{I}$ of $\overline{\alpha}$, and therefore a lift $\eta : \mathfrak{L} \longrightarrow I$ of α such that $(\eta(\mathfrak{L}), b) = I$. Note that $b \in I^2$. Applying (2.10) to the element (η, b) of $\mathfrak{L}^* \oplus R[T]$, we see that there exists $\Psi \in \mathfrak{L}^*$ such that ht$(K_b) \geq n$, where $K = (\eta + b\Psi)(\mathfrak{L})$. But $(\eta(\mathfrak{L}), b) = I$ has height n and I is a proper ideal. Therefore, by (2.10), ht$(K) = n$. Since $\eta + b\Psi$ is also a lift of α, we may replace η by $(\eta + b\Psi)$ and write

$K = \eta(\mathfrak{L})$. Note that $(K, b) = I$ and $b \in I^2$. It follows, applying (2.9), that there exists an ideal I' of $R[T]$ such that:

(1) $\eta(\mathfrak{L}) = I \cap I'$;
(2) $\eta \otimes R[T]/I = \alpha$;
(3) $\text{ht}(I') \geq n$;
(4) $I' + bR[T] = R[T]$ and therefore, $I' + C[T] = R[T]$.

If $\text{ht}(I') > n$, then $I' = R[T]$ and η is the desired lift of α. So, we assume that $ht(I') = n$.

Let $\beta : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$ be the lift of α^*. Now consider the surjection

$$\eta^* : \mathfrak{L} \otimes S[T] \overset{\eta \otimes S[T]}{\twoheadrightarrow} (I \cap I') \otimes S[T] \twoheadrightarrow (I \cap I')S[T]$$

As $I' + I = R[T]$, we have $(I \cap I')S[T] = IS[T] \cap I'S[T]$. By the subtraction principle (3.1), there exists $\gamma : \mathfrak{L} \otimes S[T] \twoheadrightarrow I'S[T]$ such that $\gamma \otimes S[T]/I' = \eta^* \otimes S[T]/I'$. Now we can apply the above proposition (taking I' in place of I) to get a surjection $\psi : \mathfrak{L} \twoheadrightarrow I'$ such that $\eta \otimes R[T]/I' = \psi \otimes R[T]/I'$.

Finally, we are going to apply the subtraction principle again to get a lift of α. We have two surjections $\eta : \mathfrak{L} \twoheadrightarrow I \cap I'$ and $\psi : \mathfrak{L} \twoheadrightarrow I'$ such that $\eta \otimes R[T]/I' = \psi \otimes R[T]/I'$. Therefore by the subtraction principle (3.1), we have a surjection $\phi : \mathfrak{L} \twoheadrightarrow I$ such that $\phi \otimes R[T]/I = \eta \otimes R[T]/I$. As $\eta \otimes R[T]/I = \alpha$, we have $\phi \otimes R[T]/I = \alpha$. Thus the proof is complete. □

Lemma 5.3. *Let $R, S, C, \mathfrak{L}, I, \alpha$ be as in the beginning of this section and assume that:*

(1) *the canonical map Spec$(S) \longrightarrow$ Spec(R) is bijective;*
(2) *for every $\mathfrak{q} \in$ Spec(S) the map $R/(\mathfrak{q} \cap R) \longrightarrow S/\mathfrak{q}$ is birational;*
(3) *dim$(R) =$ dim$(S) = n \geq 4$.*

Then α can be lifted to a surjective map $\phi : \mathfrak{L} \twoheadrightarrow I$.

Proof. Let C be the conductor ideal of R in S. By the assumptions of the lemma, $\text{ht}(C) \geq 1$. Further note that $R \hookrightarrow S$ is actually a subintegral extension and therefore by (2.17) $R[T] \hookrightarrow S[T]$ is also subintegral. Further, by (2.17) if J is any ideal of $R[T]$, then $\text{ht}(J) = \text{ht}(JS[T])$.

We prove the lemma by induction on $\dim(R/C)$. If $\dim(R/C) = 0$, then $(R/C)_{red}$ is also zero-dimensional and $(R/C)_{red}$ does not contain any non-zerodivisor. Let K be the radical of C in S, then we have $(R/C)_{red} = R/K \cap R$ and $(S/C)_{red} = R/K$. Now the total ring of fractions, $Q(S/K) = \prod k(P_i)$, where P_i are minimal prime ideals of S/K. Therefore $Q(R/K \cap R) = \prod k(P_i \cap R)$, since the canonical map $\text{Spec}(S) \longrightarrow \text{Spec}(R)$ is bijective.

Now by using $k(P_i) = k(P_i \cap R)$, we have $Q(S/K) = Q(R/K \cap R)$. But then

$$R/K \cap R \hookrightarrow S/K \hookrightarrow Q(S/K) = Q(R/K \cap R) = R/K \cap R$$

The last equality holds, since $R/K \cap R$ does not contain any non-zerodivisor.

Therefore it follows that $(R/C)_{red} = (S/C)_{red}$ and we are done by (5.2). So let us assume that $\dim(R/C) > 0$. But then by [Bh-R 2, 3.5] there exists a ring S' enjoying the following properties:

(1) $R \hookrightarrow S' \hookrightarrow S$
(2) $(R/C)_{\text{red}} = (S'/C)_{\text{red}}$
(3) $\dim(R/C) > \dim(S'/C')$ where C' is the conductor ideal of S' in S.

Now since the extension $S' \hookrightarrow S$ satisfies all the hypotheses of the lemma, by induction hypothesis there exists a surjection $\psi : \mathfrak{L} \otimes S'[T] \twoheadrightarrow IS'[T]$, which is a lift of α^*. But $(R/C)_{red} = (S'/C)_{red}$ and again applying (5.2) we obtain a surjection $\phi : \mathfrak{L} \twoheadrightarrow I$, which is a lift of α. $\square$

Remark 5.4. The above lemma is true for $n \geq 2$ if the ideal I is assumed to be comaximal with $C[T]$. To see this, first note that, in the lemma, we need $n \geq 4$ only to be able to apply the subtraction principle which was required to prove (5.2). Now if we start with $I + C[T] = R[T]$, we can apply (5.1) instead. Further note that in the proof of (5.3), the conductor of R in S' is C and C is contained in C' and therefore IS' is comaximal with $C'[T]$. For this last argument one has to go through the proof of [Bh-R 2, 3.5].

For the convenience of exposition, we make the following definition.

Definition 5.5. Let A be a ring and $\mathbb{L}$ be a projective $A[T]$-module of rank one. A ring extension $A \hookrightarrow B$ will be called *special* $\mathbb{L}$*-regular* if the following conditions are satisfied.

(1) The projective $B[T]$-module $\mathbb{L} \otimes_{A[T]} B[T]$ is extended from B,
(2) B is module-finite over A,
(3) the canonical map Spec$(B) \to$ Spec(A) is bijective, and
(4) for every $\mathfrak{P} \in$ Spec(B), the inclusion $A/(\mathfrak{P} \cap A) \longrightarrow B/\mathfrak{P}$ is birational.

Remark 5.6. Let R be a reduced ring and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Then there is a special $\mathbb{L}$-regular extension $R \hookrightarrow S$ with S reduced. The proof is essentially contained in the proof of [Bh 1, Proposition 3.3]. Further, note that a special $\mathbb{L}$-regular extension $R \hookrightarrow S$ is actually a subintegral extension. If $^+(R)$ denotes the seminormalization of R, then since $^+(R)$ is seminormal, one has Pic$(^+(R)[T]) \simeq$ Pic$(^+(R))$ and therefore $\mathbb{L} \otimes (^+(R)[T])$ is extended from $^+(R)$. As $\mathbb{L}$ is finitely generated and $^+(R)$ is the filtered direct limit of finite subintegral extensions of R, the reader will probably feel convinced that there is a special $\mathbb{L}$-regular extension of R.

The three technical lemmas we just proved above culminate in the following theorem which will be crucially used in the next section.

Theorem 5.7. *Let R be a reduced ring of dimension $n \geq 4$ and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension. Let I be an ideal of $R[T]$ and $\alpha : \mathfrak{L} \twoheadrightarrow I/I^2$ be a surjection. Suppose that the induced surjection $\alpha^* : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]/I^2S[T]$ can be lifted to a surjection $\beta : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$. Then α can also be lifted to a surjective map $\phi : \mathfrak{L} \twoheadrightarrow I$.*

Proof. Since $R \hookrightarrow S$ is a special $\mathbb{L}$-regular extension, with S reduced and it satisfies all the conditions of (5.3). $\square$

Remark 5.8. It follows from (5.4) that the above theorem is true for $n \geq 2$ if the ideal I is assumed to be comaximal with $C[T]$. We shall need this observation in Section 7.

We now demonstrate an application of (5.7) below. A result of Mandal [M1, 1.2] is improved, albeit with a stronger hypothesis on the dimension.

Theorem 5.9. *Let $A = R[T]$ be a polynomial ring over a commutative Noetherian ring R with dim$(R) = n \geq 4$. Let I be an ideal of A of height n that contains a monic polynomial. let $\mathbb{L}$ be a projective $R[T]$-module of rank 1. Write $\mathfrak{L} = \mathbb{L} \oplus R[T]^{n-1}$. Suppose that there exists $\alpha : \mathfrak{L} \twoheadrightarrow I/I^2$. Then there is a surjection $\beta : \mathfrak{L} \twoheadrightarrow I$ such that β lifts α.*

Proof. By (2.14) we may assume that R is reduced. Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension with S reduced.

Let $\alpha^* : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]/I^2S[T]$ be the surjection induced by α. Now note that $\mathfrak{L} \otimes S[T] = (\mathbb{L} \otimes S[T]) \oplus S[T]^{n-1}$. As $R \hookrightarrow S$ is a special $\mathbb{L}$-regular extension, $\mathbb{L} \otimes S[T]$ is extended from S. It then follows from [Bh-RS 4, 3.3], that α^* can be lifted to a surjection $\widetilde{\beta} : \mathfrak{L} \otimes S[T] \twoheadrightarrow I$. The result now follows from (5.7). $\square$

6. The Euler class group $E(R[T], \mathbb{L})$ for arbitrary $\mathbb{L}$

Our aim in this section is to define and study the (n-th) Euler class group of $R[T]$ with respect to a projective $R[T]$-module $\mathbb{L}$ of rank one (which is not necessarily extended from R), and extend the results of Section 4.

Remark 6.1. We keep it for the record that the top Euler class group $E^{n+1}(R[T])$ is trivial. This case falls in the domain of [Bh-RS 3]. Let $\phi : \mathbb{L} \oplus R[T]^n \twoheadrightarrow I/I^2$ be any surjection, where I is an ideal of $R[T]$ of height $n+1$. It follows from (2.13) that ϕ can be lifted to a surjective map $\Phi : \mathbb{L} \oplus R[T]^n \twoheadrightarrow I$. Therefore, $E^{n+1}(R[T], \mathbb{L})$ is trivial.

Notation. By a ring R we shall mean a commutative Noetherian ring R containing $\mathbb{Q}$ with $\dim(R) = n \geq 4$. Let us fix a projective $R[T]$-module $\mathbb{L}$ of rank one. Further, we write $\mathfrak{L} = \mathbb{L} \oplus R[T]^{n-1}$.

We now go on to define the n-th Euler class group $E^n(R[T], \mathbb{L})$ (henceforth denoted as $E(R[T], \mathbb{L})$). This definition is simply a verbatim copy of the definition of $E(R[T], L[T])$ that was given in Section 4, only replacing $L[T]$ by $\mathbb{L}$. Therefore, we just recall a few terms and point out the differences. The harder part in this section is to prove results analogous to those in Section 4.

Let $I \subset R[T]$ be an ideal of height n such that I/I^2 is surjective image of $\mathfrak{L}/I\mathfrak{L}$. Let bar denote reduction modulo I. Two surjections $\alpha, \beta : \mathfrak{L}/I\mathfrak{L} \twoheadrightarrow I/I^2$ are said to be *related* if there exists an automorphism $\sigma \in SL(\mathfrak{L}/I\mathfrak{L})$ such that $\alpha\sigma = \beta$. This defines an equivalence relation on the set of surjections from $\mathfrak{L}/I\mathfrak{L}$ to I/I^2. We call such an equivalence class a *local* $\mathbb{L}$*-orientation* of I.

We now prove

Lemma 6.2. *Let $\alpha, \beta : \mathfrak{L}/I\mathfrak{L} \twoheadrightarrow I/I^2$ be two surjections belonging to the same equivalence class. Suppose it is given that α can be lifted to a surjection $\phi : \mathfrak{L} \twoheadrightarrow I$. Then β can also be lifted to a surjection $\psi : \mathfrak{L} \twoheadrightarrow I$.*

Proof. In view of Lemma 2.14 we may assume that R is reduced.

Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension with S reduced (such an extension exists by (5.6)). Consider the two surjections $\alpha^*,\ \beta^* : (\mathfrak{L} \otimes S[T])/IS[T](\mathfrak{L} \otimes S[T]) \twoheadrightarrow IS[T]/I^2S[T]$, which are induced by α, β, respectively. By the assumption of the lemma, there exists an automorphism $\sigma \in SL(\mathfrak{L}/I\mathfrak{L})$ such that $\alpha\sigma = \beta$. This implies that $\alpha^*,\ \beta^*$ are also connected by an automorphism of determinant one. Now note that $\mathbb{L} \otimes S[T]$ is a projective $S[T]$-module of rank one which is extended from S. Therefore, as α^* has a surjective lift $\phi \otimes S[T] : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$, applying (4.2), it follows that β^* also has a surjective lift, say, $\theta : \mathfrak{L} \otimes S[T] \twoheadrightarrow IS[T]$. Now we can apply (5.2) and conclude that there is a surjection $\psi : \mathfrak{L} \twoheadrightarrow I$ which lifts β. $\square$

Definition 6.3. We call a local $\mathbb{L}$-orientation $[\alpha]$ of I a *global orientation* of I if the surjection $\alpha : \mathfrak{L}/I\mathfrak{L} \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : \mathfrak{L} \twoheadrightarrow I$.

Define the groups G and H exactly as in Section 4, by only replacing $L[T]$ with $\mathbb{L}$.

Definition 6.4. The n-th Euler class group of $R[T]$ with respect to the projective $R[T]$-module $\mathbb{L}$ is defined as $E^n(R[T], \mathbb{L}) \stackrel{\text{def}}{=} G/H$. In this paper we shall simply denote this group as $E(R[T], \mathbb{L})$.

Remark 6.5. Let A be a commutative Noetherian ring of dimension d and L be a projective A-module of rank one. For each r, $1 \leq r \leq d$, Mandal-Yang defined the r-th Euler class group $E^r(A, L)$ in [M-Y 1, M-Y 2], generalizing the definition of $E^d(A, L)$ given by Bhatwadekar-Sridharan in [Bh-RS 3]. We may note that, for $E^r(A, L)$, to define local L-orientations of an ideal J of height r, Mandal-Yang used equivalence classes given by $\mathcal{E}((L \oplus A^{r-1})/J(L \oplus A^{r-1}))$ instead of $SL((L \oplus A^{r-1})/J(L \oplus A^{r-1}))$. If $r = d$, the Mandal-Yang definition of the Euler class group $E^d(A, L)$ coincides with that of Bhatwadekar-Sridharan because then $\dim(A/J) = 0$ and $\mathcal{E}((L \oplus A^{d-1})/J(L \oplus A^{d-1})) = SL((L \oplus A^{d-1})/J(L \oplus A^{d-1}))$. Since these groups may differ if $\dim(A/J) > 0$, the reader may note that our definition of $E^n(R[T],\mathbb{L})$ given above is not obtained from the Mandal-Yang definition just by putting $A = R[T]$, $d = n+1$, $r = n$ and $L = \mathbb{L}$.

The following result is crucial for further discussions.

Proposition 6.6. *Let R be a reduced ring, $\mathbb{L}$ be a projective $R[T]$-module of rank one and let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension. Then there is a canonical injective group homomorphism $\Theta : E(R[T],\mathbb{L}) \longrightarrow E(S[T],\mathbb{L}\otimes S[T])$.*

Proof. We first note that $R \hookrightarrow S$ is a finite subintegral extension. Therefore, $R[T] \hookrightarrow S[T]$ is also subintegral. Further, if $I \subset R[T]$ is an ideal, then $\text{ht}(I) = \text{ht}(IS[T])$ (see (2.17)). Given a surjection $\omega_I : \mathfrak{L}/I\mathfrak{L} \twoheadrightarrow I/I^2$, we have the induced surjection

$$\omega_I^* : \frac{(\mathfrak{L}\otimes S[T])}{IS[T](\mathfrak{L}\otimes S[T])} \twoheadrightarrow \frac{IS[T]}{I^2S[T]},$$

as described at the beginning of Section 5. It is now easy to see that there is a canonical group homomorphism $\Theta : E(R[T],\mathbb{L}) \to E(S[T],\mathbb{L}\otimes S[T])$, which takes (I,ω_I) to $(IS[T],\omega_I^*)$. To prove that Θ is injective, let $(I,\omega_I) \in E(R[T],\mathbb{L})$ be such that $\Theta((I,\omega_I)) = 0$ in $E(S[T],\mathbb{L}\otimes S[T])$. In other words, $(IS[T],\omega_I^*) = 0$ in $E(S[T],\mathbb{L}\otimes S[T])$, where ω_I^* is induced by ω_I. As $\mathbb{L}\otimes S[T]$ is extended from S, it follows from (4.10), that ω_I^* has a surjective lift $\eta : \mathfrak{L}\otimes S[T] \twoheadrightarrow IS[T]$. But then (5.2) implies that there is a surjection $\zeta : \mathfrak{L} \twoheadrightarrow I$ lifting ω_I. Therefore ω_I is a global $\mathbb{L}$-orientation and consequently, $(I,\omega_I) = 0$ in $E(R[T],\mathbb{L})$. $\square$

We now prove the following results on the Euler class group $E(R[T],\mathbb{L})$.

Theorem 6.7. *Let R be a reduced ring of dimension $n \geq 4$. Let $I \subset R[T]$ be an ideal of height n such that I/I^2 is surjective image of $\mathfrak{L}$ and let ω_I be a local $\mathbb{L}$-orientation of I. Suppose that the image of (I,ω_I) is zero in $E(R[T],\mathbb{L})$. Then ω_I can be lifted to a surjection $\theta : \mathfrak{L} \twoheadrightarrow I$.*

Proof. This is a direct consequence of (4.10) and (5.2). $\square$

So far we kept assuming that the ring R is reduced. To extend the theory to non-reduced rings, the following proposition is in order.

Proposition 6.8. *Let R be a ring and let $R_{red} = R/\mathfrak{n}(R)$, where $\mathfrak{n}(R)$ is the nil-radical of R. Let $\mathbb{L}$ be a projective $R[T]$-module of rank one. Then there is a canonical isomorphism $\eta : E(R[T],\mathbb{L}) \xrightarrow{\sim} E(R_{red}[T],\mathbb{L}\otimes R_{red}[T])$.*

Proof. The proof is along the same line as [D 3, 2.15] and [Bh-RS 3, 4.6] and therefore omitted. The reader may also consult [K, 4.13]. $\square$

Remark 6.9. As a consequence of the above proposition, (6.7) is now valid for non-reduced R and therefore, throughout this section we may assume R to be reduced.

We now define the Euler class of a projective $R[T]$-module with determinant $\mathbb{L}$.

Let P be a projective $R[T]$-module of rank n whose determinant is isomorphic to $\mathbb{L}$. Let $\chi : \mathbb{L} \overset{\sim}{\to} \wedge^n(P)$ be an isomorphism. To the pair (P, χ), we associate an element $e(P, \chi)$ of $E(R[T], \mathbb{L})$ as follows:

Let $\alpha : P \twoheadrightarrow I$ be a surjection, where I is an ideal of $R[T]$ of height n. Let bar denote reduction modulo I. Note that, since $\dim(R[T]/I) \leq 1$, by Serre's splitting theorem (2.2) we have $P/IP \simeq \mathfrak{L}/I\mathfrak{L}$. We choose an isomorphism $\overline{\gamma} : \mathfrak{L}/I\mathfrak{L} \overset{\sim}{\to} P/IP$ such that $\wedge^n \overline{\gamma} = \overline{\chi}$. Let ω_I be the composite surjection

$$\mathfrak{L}/I\mathfrak{L} \overset{\overline{\gamma}}{\overset{\sim}{\to}} P/IP \overset{\overline{\alpha}}{\twoheadrightarrow} I/I^2.$$

Let $e(P, \chi)$ be the image in $E(R[T], \mathbb{L})$ of the element (I, ω_I). We say that (I, ω_I) is obtained from the pair (α, χ).

Definition 6.10. We define the Euler class of (P, χ) to be $e(P, \chi)$.

Lemma 6.11. *The assignment sending the pair (P, χ) to the element $e(P, \chi)$, as described above, is well defined.*

Proof. Let $\alpha : P \twoheadrightarrow I$ and $\beta : P \twoheadrightarrow J$ be two surjection, where $I, J \subset R[T]$ be two ideals of height n. Let (I, ω_I) and (J, ω_J) be obtained from (α, χ) and (β, χ), respectively.

Applying (2.11), we can find an ideal $K \subset R[T]$ of height n such that K is comaximal with I, J and there is a surjection $\gamma : \mathfrak{L} \twoheadrightarrow I \cap K$ such that $\gamma \otimes R[T]/I = \omega_I$. Since K and I are comaximal, γ induces a local $\mathbb{L}$-orientation ω_K of K. Clearly, $(I, \omega_I) + (K, \omega_K) = 0$ in $E(R[T], \mathbb{L})$.

Let $M = K \cap J$. Note that ω_K and ω_J together will induce a local $\mathbb{L}$-orientation of M. Call it ω_M. Then, $(M, \omega_M) = (K, \omega_K) + (J, \omega_J)$. Therefore, showing $(M, \omega_M) = 0$ in $E(R[T], \mathbb{L})$ is enough to prove the lemma.

We may assume that R is reduced. Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension. Then $\mathbb{L} \otimes S[T]$ is extended from S. As the Euler class of a projective $S[T]$-module (in $E(S[T], \mathbb{L} \otimes S[T])$) is well defined, it follows that $(MS[T], \omega_M^*) = 0$ in $E(S[T], \mathbb{L} \otimes S[T])$, where ω_M^* is induced by ω_M. Therefore, by (4.10), it follows that ω_M^* can be lifted to a surjective map $\theta : \mathfrak{L} \otimes S[T] \twoheadrightarrow MS[T]$. Applying (5.2), we obtain a surjective lift $\phi : \mathfrak{L} \twoheadrightarrow M$ of ω_M. In other words, $(M, \omega_M) = 0$ in $E(R[T], \mathbb{L})$. $\square$

Theorem 6.12. *Let R be a ring and $\mathbb{L}, P, \chi$ as above. Then, $e(P, \chi) = 0$ in $E(R[T], \mathbb{L})$ if and only if P has a unimodular element.*

Proof. Without loss of generality we may assume that R is reduced. Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension with S reduced. Let $\alpha : P \twoheadrightarrow I$ be a surjection, where I is an ideal in $R[T]$ of height n. Let $e(P, \chi) = (I, \omega_I)$ in $E(R[T], \mathbb{L})$, where (I, ω_I) is obtained from the pair (α, χ).

We first assume that $e(P, \chi) = 0$ in $E(R[T], \mathbb{L})$. Then, $e(P \otimes S[T], \chi \otimes S[T]) = 0$ in $E(S[T], \mathbb{L} \otimes S[T])$. As $\mathbb{L} \otimes S[T]$ is extended from S, it follows from (4.15) that $P \otimes S[T]$ has a unimodular element. But then by [Bh 1, 3.2], P has a unimodular element.

Conversely, assume that P has a unimodular element. Therefore, $P \otimes S[T]$ also has a unimodular element. As $\mathbb{L} \otimes S[T]$ is extended from S, it follows from (4.15) that $e(P \otimes S[T], \chi \otimes S[T]) = (IS[T], \omega_I^*) = 0$ in $E(S[T], \mathbb{L} \otimes S[T])$. But then by (6.6), $e(P, \chi) = 0$ in $E(R[T], \mathbb{L})$. $\square$

Theorem 6.13. *Let R be a ring and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Let P be a projective $R[T]$-module of rank n which is stably isomorphic to $\mathbb{L} \oplus R[T]^{n-1}$. Let $\chi : \mathbb{L} \xrightarrow{\sim} \wedge^n P$ be an isomorphism. Let $I \subset R[T]$ be an ideal of height n such that I/I^2 is surjective image of $\mathfrak{L}$ and ω_I be a local $\mathbb{L}$-orientation of I. Suppose that $e(P, \chi) = (I, \omega_I)$ in $E(R[T], \mathbb{L})$. Then, there exists a surjection $\alpha : P \twoheadrightarrow I$ such that (I, ω_I) is obtained from (α, χ).*

Proof. By (2.2), P/IP is isomorphic to $\mathfrak{L}/I\mathfrak{L}$. Choose an isomorphism $\sigma : \mathfrak{L}/I\mathfrak{L} \xrightarrow{\sim} P/IP$ such that $\wedge^n \sigma = \chi \otimes R[T]/I$. Let $\psi : P/IP \twoheadrightarrow I/I^2$ be the composite surjection:

$$P/IP \xrightarrow{\sigma^{-1}} \mathfrak{L}/I\mathfrak{L} \overset{\omega_I}{\twoheadrightarrow} I/I^2$$

Applying (2.11) we obtain a lift of ψ, say, $\phi \in \text{Hom}_{R[T]}(P, I)$ such that $\phi(P) = I \cap I'$, where I' is an ideal of $R[T]$ of height $\geq n$ and $I + I' = R[T]$. If $I' = R[T]$, then obviously ϕ is surjective and we are done in this case. Therefore, we assume that I' is proper and $\text{ht}(I') = n$.

Choose an isomorphism $\delta : \mathfrak{L}/I'\mathfrak{L} \xrightarrow{\sim} P/I'P$ such that $\wedge^n(\delta) = \chi \otimes R[T]/I'$. By the Chinese Remainder Theorem, we have $P/(I \cap I')P \simeq P/IP \oplus P/I'P$. Therefore, σ and δ together will induce an isomorphism $\tau : \mathfrak{L}/(I \cap I')\mathfrak{L} \xrightarrow{\sim} P/(I \cap I')P$ such that $\wedge^n(\tau) = \chi \otimes R[T]/(I \cap I')$. Composing τ with $\phi \otimes R[T]/(I \cap I')$ (as in the definition of the Euler class) one obtains a local orientation of $I \cap I'$, say, $\omega_{I \cap I'}$, and therefore, $e(P, \chi) = (I \cap I', \omega_{I \cap I'})$. Again note that as I and I' are comaximal, $\omega_{I \cap I'}$ induces local $\mathbb{L}$-orientations of I and I' and it is easy to see that the induced local orientation for I is precisely ω_I. If we call the one induced for I' as $\omega_{I'}$ then we have:

$$e(P, \chi) = (I, \omega_I) + (I', \omega_{I'}) \text{ in } E(R[T], \mathbb{L}).$$

From the hypothesis of the theorem it now follows that $(I', \omega_{I'}) = 0$, and therefore by (6.7) there exists a surjection $\beta : \mathfrak{L} \twoheadrightarrow I'$ such that $\beta \otimes R[T]/I' = \phi \otimes R[T]/I'$. Now we can apply (3.2) and conclude the proof of the corollary. $\square$

Theorem 6.14. *Let R be a regular ring of dimension n which is essentially of finite type over a field k such that R has infinite residue fields. Let $\mathbb{L}$ be a projective $R[T]$-module of rank one. Then $E(R[T],\mathbb{L}) \simeq E(R,\mathbb{L}/T\mathbb{L})$.*

Proof. By a result of Lindel [Li 1], the projective $R[T]$-module $\mathbb{L}$ is extended from R. Therefore, there exists projective R-module L of rank one such that $\mathbb{L} \simeq L[T]$ and $\mathbb{L}/T\mathbb{L} \simeq L$. Therefore, we need to prove that $E(R[T],L[T]) \simeq E(R,L)$ and we are done by (4.19). $\square$

Remark 6.15. A result similar to the above has been proved in [M-Y 2] using a different technique. However, as remarked in (6.5), the group $E(R[T],\mathbb{L})$ is not exactly the same as the one appearing in [M-Y 2].

In the following theorem we extend some results from [D-Z]. In [D-Z], it has been proved that if $R \hookrightarrow S$ is a subintegral extension then the Euler class groups $E(R[T])$ and $E(S[T])$ are isomorphic. The proofs given below are natural extensions of arguments from [D-Z].

Theorem 6.16. *Let R be a ring and S be an extension ring. Let $\mathbb{L}$ be a projective $R[T]$-module of rank one. Then $E(R[T],\mathbb{L}) \simeq E(S[T],\mathbb{L}\otimes S[T])$ in the following cases:*

(1) *$R \hookrightarrow S$ is elementarily subintegral.*
(2) *$R \hookrightarrow S$ is finite subintegral. In particular, when S is a special $\mathbb{L}$-regular extension.*
(3) *$R \hookrightarrow S$ is subintegral.*
(4) *$S = {}^+(R_{red})$, the seminormalization of R_{red}.*

Proof. We may assume by (6.8) that the ring R is reduced to start with. Also note that if $R \hookrightarrow S$ is subintegral, then we have a natural group homomorphism $\Theta : E(R[T],\mathbb{L}) \to E(S[T],\mathbb{L}\otimes S[T])$, which sends (J,ω_J) to (JS,ω_J^*), where ω_J^* is the local orientation induced by ω_J.

(1) Let $R \hookrightarrow S$ be elementarily subintegral. Let C be the conductor of R in S. Then by [D-Z, 3.5] we have $(R/C)_{\text{red}} = (S/C)_{\text{red}}$. It now follows from (5.1) that Θ is injective.

To prove that Θ is surjective, let $(I,\sigma) \in E(S[T],\mathbb{L}\otimes S[T])$, where $I \subset S[T]$ is an ideal of height n and $\sigma : (\mathfrak{L}\otimes S[T])/I(\mathfrak{L}\otimes S[T]) \twoheadrightarrow I/I^2$ is a surjection. By using the moving lemma (2.11), we can find an ideal $K \subseteq S[T]$ and a surjection $\tau : \mathfrak{L}\otimes S[T] \twoheadrightarrow I\cap K$ such that: (i) $\text{ht}(K) \geq n$, (ii) $K + I\cap C[T] = S[T]$, and (iii) $\tau\otimes S[T]/I = \sigma$.

If $\text{ht}(K) > n$, then $K = S[T]$ and we have $(I,\sigma) = 0$ in $E(S[T])$. Therefore we assume that $\text{ht}(K) = n$. Let $\eta = \tau\otimes S[T]/K$ be the local $\mathbb{L}$-orientation of K. Then we have, $(I,\sigma)+(K,\eta) = 0$ in $E(S[T],\mathbb{L}\otimes S[T])$. It is now enough to prove that (K,η) has a preimage in $E(R[T],\mathbb{L})$.

Let $K\cap R[T] = J$. As $K+C[T] = S[T]$, we have $J+C[T] = R[T]$, and therefore there exists $f \in C[T]$ such that $g = 1-f \in J$. We can assume that $\text{ht}(g) = 1$. (If $\text{ht}(g) = 0$, choose $g' \in J$ such that g' does not belong to any minimal prime ideal of $R[T]$. Then $\text{ht}(g+g'-gg')=1$. Now $(1-g)(1-g')=1-g-g'+gg'$. If we write $g''=g+g'-gg'$, then we have $1-g' \in C[T]$ and $\text{ht}(g'')=1$ and we can work with g''). Since $f \in C[T]$, we have $R[T]_f = S[T]_f$. Therefore $R[T]/(1-f)=S[T]/(1-f)$ and $R[T] \hookrightarrow S[T]$ is an analytic isomorphism along $g \in J$. Therefore using [N, proposition 1.3], we have

(a) $R[T]/J \simeq S[T]/K$.
(b) $K = JS[T]$
(c) As $g \in J$, we have $J/J^2 \simeq K/K^2$.

As a consequence of (a) we have, $\mathfrak{L}/J\mathfrak{L} \simeq (\mathfrak{L}\otimes S[T])/K(\mathfrak{L}\otimes S[T])$. It is now easy to see from (c) that η is induced from a surjection $\omega_J : \mathfrak{L}/J\mathfrak{L} \twoheadrightarrow J/J^2$. Therefore, $\Theta((J,\omega_J))=(K,\eta)$.

(2) Let $R \hookrightarrow S$ be finite subintegral. Then S is obtained from R by a finite sequence of elementarily subintegral extensions and we are done by (1) above.

(3) Here S is the filtered direct limit of subrings S_α such that each S_α can be obtained from R by a finite number of elementarily subintegral extensions. A direct limit argument as in [D-Z] can easily be given to conclude the result.

(4) Obvious from (3). □

7. LOW DIMENSIONAL RINGS

In this section we treat the cases when $\dim(R)=2,3$. The methods of previous sections do not naturally extend to three dimensional rings due to the lack of a suitable subtraction principle and we need to handle this carefully. The case of two dimensional rings is much simpler but the method is different.

7.1. **Three dimensional rings.** Let R be a ring of dimension 3 (containing $\mathbb{Q}$) and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Assume for the time being that R is reduced. Let $R \hookrightarrow S$ be a special $\mathbb{L}$-regular extension with S reduced and C be the conductor of R in S. We fix S for the following discussion. We have, $\text{ht}(C) \geq 1$. A careful inspection of the theory and the results in Section 4 would reveal that if we had $\text{ht}(C) \geq 2$, or more generally, $\text{ht}(J(R,\mathbb{L})) \geq 2$, where $J(R,\mathbb{L})$ is the Quillen ideal of $\mathbb{L}$ in R, then one can similarly develop the theory of $E(R[T],\mathbb{L})$ and prove all the results of Sections 4, 6. However, we now assume that $\text{ht}(C) \geq 1$ and define a "restricted" Euler class group $\widetilde{E}(R[T],\mathbb{L})$ below, which will serve most of our purposes. The definition is exactly the same as those in Sections 4 and 6, only with one restriction imposed on the ideals concerned. We shall not repeat the whole definition in detail and we shall freely use terms defined in Section 6.

Definition 7.1. (*The "restricted" Euler class group* $\widetilde{E}(R[T],\mathbb{L})$): Let R be reduced. Let $\widetilde{G}$ be the free abelian group on pairs $(\mathcal{I},\omega_{\mathcal{I}})$, where $\mathcal{I}\subset R[T]$ is an ideal of height n such that $\operatorname{Spec}(R[T]/\mathcal{I})$ is connected and $\mathcal{I}+C[T]=R[T]$ (here is the restriction), and

$$\omega_{\mathcal{I}}:\frac{(\mathbb{L}\oplus R[T]^2)}{\mathcal{I}(\mathbb{L}\oplus R[T]^2)}\twoheadrightarrow\frac{\mathcal{I}}{\mathcal{I}^2}$$

is a local $\mathbb{L}$-orientation of $\mathcal{I}$. Given any ideal I of $R[T]$ such that $I+C[T]=R[T]$ and any local $\mathbb{L}$-orientation ω_I, one can easily associate an element in $\widetilde{G}$, as it was done before. We denote this element as (I,ω_I). Take $\widetilde{H}$ to be the subgroup of $\widetilde{G}$ generated by all those (I,ω_I) of $\widetilde{G}$ such that ω_I is a global $\mathbb{L}$-orientation. Define $\widetilde{E}(R[T],\mathbb{L})=\widetilde{G}/\widetilde{H}$.

We write $\mathfrak{L}=\mathbb{L}\oplus R[T]^2$.

Theorem 7.2. *Let R be a reduced ring of dimension 3, $\mathbb{L}$ be a projective $R[T]$-module of rank one and $R\hookrightarrow S$ be a special $\mathbb{L}$-regular extension. Then there is an injective group homomorphism $\Theta:\widetilde{E}(R[T],\mathbb{L})\longrightarrow E(S[T],\mathbb{L}\otimes S[T])$.*

Proof. The definition of Θ is the same as (6.6). Obviously it is a group homomorphism. The injectivity of Θ follows from (5.8). □

Corollary 7.3. *Let $R,\mathbb{L},S$ be as above. Let $(I,\omega_I)=0$ in $\widetilde{E}(R[T],\mathbb{L})$. Then ω_I is a global $\mathbb{L}$-orientation of I, i.e., there is a surjective map $\alpha:\mathfrak{L}\twoheadrightarrow I$ such that α lifts ω_I.*

Proof. Clearly follows from the above theorem because Θ is injective. □

Now let P be a projective $R[T]$-module of rank 3 with determinant $\mathbb{L}$ and let $\chi:\mathbb{L}\xrightarrow{\sim}\wedge^3(P)$ be an isomorphism. We can associate an element $e(P,\chi)$, called the Euler class of (P,χ), in the group $\widetilde{E}(R[T],\mathbb{L})$ so that it serves as the precise obstruction for P to split off a free summand of rank one. We describe it now.

Let $R\hookrightarrow S$ be a special $\mathbb{L}$-regular extension as above and C be the conductor of R in S. Since $\dim(R/C)\le 2$, it follows from (2.2) that the projective $(R/C)[T]$-module $P/C[T]P$ has a unimodular element. Applying (2.10) it is easy to see that there is an ideal $I\subset R[T]$ of height 3 which is comaximal with $C[T]$ such that there is a surjection $\alpha:P\twoheadrightarrow I$. Choose an isomorphism $\overline{\gamma}:\mathfrak{L}/I\mathfrak{L}\xrightarrow{\sim}P/IP$ such that $\wedge^n\overline{\gamma}=\overline{\chi}$, where bar denotes reduction modulo I. Let ω_I be the composite surjection

$$\mathfrak{L}/I\mathfrak{L}\overset{\overline{\gamma}}{\xrightarrow{\sim}}P/IP\overset{\overline{\alpha}}{\twoheadrightarrow}I/I^2.$$

We define the Euler class of (P,χ) as $e(P,\chi)=(I,\omega_I)\in\widetilde{E}(R[T],\mathbb{L})$. Following the same method as in (6.11) and using (7.2) it is easy to prove that the Euler class is well defined.

Theorem 7.4. *Let $R, \mathbb{L}$ be as above. Let P be a projective $R[T]$-module of rank 3 with determinant $\mathbb{L}$ and let $\chi : \mathbb{L} \xrightarrow{\sim} \wedge^3(P)$ be an isomorphism. Then $e(P,\chi) = 0$ in $\widetilde{E}(R[T],\mathbb{L})$ if and only if P has a unimodular element.*

Proof. Let $R \hookrightarrow S$ be the special $\mathbb{L}$-regular extension fixed above and $\Theta : \widetilde{E}(R[T],\mathbb{L}) \longrightarrow E(S[T], \mathfrak{L} \otimes S[T])$ be the group homomorphism from the above theorem. Let $e(P,\chi) = (I, \omega_I)$ in $\widetilde{E}(R[T],\mathbb{L})$.

First assume that $e(P,\chi) = 0$ in $\widetilde{E}(R[T],\mathbb{L})$. This will imply that $e(P \otimes S[T], \chi \otimes S[T]) = (IS[T], \omega_I^*) = 0$ in $E(S[T], \mathfrak{L} \otimes S[T])$, where ω_I^* is the local orientation of $IS[T]$ induced by ω_I. As $\mathbb{L} \otimes S[T]$ is extended from S, it follows from Corollary 4.15 that $P \otimes S[T]$ has a unimodular element. Then by [Bh 1, Lemma 3.1], P has a unimodular element.

Conversely, if P has a unimodular element then the same is true for $P \otimes S[T]$, and then $(IS[T], \omega_I^*) = 0$ in $E(S[T], \mathfrak{L} \otimes S[T])$. As Θ is injective, it follows that $(I, \omega_I) = 0$ in $E(S[T], \mathfrak{L} \otimes S[T])$. Therefore, $e(P,\chi) = 0$ in $E(S[T], \mathfrak{L} \otimes S[T])$. $\square$

Remark 7.5. Now let R be a ring of dimension 3 which is not necessarily reduced. Let P be a projective $R[T]$-module of rank 3 with determinant $\mathbb{L}$. Let $R_{\text{red}} = R/\mathfrak{n}(R)$, where $\mathfrak{n}(R)$ is the nil radical of R. It is easy to derive that P has a unimodular element if and only if $P \otimes R_{\text{red}}$ has a unimodular element. Fix an isomorphism $\chi : \mathbb{L} \xrightarrow{\sim} \wedge^3(P)$. Consider the Euler class $e(P \otimes R_{\text{red}}, \chi \otimes R_{\text{red}}) \in \widetilde{E}(R_{\text{red}}, \mathbb{L} \otimes R_{\text{red}})$. Then P has a unimodular element if and only if $e(P \otimes R_{\text{red}}, \chi \otimes R_{\text{red}}) = 0$ in $\widetilde{E}(R_{\text{red}}, \mathbb{L} \otimes R_{\text{red}})$.

7.2. **Two dimensional rings.** Let R be a ring of dimension 2 and $\mathbb{L}$ be a projective $R[T]$-module of rank one. The theory of the Euler class group $E(R[T],\mathbb{L})$ is very much similar to the two-dimensional case developed in [D 1, Section 7]. Unlike the higher dimensional cases treated so far in this paper, most of the results for two dimensional rings from [D 1, Section 7] can be extended without much hurdle. We first prove the following easy lemma.

Lemma 7.6. *Let R be a ring of dimension 2 and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Let $J \subset R[T]$ be an ideal such that there is a surjection $\overline{\alpha} : \mathbb{L} \oplus R[T] \twoheadrightarrow J/J^2$. Then there exists a projective $R[T]$-module P of determinant $\mathbb{L}$ such that P maps onto J.*

Proof. Write $\mathfrak{L} = \mathbb{L} \oplus R[T]$. Let $\alpha : \mathfrak{L} \to J$ be a lift of $\overline{\alpha}$. Then $\alpha(\mathfrak{L}) + J^2 = J$. Therefore, by (2.9) there exits $e \in J$ such that $J = (\alpha(\mathfrak{L}), e)$ with $e(1-e) \in \alpha(\mathfrak{L})$. This implies that $\alpha' = \alpha_{1-e} : \mathfrak{L}_{1-e} \twoheadrightarrow J_{1-e}$ is a surjection.

On the other hand, we have a surjection $\beta : \mathfrak{L}_e \twoheadrightarrow J_e = R[T]_e$ which is projection onto the second factor.

Thus we obtain two surjections α'_e, β_{1-e} from $\mathfrak{L}_{e(1-e)}$ to $J_{e(1-e)} = R[T]_{e(1-e)}$, and exact sequences:

$$0 \to \ker(\alpha'_e) \to \mathfrak{L}_{e(1-e)} \to J_{e(1-e)} = R[T]_{e(1-e)} \to 0$$

$$0 \to \ker(\beta_{1-e}) \to \mathfrak{L}_{e(1-e)} \to J_{e(1-e)} = R[T]_{e(1-e)} \to 0$$

As projective modules of rank one are always cancellative, we have $\ker(\alpha'_e) \simeq \mathbb{L}_{e(1-e)} \simeq \ker(\beta_{1-e})$. Therefore, there is an automorphism ϕ of $\mathfrak{L}_{e(1-e)}$ such that $\det(\phi) = 1$ and $(\beta_{1-e})\phi = \alpha'_e$. By a standard patching argument we obtain a projective $R[T]$-module P of rank 2 and a surjection from P to J. As $\det(\phi) = 1$, it is easy to see that $\wedge^2(P) \simeq \mathbb{L}$. $\square$

Armed with the above lemma one can now easily extend [D 1, 7.1] in the following manner. We omit the proof as it can be worked out modifying the proof of [D 1, 7.1].

Theorem 7.7. *Let R be a ring of dimension 2 and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Write $\mathfrak{L} = \mathbb{L} \oplus R[T]$. Let $J \subset R[T]$ be an ideal such that there is a surjection $\overline{\alpha} : \mathfrak{L} \twoheadrightarrow J/J^2$. Suppose that there is a surjection $\Gamma : \mathfrak{L} \otimes R(T) \twoheadrightarrow JR(T)$ such that Γ lifts $\alpha \otimes R(T)$. Then there is a surjective map $\beta : \mathfrak{L} \twoheadrightarrow J$ and $\theta \in SL(\mathfrak{L}/J\mathfrak{L})$ such that $\alpha\theta = \beta \otimes R[T]/J$.*

Remark 7.8. Let R be a ring of dimension 2 and $\mathbb{L}$ be a projective $R[T]$-module of rank one. Applying (7.7) one can easily extend [D 1, 7.2, 7.3]. The Euler class group $E(R[T], \mathbb{L})$ can be defined exactly as it has been done in this paper. The only difference is that, a local $\mathbb{L}$-orientation $\alpha : \mathfrak{L}/J\mathfrak{L} \twoheadrightarrow J/J^2$ will be called global if there is a surjection $\theta : \mathfrak{L} \twoheadrightarrow J$ and some $\sigma \in SL(\mathfrak{L}/J\mathfrak{L})$ such that $\alpha\sigma = \theta \otimes R[T]/J$. The Euler class of a projective $R[T]$-module P of rank 2, together with an isomorphism $\chi : \mathbb{L} \xrightarrow{\sim} \wedge^2(P)$ can also be defined as it has been done in Section 6. It can be easily checked that the Euler class $e(P, \chi)$ is trivial in $E(R[T], \mathbb{L})$ if and only if $P \simeq \mathbb{L} \oplus R[T]$. We leave all the details for the reader as no new technique is involved here. Only result that we could not extend from [D 1] is [D 1, 7.6]. Note that in the proof of [D 1, 7.6], the "Symplectic" cancellation theorem of Bhatwadekar [Bh 2, 4.8] is crucially used, which is not available in this case.

References

[B-M] J. Barge and F. Morel, Groupe de cycles orientés et classe d'Euler des fibrés vectoriels, *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000), 287-290.

[Ba] H. Bass, K-theory and stable algebra, *I.H.E.S.* **22** (1964), 5-60.

[Bh 1] S. M. Bhatwadekar, Inversion of monic polynomials and existence of unimodular elements (II), *Math. Z.* **200** (1989), 233-238.

[Bh 2] S. M. Bhatwadekar, Cancellation theorems for projective modules over a two dimensional ring and its polynomial extensions, *Compositio Math.* **128** (2001), 339-359.

[Bh-K] S. M. Bhatwadekar and Manoj Keshari, A Question of Nori: projective generation of ideals, *K-Theory* **28** (2003), 329-351.

[Bh-R 1] S. M. Bhatwadekar and Amit Roy, Some theorems about projective modules over polynomial rings, *J. Algebra* **86** (1984), 150-158.

[Bh-R 2] S. M. Bhatwadekar and Amit Roy, Some cancellation theorems about projective modules over polynomial rings, *J. Algebra* **111** (1987), 166-176.

[Bh-RS 1] S. M. Bhatwadekar and Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain and a question of Nori, *Invent. Math.* **133** (1998), 161-192.

[Bh-RS 2] S. M. Bhatwadekar and Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain (II), *K-Theory* **15** (1998), 293-300.

[Bh-RS 3] S. M. Bhatwadekar and Raja Sridharan, Euler class group of a Noetherian ring, Compositio Math. 122 (2000), 183-222.

[Bh-RS 4] S. M. Bhatwadekar and Raja Sridharan, On a question of Roitman, *J. Ramanujan Math. Soc.* **16** (2001), 45-61.

[Bh-RS 5] S. M. Bhatwadekar and Raja Sridharan, On Euler classes and stably free projective modules, in: Algebra, arithmetic and geometry, Part I, II(Mumbai, 2000), 139–158, Tata Inst. Fund. Res. Stud. Math., 16, Tata Inst. Fund. Res., Bombay, 2002.

[D 1] M. K. Das, The Euler class group of a polynomial algebra, *J. Algebra* **264** (2003), 582-612.

[D 2] M. K. Das, The Euler class group of a polynomial algebra II, *J. Algebra* **299** (2006), 94-114.

[D 3] M. K. Das, Revisiting Nori's question and homotopy invariance of Euler class groups, *J. K-Theory* **8** (2011), 451-480.

[D 4] M. K. Das, On triviality of the Euler class group of a deleted neighbourhood of a smooth local scheme, *Transactions of American Mathematical Society*, **365** (2013), 3397-3411.

[D-RS 1] M. K. Das and Raja Sridharan, Good invariants for bad ideals, *J. Algebra* **323** (2010), 3216-3229.

[D-RS 2] M. K. Das and Raja Sridharan, Euler class groups and a theorem of Roitman, *J. Pure and Applied Algebra* **215** (2011), 1340-1347.

[D-Z] M. K. Das and Md. Ali Zinna, On invariance of the Euler class group under a subintegral base change, *Preprint*.

[E-E] D. Eisenbud and E. G. Evans, Generating modules efficiently: Theorems from algebraic K-Theory, *J. Algebra* **27** (1973), 278-305.

[F] J. Fasel, Groupes de Chow-Witt, *Mém. Soc. Math. Fr. (N.S.)* **113** (2008): viii+197.

[F-Sr] J. Fasel and V. Srinivas, Chow-Witt groups and Grothendieck-Witt groups of regular schemes, *Adv. Math.* **221** (2009), 302-329.

[K] M. Keshari, Euler Class group of a Noetherian ring, M.Phil. thesis, available at : http://www.math.iitb.ac.in/˜keshari/acad.html

[La] T. Y. Lam, Serre conjecture, in: Lecture Notes in Mathematics, Vol. 635, Springer, Berlin, 1978.

[Li 1] H. Lindel, On the Bass-Quillen conjecture concerning projective modules over polynomial rings, *Invent. Math.* **65** (1981), 319-323.

[Li 2] H. Lindel, Unimodular elements in projective modules, *J. Algebra* **172** (1995), 301-319.

[M1] S. Mandal, On efficient generation of ideals, *Invent. Math.* **75** (1984), 59-67.

[M 2] S. Mandal, Homotopy of sections of projective modules, *J. Algebraic Geometry* **1** (1992), 639-646.

[M-RS] S. Mandal and Raja Sridharan, Euler classes and complete intersections, *J. Math. Kyoto Univ.* **36** (1996), 453-470.

[M-V] S. Mandal and P. L. N. Varma, On a question of Nori: the local case, *Communications in Algebra* **25** (1997), 451-457.

[M-Y 1] S. Mandal and Yong Yang, Intersection theory of algebraic obstructions, *J. Pure Appl. Algebra* **214** (2010), 2279-2293.

[M-Y 2] S. Mandal and Yong Yang, Excision in algebraic obstruction theory, *J. Pure Appl. Algebra* **216** (2012), 2159-2159.

[MK-M] N. Mohan Kumar and M. P. Murthy, Algebraic cycles and vector bundles over affine three-folds, *Ann. Math.* **116** (1982), 579-591.

[MK-M-R] N. Mohan Kumar, M. P. Murthy and Amit Roy, A cancellation theorem for projective modules over finitely generated rings, in: Hijikata H., et al. (Eds.), Algebraic geometry and commutative algebra in honor of Masayoshi Nagata, vol. 1, (1987), 281-287.

[Mo] F. Morel, $\mathbb{A}^1$-algebraic topology over a field, To appear in Lecture Notes in Mathematics. Preprint available at `http://www.mathematik.uni-muenchen.de/~morel/preprint.html`.

[Mu] M. P. Murthy, Zero cycles and projective modules, *Ann. Math.* **140** (1994), 405-434.

[N] B. S. Nashier, Efficient generation of ideals in polynomial rings, *J. Algebra* **85** (1983), 287-302.

[P] B. R. Plumstead, The Conjectures of Eisenbud and Evans, *American Journal of Mathematics* **Vol 105 No 6** (1983), 1417-1433.

[Ra] R. A. Rao, The Bass-Quillen conjecture in dimension three but characteristic $\neq 2,3$ via a question of A. Suslin, *Invent. Math.* **93** (1988), 609-618.

[Ro] M. Roitman, On projective modules over polynomial rings, *J. Algebra* **58** (1979), 51-63.

[Se] Serre, J-P, Sur les modules projectifs, *Semin. Dubreil-Pisot* **14** (1960-61).

[Su 1] A. A. Suslin, On the structure of the special linear group over polynomial rings, *Math. USSR-Izv* **11** (1977), 221-238.

[Su 2] A. A. Suslin, Cancellation over affine varieties, *J. Sov. Math.* **27** (1984), 2974-2980.

[Sw] Richard G. Swan, On seminormality, *J. Algebra* **67** (1980), 210-229.

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