EFFICIENT GENERATION OF IDEALS IN OVERRINGS OF POLYNOMIAL RINGS

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ABSTRACT. Let R be a commutative Noetherian ring. This paper examines the question of efficient generation of ideals in a birational overring of $R[X]$. Also, we study similar questions for rings of the type $R[X,Y]/(XY)$. These studies enable us to compute the Euler class groups for those two types of rings.

1. INTRODUCTION

Let k be a field and $f \in R = k[T_1, \cdots, T_d]$. Consider $A = R[X,Y]/(f - XY)$. In the case when f defines a nonsingular hypersurface in k^d, Murthy [Mu] proved that $K_0(A) \simeq K_0(R/fR)$ and $\text{Pic}(A)$ is trivial. Further, taking a suitable plane curve (namely, taking $f(T_1, T_2) = T_1^2 + T_2^3 - 1$), Murthy comments that $K_0(A)$ may not be even finitely generated. In such a situation, the triviality of $\text{Pic}(A)$ forces that a non-zero element of $\widetilde{K}_0(A)$ must be represented by a projective A-module P which is not a direct sum of rank one modules, in particular, P is not free. This is a very interesting phenomenon if we note that $k[T_1, \cdots, T_d, X] \subset A \subset k[T_1, \cdots, T_d, X, X^{-1}]$ and by [Q, Su1] projective modules defined over the ring on the left are free, whereas the same is true for projective modules defined over the ring on the right by [Su2].

In the case when f defines an integral (possibly singular) hypersurface in k^d, Swan, Weibel carried out detailed K-theoretic studies of these type of rings in [W].

Later, motivated by the works of Murthy, Swan and Weibel mentioned above, Bhatwadekar and Roy [B-R] considered a d-dimensional commutative Noetherian ring R and proved some remarkable results for projective modules of rank $d+1$ defined over a ring A where $R[X] \hookrightarrow A \hookrightarrow R[X, X^{-1}]$. For instance, they proved that a projective A-module of rank $d+1$ always splits as $P \simeq Q \oplus A$, where Q is an A-module. We may note that this result is consistent with similar results on projective modules of rank $d+1$ defined over the rings $R[X]$ and $R[X, X^{-1}]$, proved in [P] and [M], respectively.

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Very soon after [B-R] appeared, Rao [R] proved that the results of Bhatwadekar and Roy can be generalized by taking A as: $R[X] \hookrightarrow A \hookrightarrow S^{-1}R[X]$, where S is a multiplicative set of non-zerodivisors in $R[X]$.

Inspired by the results discussed above, we decided to investigate these rings from a different perspective, namely to study the behaviour of ideals in such rings. As a starting point, let us consider the following question. For a moment let $f \in R = k[T_1, \cdots, T_d]$, $A = R[X,Y]/(f - XY)$ and assume that f defines a nonsingular hypersurface in k^d. It can be easily argued (see [Mu]) to conclude that A is regular of dimension $d+1$. Let $\mathfrak{m}$ be a maximal ideal of A. We ask: *Is* $\mathfrak{m}$ *a complete intersection?* Note that $\operatorname{ht}(\mathfrak{m}) = d+1 = \mu(\mathfrak{m}/\mathfrak{m}^2)$. Therefore, we may as well ask, if $I \subset A$ is an ideal of height $d+1$ such that $\mu(I/I^2) = d+1$, is I generated by $d+1$ elements? One is tempted to go one step further and inquire whether a given set of $d+1$ generators of I/I^2 can be lifted to a set of $d+1$ generators of I. Taking cues from [B-R] and [R], we answer all these questions affirmatively in a much more general setup in the form of the following theorem (3.2 below).

Theorem 1.1. *Let R be a commutative Noetherian ring of dimension $d \geq 1$ and S be a multiplicative set of non-zerodivisors in $R[X]$. Let A be a ring such that $R[X] \hookrightarrow A \hookrightarrow S^{-1}R[X]$. Let $I \subset A$ be an ideal of height ≥ 2. Suppose that $I = (f_1, \cdots, f_n) + I^2$ where $n \geq d+1$. Then there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$ for $1 \leq i \leq n$.*

The above result has been proved for $A = R[X]$ and $R[X, X^{-1}]$ by Mandal in [M 1]. For $S^{-1}R[X]$, the result can be deduced from [M-P]. Our method of proof is quite simple and similar in principle to the methods adopted in [B-R] and [R], namely, to reduce the problem to one over $R[X]$ and then appeal to the existing results. An interesting consequence of Theorem 1.1 is that any ideal of A of height at least two is set theoretically generated by $d+1$ elements. We prove this result in Theorem 3.12.

A reader familiar with the Euler class theory will readily understand that Theorem 1.1 implies that the "top" Euler class group $E^{d+1}(A)$ of A is trivial. Consequently, (if $\mathbb{Q} \subset A$) any projective A-module P of rank $d+1$ with trivial determinant will have a trivial Euler class and therefore split as $P \simeq Q \oplus A$ for some A-module Q. This recovers [B-R, Theorem 4.2] of Bhatwadekar-Roy (indeed under some restrictions), where they proved that any projective A-module of rank $d+1$ splits off a free summand of rank one.

Now let us go back to the setup at the beginning of the introduction. Recall that we started our discussion with $f \in R = k[T_1, \cdots, T_d]$ and $A = R[X,Y]/(f - XY)$. Unlike the case when f defines a nonsingular hypersurface (when Murthy showed that there can be non-free projective A-modules), the projective A-modules are all free if we take

f to be zero (i.e., $R = k[T_1, \cdots, T_d]$ and $A = R[X,Y]/(XY)$). This was proved in [B-R, Theorem 5.1]. In a spirit similar to Theorem 1.1 we prove the following theorem (4.2 below) and derive similar consequences.

Theorem 1.2. *Let R be a Noetherian ring of dimension $d \geq 1$ and $D = R[X,Y]/(XY)$. Let $I \subset D$ be an ideal of height ≥ 2. Suppose that $I = (f_1, \cdots, f_n) + I^2$ where $n \geq d+1$. Then there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$ for $1 \leq i \leq n$.*

Again we remark that $\dim(D) = d+1$ and Theorem 1.2 implies that the Euler class group $E^{d+1}(D) = 0$. As a consequence, we observe that if $\mathbb{Q} \subset A$, any projective D-module P of rank $d+1$ (with trivial determinant) will have a trivial Euler class and therefore split as $P \simeq Q \oplus D$ for some D-module Q. The restriction on the determinant of P can be removed. Therefore we obtain (Corollary 4.6) an Euler class theoretic proof of [B-R, Theorem 5.3(ii)] with the assumption that $\mathbb{Q} \subset A$.

Interestingly, we have been able to carry out our analysis further for a ring D as above to detect the precise obstruction for a projective D-module of rank d to split off a free summand of rank one. Let us give a brief exposition.

Let P be a projective D-module of rank d and for simplicity assume that the determinant of P is trivial. Fix $\chi : D \xrightarrow{\sim} \wedge^d(P)$. By a theorem of Eisenbud-Evans [E-E], there exists a surjective map $\alpha : P \twoheadrightarrow J$, where $J \subset D$ is an ideal of height d. The map α will induce a surjection $\overline{\alpha} : P/JP \twoheadrightarrow J/J^2$. As $\dim(D/J) \leq 1$ and the determinant of P is trivial, the D/J-module P/JP is free. Choose an isomorphism $\sigma : (D/J)^d \xrightarrow{\sim} P/JP$ such that $\wedge^d(\sigma) = \chi \otimes D/J$ and take the composite surjection $\omega_J := \overline{\alpha}\sigma : (D/J)^d \xrightarrow{\sim} P/JP \twoheadrightarrow J/J^2$. We prove that if $\mathbb{Q} \subset D$, then $P \simeq Q \oplus D$ for some D-module Q if and only if ω_J can be lifted to a surjection $\theta : D^d \twoheadrightarrow J$.

The above phenomenon has been formalized in the form of the theory of the d-th Euler class group $E^d(D)$ of D and the Euler class of (P, χ) taking values in $E^d(D)$. The result can be found in Corollary 5.11 which asserts that $P \simeq Q \oplus D$ if and only if the Euler class of (P, χ) is trivial in $E^d(D)$. The preceding paragraph is an exposition of the same in simpler terms.

2. Preliminaries

Throughout this paper, rings are assumed to be commutative and Noetherian and all modules are assumed to be finitely generated. Projective modules are assumed to have constant rank. For a ring A, dim(A) will denote the Krull dimension of A. For an A-module M, the notation $\mu(M)$ stands for the minimal number of generators of M as an A-module.

Definition 2.1. An ideal I of a ring A is said to be *efficiently generated* if $\mu(I) = \mu(I/I^2)$.

Definition 2.2. Let A be a ring and $I \subset A$ be an ideal of height r. We shall call I to be a *complete intersection* if $\mu(I) = r$. We shall say that I is a *local complete intersection* if I is locally generated by r elements.

We now collect some results for later use. We start with a lemma from [D-K, Lemma 3.1].

Lemma 2.3. (Moving Lemma) *Let R be a Noetherian ring and $J \subset R$ be an ideal. Let P be a projective R-module of rank $n \geq dim(R/J) + 1$ and let $\alpha : P/JP \twoheadrightarrow J/J^2$ be a surjection. Given any ideal $K \subset R$ with $dim(R/K) \leq n - 1$, the map α can be lifted to a surjection $\beta : P \twoheadrightarrow J \cap J'$ such that:*

(1) $(J^2 \cap K) + J' = R$.
(2) $ht(J') \geq n$.

The next result, due to Bhatwadekar-Roy [B-R, Lemma 4.1], will be crucially used in Section 3.

Lemma 2.4. *Let $B \subset C$ be rings of dimension d and let x be an element of B such that $B_x = C_x$. Then*

(1) *$B/(1 + bx) = C/(1 + bx)$ for all $b \in B$.*
(2) *If J is an ideal of C such that $ht(J) \geq d$, and $J + xC = C$, then there exists an element $b \in B$ such that $1 + xb \in J$.*

The following proposition is proved in ([B-R, Proposition 3.2]).

Proposition 2.5. *Let B and C be reduced rings such that $B \hookrightarrow C \hookrightarrow Q(B)$, where $Q(B)$ denotes the total quotient ring of B. Assume that B is Noetherian. Then $dim(C) \leq dim(B)$.*

The following theorem is due to Mandal [M 2, Theorem 2.1].

Theorem 2.6. *Let A be a Noetherian ring. Let $I \subset A[T]$ be an ideal containing a monic polynomial. Suppose that $I = (f_1, \cdots, f_n) + (I^2T)$, where $n \geq dim(A[T]/I) + 2$. Then, there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ and $g_i - f_i \in (I^2T)$ for $i = 1, \cdots, n$.*

The following result is due to Das ([D 1, Theorem 3.10]).

Theorem 2.7. *Let R be a Noetherian ring of dimension $d \geq 3$, containing the field of rationals. Let $I \subset R[T]$ be an ideal of height d. Suppose that $I = (f_1, \cdots, f_d) + (I^2T)$ and there exist $F_1, \cdots, F_d \in IR[T]_f$ such that $IR[T]_f = (F_1, \cdots, F_d)$ and $F_i = f_i$ mod $I^2R[T]_f$ for some monic polynomial $f \in R[T]$. Then, there exist $g_1, \cdots, g_d$ such that $I = (g_1, \cdots, g_d)$ and $g_i = f_i$ mod (I^2T) for $i = 1, \cdots, d$.*

We shall frequently use the following lemma from [B-RS 1, Remark 3.9].

Lemma 2.8. *Let R be a ring, $I \subset R[T]$ be an ideal such that $I = (f_1, \cdots, f_n) + I^2$. Assume further that either $I(0) = R$ or $I(0) = (a_1, \cdots, a_n)$ such that $f_i(0) - a_i \in I(0)^2$. Then we can find $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n) + (I^2T)$ with $g_i - f_i \in I^2$ and $g_i(0) = a_i$ for $i = 1, \cdots, n$.*

The above lemma was generalized slightly in [D 1, Lemma 4.9]. In this paper we shall also need a more general version, which we state below. A proof can be easily given by mimicking the proofs from [B-RS 1, D 1]. Note that taking $A = R[T]$, $x = T$, and $P = R[T]^n$ in the following lemma one obtains Lemma 2.8.

Lemma 2.9. *Let A be a ring and I, J be two ideals in A such that $J \subset I^2$. Let P be a projective A-module and $x \in A$. Suppose that we are given surjections $\alpha : P \twoheadrightarrow I/J$ and $\beta : P \twoheadrightarrow \overline{I}$ such that $\alpha \equiv \beta$ mod $\overline{J}$, where bar denotes reduction modulo x. Then the map α can be lifted to surjection $\phi : P \twoheadrightarrow I/(Jx)$.*

The following proposition is implicit in the proof of [B-RS 2, Proposition 2.5].

Proposition 2.10. *Let R be a commutative Noetherian ring of dimension $d \geq 1$ and I be an ideal of $R[T]$ of height ≥ 2. Assume that $I = (f_1, \cdots, f_n) + I^2$, where $n \geq d + 1$. Then there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$ for $i = 1, \cdots, n$.*

Definition 2.11. Let R be a ring and P be a projective R-module. An element $p \in P$ is called *unimodular* if there is a surjective R-linear map $\varphi : P \twoheadrightarrow R$ such that $\varphi(p) = 1$. Note that P has a unimodular element if and only if $P \simeq Q \oplus R$ for some R-module Q.

The following result is due to Serre [Se].

Theorem 2.12. *Let R be a ring and P be a projective R-module. If rank$(P) >$ dim(R), then $P \simeq Q \oplus R$ for some R-module Q.*

Let A be a ring of dimension n. We record the definition of the n-th Euler class group $E^n(A)$ below for the convenience of the reader. The following is a simpler form of the definition given in [B-RS 3].

Definition 2.13. (**The n-th Euler class group $E^n(A)$**) Let A be a commutative Noetherian ring of dimension $n \geq 2$. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Two surjections α, β from $(A/J)^n$ to J/J^2 are said to be related if there exists $\sigma \in SL_n(A/J)$ such that $\alpha\sigma = \beta$. Clearly this is an equivalence relation on the set of surjections from $(A/J)^n$ to J/J^2. Let $[\alpha]$ denote the equivalence class of α. Such an equivalence class $[\alpha]$ is called a *local orientation* of J. By abuse of notation, we shall identify an equivalence class $[\alpha]$ with α. A local orientation α is called a *global orientation* if $\alpha : (A/J)^n \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : A^n \twoheadrightarrow J$. Let G be the free abelian group on the set of pairs $(\mathcal{N}, \omega_{\mathcal{N}})$ where $\mathcal{N}$ is an $\mathcal{M}$-primary ideal for some maximal ideal $\mathcal{M}$ of height n such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements and $\omega_{\mathcal{N}}$ is a local orientation of $\mathcal{N}$. Now let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements and ω_J be a local orientation of J. Let $J = \cap_i \mathcal{N}_i$ be the (irredundant) primary decomposition of J. We associate to the pair (J, ω_J), the element $\sum_i (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ of G where $\omega_{\mathcal{N}_i}$ is the local orientation of $\mathcal{N}_i$ induced by ω_J. By abuse of notation, we denote $\sum_i (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ by (J, ω_J). Let H be the subgroup of G generated by the set of pairs (J, ω_J), where J is an ideal of height n and ω_J is a global orientation of J. The n-th Euler class group of A is $E^n(A) \stackrel{\text{def}}{=} G/H$.

Definition 2.14. (**The Euler class of a projective A-module of top rank**) Let P be a projective A-module of rank n such that $A \simeq \wedge^n(P)$ and let $\chi : A \xrightarrow{\sim} \wedge^n P$ be an isomorphism. Let $\varphi : P \twoheadrightarrow J$ be a surjection where J is an ideal of height n. Therefore we obtain an induced surjection $\overline{\varphi} : P/JP \twoheadrightarrow J/J^2$. Let $\overline{\gamma} : (A/J)^n \simeq P/JP$, be an isomorphism such that $\wedge^n(\overline{\gamma}) = \overline{\chi}$. Let ω_J be the local orientation of J given by $\overline{\varphi}\,\overline{\gamma} : (A/J)^n \twoheadrightarrow J/J^2$. Let $e(P, \chi)$ be the image in $E^n(A)$ of the element (J, ω_J) of G. If $\mathbb{Q} \subset A$, the assignment sending the pair (P, χ) to the element $e(P, \chi)$ of $E^n(A)$ is well defined (see [B-RS 3]). The *Euler class* of (P, χ) is defined to be $e(P, \chi)$.

Theorem 2.15. [B-RS 3] *Let A be a commutative Noetherian ring of dimension $n \geq 2$.*

(1) *An element (I, ω_I) is zero in $E^n(A)$ if and only if ω_I is a global orientation.*
(2) *Let P be a projective A-module of rank n together with an isomorphism $\chi : A \xrightarrow{\sim} \wedge^n P$. Then $e(P, \chi) = 0$ in $E^n(A)$ if and only if P has a unimodular element.*

Remark 2.16. The Euler class group $E^n(A)$ captures the projective A-modules of rank n with *trivial determinant*. This restriction is removed in [B-RS 3] by defining $E^n(A, L)$ where L is a projective A-module of rank one (the Euler class of a projective module with determinant L takes values in this group).

Remark 2.17. Let $\dim(A) = n$. Then it can be easily argued using a theorem of Mandal [M 1] that the top Euler class group $E^{n+1}(A[T])$ is trivial. The n-th Euler class groups of $A[T]$ (quite different from the n-th Euler class group of A) have been defined and studied in [D 1, D-Z].

3. Birational overrings of $R[X]$

Let R be a Noetherian ring. A *birational overring* of $R[X]$ is a ring A such that A contains $R[X]$ as a subring and is contained in the total quotient ring of $R[X]$.

Remark 3.1. In this section we are considering ideals of A and modules defined over A. For our purposes, it can be easily deduced that we can take A to be Noetherian and of the form $R[X] \hookrightarrow A \hookrightarrow R[X, \frac{1}{g}]$, where $g \in R[X]$ is a non-zerodivisor. Further, if necessary, R and A can be taken to be reduced. It follows from Proposition 2.5 that dim$(A) \leq$ dim$(R[X])$. In all the questions addressed in this section, the reader will observe that in the case when dim$(A) <$ dim$(R[X])$, the results are either well-known or can be easily proved. Therefore, we shall always assume that dim$(A) =$ dim$(R[X])$. Further, it is assumed throughout this section that $A \neq R[X]$ and $A \neq R[X, \frac{1}{g}]$.

3.1. **Efficient generation of ideals and the triviality of the 'top' Euler class group.** We start with the main theorem of this section.

Theorem 3.2. *Let R be a ring with dim$(R) = d \geq 1$ and A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, \frac{1}{g}]$, where $g \in R[X]$ is a non-zerodivisor. Let $I \subset A$ be an ideal of height ≥ 2. Suppose that $I = (f_1, \cdots, f_n) + I^2$ where $n \geq d+1$. Then there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$ for $1 \leq i \leq n$.*

Proof. As remarked above, we assume that dim$(A) = d+1$. If $n \geq d+2$, then the theorem can be easily proved using some standard general position argument. Therefore we further assume that $n = d+1$.

Since $I = (f_1, \cdots, f_n) + I^2$, applying the moving lemma (2.3), we can find $h_1, \cdots, h_n \in I$ and an ideal $I' \subset A$ such that

(1) $I \cap I' = (h_1, \cdots, h_n)$ with $f_i - h_i \in I^2$.
(2) $I + I' = A$.
(3) $I' + gA = A$.
(4) ht$(I') \geq n$.

Now, if ht$(I') > n$, then $I' = A$, and hence by (1), $I = (h_1, \cdots, h_n)$ with $f_i - h_i \in I^2$ and we are done. Therefore we assume that ht$(I') = n$.

We have $R[X]_g = A_g = R[X, \frac{1}{g}]$ and ht$(I') = n$. Also from (3) we have $I' + gA = A$. By Lemma 2.4, there exists $f \in R[X]$ such that $1 + gf \in I'$ and $R[X]/(1+gf) = A/(1+gf)$.

Let $J = I' \cap R[X]$. Then as $f, g \in R[X]$, $1 + gf \in J$.

Therefore, using [N, Proposition 1.3] , we have

(1) $R[X]/J \simeq A/I'$.
(2) $JA = I'$.
(3) $J/J^2 \simeq I'/I'^2$.

Therefore, corresponding to $h_1, \cdots, h_n$ we have a set of generators of J/J^2, say, given by

$$J = (\alpha_1, \cdots, \alpha_n) + J^2,$$

and we proceed to prove that there exist $\beta_1, \cdots, \beta_n$ in J such that $J = (\beta_1, \cdots, \beta_n)$ with $\alpha_i - \beta_i \in J^2$.

Here we note that $J \subset R[X]$ with $\text{ht}(J) \geq 2$ and $n \geq d+1 = \dim(R)+1$. Therefore, Proposition 2.10 ensures that we indeed can find $\beta_1, \cdots, \beta_n \in J$ such that $J = (\beta_1, \cdots, \beta_n)$ with $\alpha_i - \beta_i \in J^2$.

As $JA = I'$, we can now use the subtraction principle [D-RS, Proposition 2.2] to find $g_1, \cdots, g_n \in I$ with $I = (g_1, \cdots, g_n)$ such that $f_i - g_i \in I^2$. □

Modifying the above proof, it is easy to establish the following theorem.

Theorem 3.3. *Let R be a ring with $\dim(R) = d \geq 1$ and A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, \frac{1}{g}]$, where $g \in R[X]$ is a non-zerodivisor. Let L be any rank one projective A-module which is extended from $R[X]$ and $I \subset A$ be an ideal of height $n \geq 3$. Suppose that there is a surjection $\alpha : L/IL \oplus (A/I)^{n-1} \twoheadrightarrow I/I^2$ where $n \geq d+1$. Then α can be lifted to a surjection $\beta : L \oplus A^{n-1} \twoheadrightarrow I$.*

Now we deduce some consequences of Theorem 3.3.

Corollary 3.4. *Let R be a ring of dimension $d \geq 1$. Let A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, \frac{1}{g}]$, where $g \in R[X]$ is a non-zerodivisor. Let L be any rank one projective A-module which is extended from $R[X]$. Then,*

(1) *The $(d+1)$-th Euler class group $E^{d+1}(A, L)$ is trivial.*
(2) *The weak Euler class group $E_0^{d+1}(A, L)$ is trivial.*

Proof. The first assertion follows from Theorem 3.3 and the definition of the Euler class group as given in [B-RS 3]. Since $E_0^{d+1}(A, L)$ is surjective image of the Euler class group $E^{d+1}(A, L)$ (see [B-RS 3]), the second assertion follows trivially from the first. □

Remark 3.5. In Theorem 3.3 and Corollary 3.4 above we had to put the restriction that the rank one projective A-module L is extended from $R[X]$. To see that there could be rank one projective A-modules which are not extended from $R[X]$, take $R = k[T]$, where k is a field and let $f \in k[T]$ be any nonconstant reducible polynomial which is not associated to a power of an irreducible in $k[T]$. Take $A = k[T, X, Y]/(f - XY)$. Then $\text{Pic}(R[X])$ is trivial but by [B-R, Theorem 7.1], $\text{Pic}(A)$ is not so. We do not know how to remove the restriction in (3.3, 3.4) in the general setup. However, there are important class of rings where this can be done. We prove a couple of corollaries below to illustrate this point.

Corollary 3.6. *Let R be a normal domain of dimension $d \geq 1$ and $b(\neq 0) \in R$ be such that R/bR is also a normal domain. Let $A = R[X, Y]/(XY - b)$ and L be any rank one projective A-module. Then the $(d+1)$-th Euler class group $E^{d+1}(A, L)$ is trivial.*

Proof. Let x be the image of X in A. Then we have

(1) $A[1/x] = R[X, X^{-1}]$.
(2) $A/xA = (R/bR)[Y]$.

Now it follows from (1) and (2), respectively, that $A[1/x]$ and A/xA are normal and hence, so is A. Also A is an integral domain.

Now $R[X]$ is a normal domain and by Nagata's theorem (see [Sa]) it follows that $\text{Pic}(R[X]) \simeq \text{Pic}(R[X, X^{-1}])$. Again, as x is a prime element in A, it follows from Nagata's theorem that $\text{Pic}(A) \simeq \text{Pic}(A[1/x]) = \text{Pic}(R[X, X^{-1})$. Therefore, $\text{Pic}(R[X]) \simeq \text{Pic}(A)$, which means that any projective A-module of rank one is extended from $R[X]$.

Since $R[X] \hookrightarrow A \hookrightarrow R[X, X^{-1}]$, the corollary now follows from Corollary 3.4. □

Corollary 3.7. *Let R be a UFD of dimension $d \geq 1$. Let A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, \frac{1}{g}]$ and assume that g is a prime in A. Let L be any rank one projective A-module. Then the $(d+1)$-th Euler class group $E^{d+1}(A, L)$ is trivial.*

Proof. We have $A[1/g] = R[X, \frac{1}{g}]$. Therefore $A[1/g]$ is a UFD. As g is a prime, by Nagata's criterion for UFD, it follows that A is a UFD. Then applying Nagata's theorem we have $\text{Pic}(R[X]) \simeq \text{Pic}(A)$. And again we are done by Corollary 3.4. □

Theorems proved by Bhatwadekar-Roy [B-R, Theorem 4.2] and Rao [R] can now be deduced (for any projective A-module with extended determinant), with the assumption that $\mathbb{Q} \subset A$, in the following form.

Corollary 3.8. *Let R be a ring of dimension $d \geq 1$, containing $\mathbb{Q}$ and A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, \frac{1}{g}]$, where $g \in R[X]$ is a non-zerodivisor. Let P be a projective A-module of determinant L which is extended from $R[X]$ with rank$(P) = d+1$. Then P has a unimodular element.*

Proof. We may assume that $\dim(A) = \dim(R) + 1 \geq 2$. If $\mathbb{Q} \subset A$, then the Euler class of P is defined and hence the result follows from Corollary 3.4 and [B-RS 3, Corollary 4.4]. □

Let R be a ring and I be an ideal of $R[X]$ such that I contains a monic polynomial. Let $\mu(I/I^2) = n \geq \dim(R[X]/I) + 2$. Then Mandal [M 1] proved that any set of n generators of I/I^2 can be lifted to a set of n generators of I. In the same paper, a similar result was proved for ideals in $R[X, X^{-1}]$ which contain a doubly monic polynomial (i.e., a polynomial which is monic in X as well as X^{-1}). Motivated by these two results we prove the following theorem. Recall that a polynomial in $R[X]$ is called a *special monic* if the coefficient of the highest degree term and the constant term are both 1.

Theorem 3.9. *Let R, A be rings such that $R[X] \hookrightarrow A \hookrightarrow R[X, X^{-1}]$. Let $I \subset A$ be an ideal containing a special monic polynomial. Suppose that $I = (f_1, \cdots, f_n) + I^2$, where $n \geq dim(A/I) + 2$. Then $I = (g_1, \cdots, g_n)$ such that $f_i - g_i \in I^2$ for $i = 1, \cdots, n$.*

Proof. Let $h(X) \in I$ be a special monic polynomial. Therefore $h(X)$ can be written in the form $1 + Xf(X)$, where $f(X) \in R[X]$. Let $J = I \cap R[X]$. Then $h(X) = 1 + Xf(X) \in J$. Since $\dim(R[X]) = \dim(A)$ and $R[X]_X = A_X$, applying Lemma 2.4 we have $R[X]/(1 + Xf) = A/(1 + Xf)$.

Therefore using [N, Proposition 1.3] , we have

(1) $R[X]/J \simeq A/I$.
(2) $JA = I$.
(3) $J/J^2 \simeq I/I^2$.

Recall that we have $I = (f_1, \cdots, f_n) + I^2$ and $J/J^2 \simeq I/I^2$. Therefore corresponding to $f_1, \cdots, f_n$ we have a set of generators of J/J^2 say, give by

$$J = (h_1, \cdots, h_n) + J^2,$$

and it is enough to prove that $J = (k_1, \cdots, k_n)$ with $h_i - k_i \in J^2$. Now J contains a monic polynomial, namely, $h(X)$. By using [M 1, Theorem 1.2], we can find $k_1, \cdots, k_n \in J$ such that $J = (k_1, \cdots, k_n)$ with $h_i - k_i \in J^2$. □

3.2. **Set-theoretic generation of ideals.** As an application of our main theorem in this section, we can easily prove some results on set-theoretic generation of ideals in the ring A. As preparation we need to recall the definition of a reduction of an ideal and a result due to Katz ([Ka]).

Definition 3.10. Let R be a ring. Let $J \subseteq I$ be ideals. J is said to be a *reduction* of I if there exists a non-negative integer m such that $I^{m+1} = JI^m$.

Theorem 3.11. [Ka] *Let R be a Noetherian ring and $I \subset R$ an ideal. Let d be the maximum of the heights of maximal ideals containing I, and suppose that $d < \infty$. Then some power of I admits a reduction J satisfying $\mu(J/J^2) \leq d$.*

Theorem 3.12. *Let R be a ring of dimension $d \geq 1$ and A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, \frac{1}{g}]$. Let $I \subset A$ be an ideal of height ≥ 2. Then I is set theoretically generated by $d + 1$ elements.*

Proof. It follows from the above theorem of Katz that for some k, the ideal I^k has a reduction J such that $\mu(J/J^2) \leq d+1$. If $\mu(J/J^2) \leq d$, then clearly J is generated by at most $d+1$ elements. Therefore we assume that $\mu(J/J^2) = d+1$. Since J is a reduction of I^k, it is easy to see that $\sqrt{I} = \sqrt{I^k} = \sqrt{J}$ and $\text{ht}(I) = \text{ht}(J)$. Then applying Theorem 3.2 we see that J is generated by $d + 1$ elements. Therefore, I is set-theoretically generated by $d + 1$ elements. □

Corollary 3.13. *Let R be a Cohen-Macaulay ring of dimension d and $f \in R$ be such that R/fR is also Cohen-Macaulay. Let $A = R[X,Y]/(f - XY)$. Then any ideal I of height $d+1$ is a set-theoretic complete intersection. If both R and R/fR are regular rings, then any maximal ideal $\mathfrak{m}$ of A (and therefore any reduced ideal of height $d+1$) is a complete intersection.*

Proof. We only remark that if both R and R/fR are Cohen-Macaulay, then so is the ring A. Similarly, if both R and R/fR are regular, then A is also regular. The rest of the corollary now follows from Theorem 3.12. □

The following corollary is a derivative of Theorem 3.9.

Corollary 3.14. *Let R be a ring of dimension d and A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, X^{-1}]$. Let I be a locally complete intersection ideal in A and I contains a special monic polynomial. Then, we have the following assertions.*

(1) *If dim$(A/I) \leq 1$ and ht$(I) = r$, then I is set theoretically generated by r elements.*
(2) *If dim$(A/I) > 1$, then I is set theoretically generated by d elements.*

Proof. (1) Let $J = I \cap R[X]$. Since I contains a special monic polynomial, therefore we have (i) $R[X]/J \simeq A/I$, (ii) $JA = I$ and (iii) $J/J^2 \simeq I/I^2$.

From (i), (ii) and (iii), we have $\dim(R[X]/J) = \dim(A/I) \leq 1$ and J is a locally complete intersection ideal. Also $\text{ht}(I) = \text{ht}(J) = r$.

Since J contains a special monic polynomial, a theorem of Mandal [M 5, Theorem 1.1] ensures that J is set theoretically generated by r elements. Therefore, I is set theoretically generated by r elements and the result follows.

(2) Since $\dim(R[X]/J) = \dim(A/I) > 1$, it follows from ([M 4, Theorem 6.2.6] or [M 5, Theorem 1.4]) that J is set theoretically generated by d elements. Therefore, I is set theoretically generated by d elements. □

3.3. **Polynomial extensions of A.** We end this section by extending Theorem 3.2 to the polynomial ring $A[T]$, and deriving some consequences.

Theorem 3.15. *Let R be a Noetherian ring of dimension $d \geq 2$ containing $\mathbb{Q}$ and A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, \frac{1}{g}]$, where $g \in R[X]$ is a non-zerodivisor. Let $I \subset A[T]$ be an ideal of height ≥ 3. Suppose that $I = (f_1, \cdots, f_n) + I^2$ where $n \geq d+1$. Then $I = (g_1, \cdots, g_n)$ such that $f_i - g_i \in I^2$ for $i = 1, \cdots, n$.*

Proof. Without loss of generality we may assume that $n = d+1 = \dim(A)$. Let ht$(I) = r$. Since R contains $\mathbb{Q}$, by [B-RS 1, Lemma 3.3], we can find some $\lambda \in \mathbb{Q}$ such that $I(\lambda) = A$ or $I(\lambda)$ is an ideal of A of height r. If necessary, we can replace T by $T - \lambda$ and assume that either $I(0) = A$ or $\text{ht}(I(0)) = r$.

If $I(0) = A$, by Lemma 2.8, we can lift $f_1, \cdots, f_n$ to a set of n generators of $I/(I^2T)$, say, $I = (l_1, \cdots, l_n) + (I^2T)$ with $f_i - l_i \in I^2$ for $i = 1, \cdots, n$.

Now assume that $\text{ht}(I(0)) = r$. We have $I(0) = (f_1(0), \cdots, f_n(0)) + I(0)^2$. By Theorem 3.2, there exist $a_1, \cdots, a_n \in I(0)$ such that $I(0) = (a_1, \cdots, a_n)$ with $f_i(0) - a_i \in I(0)^2$. Therefore, again by Lemma 2.8, we can lift $f_1, \cdots, f_n$ to a set of n generators of $I/(I^2T)$, say, $I = (l_1, \cdots, l_n) + (I^2T)$ with $f_i - l_i \in I^2$ for $i = 1, \cdots, n$.

Therefore, in any case, we can lift the given set of n generators of I/I^2 to a set of n generators of $I/(I^2T)$.

Let S be the multiplicatively closed set of all monic polynomials in $R[T]$. Write $R(T) = S^{-1}R[T]$.

Then we have $\dim(R(T)) = \dim(R) = d$ and $R(T)[X] \hookrightarrow S^{-1}A[T] \hookrightarrow R(T)[X, \frac{1}{g}]$. It follows from Proposition 2.5 that $\dim(S^{-1}A[T]) \leq d + 1$. Clearly, the case when $\dim(S^{-1}A[T]) \leq d$ can be ignored. Now we move to the ring $S^{-1}A[T]$. Applying Theorem 3.2, we get $IS^{-1}A[T] = (k_1, \cdots, k_n)$ such that $f_i - k_i \in I^2S^{-1}A[T]$.

Now we can apply Theorem [D-RS, Theorem 4.2] and obtain the desired set of generators for I. $\square$

Corollary 3.16. *Let R be a ring of dimension $d \geq 2$ containing $\mathbb{Q}$ and let A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, \frac{1}{g}]$. Then $E^{d+1}(A[T]) = 0$ and $E_0^{d+1}(A[T]) = 0$*

Proof. Since there is a surjective map $E^{d+1}(A[T]) \twoheadrightarrow E_0^{d+1}(A[T])$ (see [D 1]), we only need to prove $E^{d+1}(A[T]) = 0$, which follows from Theorem 3.15 and the definition of the Euler class group as given in [D 1]. $\square$

Corollary 3.17. *Let R be a ring of dimension $d \geq 2$, containing $\mathbb{Q}$ and A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, \frac{1}{g}]$. Let P be a projective $A[T]$-module of trivial determinant with rank$(P) = d + 1$. Then P contains a unimodular element.*

Proof. By Proposition 2.5, $\dim(A) \leq \dim(R) + 1 = d + 1$. In view of Theorem 2.12, the only relevant case here is when $\dim(A) = \dim(R) + 1 = d + 1 \geq 3$. Since $\mathbb{Q} \subset A$, the Euler class of P is defined and hence the result follows from Corollary 3.16 and [D 1, Corollary 4.11]. $\square$

Corollary 3.18. *Let R be a ring of dimension $d \geq 2$ containing $\mathbb{Q}$ and A be a ring such that $R[X] \hookrightarrow A \hookrightarrow R[X, \frac{1}{g}]$. Let $I \subset A[T]$ be a local complete intersection ideal of height ≥ 3 such that dim$(A[T]/I) \leq 1$. Then I is set theoretically generated by $d + 1$ or less elements.*

Proof. Let $\text{ht}(I) = r$. It follows from the Ferrand-Szpiro construction (see [M 4, Theorem 6.1.3]) that there exists an ideal $J \subset A[T]$ such that

(1) $\sqrt{I} = \sqrt{J}$.
(2) J/J^2 is a free $A[T]/J$-module of rank r.

If $r \leq d$, then J is generated by at most $d+1$ elements. So, take $r = d+1$. Let $J = (f_1, \cdots, f_{d+1}) + J^2$. Applying Theorem 3.15, we see that J is generated by $d+1$ elements. From (1), it follows that I is set-theoretically generated by $d+1$ elements. □

4. Rings of the type $R[X,Y]/(XY)$

Let R be a Noetherian ring and let D denote the ring $R[X,Y]/(XY) = R[x,y]$, where x, y are the images of X, Y in $R[X,Y]/(XY)$, respectively. Note that D/yD is the polynomial ring $R[X]$ and D/xD is the polynomial ring $R[Y]$, whereas $D/(x,y)D$ is R.

We now set up some notations which we intend to use for the rest of this paper.

Notation. Let I be an ideal of $D = R[x,y]$ and P be a projective D-module. In general, we shall use 'bar' when we move modulo (y) and 'tilde' when we move modulo (x). For instance, we have:

(1) $\overline{I} = I/I \cap (y)$, an ideal of $R[X]$,
(2) $\widetilde{I} = I/I \cap (x)$, an ideal of $R[Y]$,
(3) $\overline{\overline{I}} = I/(I \cap (x,y))$, an ideal of R.

Similarly, if $\alpha : P \to I$ is a map then $\overline{\alpha}$ denotes the induced map $P/yP \to \overline{I}$ and $\widetilde{\alpha}$ denotes the induced map $P/xP \to \widetilde{I}$.

Remark 4.1. The following remarks are in order for subsequent discussions.

(1) Since X is a zero-divisor in D, the canonical map $D \longrightarrow R[X, X^{-1}]$ is not injective whereas $R[X] \longrightarrow D$ is injective. This is an important difference between D and the rings A considered in the previous section.
(2) Let $I \subset D$ be an ideal. Then, either $\overline{I} = R[X]$ or ht $(\overline{I})$ = ht (I). Same for $\widetilde{I}$.
(3) If $I \subset D$ is a proper ideal, then both $\overline{I} = R[X]$ and $\widetilde{I} = R[Y]$ cannot hold simultaneously.

Most of our arguments in the rest of this paper depend crucially on the following two theorems. Theorem 4.2 is the 'free' version of Theorem 4.3. Essentially the same idea is used to prove these results but we decided to treat the free case separately as it will be more comprehensible to the reader.

Theorem 4.2. *Let R be a Noetherian ring of dimension d and $D = R[X,Y]/(XY)$. Let I be an ideal in D. Suppose $I = (f_1, \cdots, f_n) + I^2$. Then $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$ in each of the following cases:*

(1) *$ht(I) = d+1 = n \geq 2$.*
(2) *$ht(I) = d = n \geq 3$ and $\overline{I} = (u_1, \cdots, u_d)$ such that $u_i - \overline{f_i} \in \overline{I}^2$ and $\widetilde{I} = (v_1, \cdots, v_d)$ such that $v_i - \widetilde{f_i} \in \widetilde{I}^2$. We also need the additional assumption that $\mathbb{Q} \subset R$.*

Proof. *Case 1:* Note that $\text{ht}(\overline{I}) \geq d+1 = n$. We first go modulo x, i.e., to the ring $R[Y]$ and consider the equation $\widetilde{I} = (\widetilde{f}_1, \cdots, \widetilde{f}_n) + \widetilde{I}^2$ in $R[Y]$. Since $n \geq \dim(R)+1$, applying Proposition 2.10 we can find $v_1, \cdots, v_n$ such that $\widetilde{I} = (v_1, \cdots, v_n)$ with $\widetilde{f}_i - v_i \in \widetilde{I}^2$. Therefore, it follows from Lemma 2.9 that there exist $\lambda_1, \cdots, \lambda_n$ such that $I = (\lambda_1, \cdots, \lambda_n) + (I^2x)$ with $f_i - \lambda_i \in I^2$.

We now go modulo y, i.e., to the ring $R[X]$ and consider the equation

$$\overline{I} = (\overline{\lambda}_1, \cdots, \overline{\lambda}_n) + (\overline{I}^2 X)$$

(Note that x becomes X in $R[X]$.) Since $\text{ht}(\overline{I}) \geq d+1$, therefore $\overline{I}$ contains a monic polynomial in X. Also $n \geq 2 = \dim(R[X]/\overline{I}) + 2$. By Theorem 2.6, there exist $\delta_1, \cdots, \delta_n$ such that $\overline{I} = (\delta_1, \cdots, \delta_n)$ with $\overline{\lambda}_i - \delta_i \in (\overline{I}^2 X)$.

Therefore, it follows from Lemma 2.9 that there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n) + (I^2xy)$ with $\lambda_i - g_i \in (I^2x)$. As $xy = 0$ in $D = R[x, y]$, we obtain $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$.

Case 2: As $\widetilde{I} = (v_1, \cdots, v_n)$ with $v_i - \widetilde{f}_i \in \widetilde{I}^2$, by Lemma 2.9 there exist $\lambda_1, \cdots, \lambda_n$ such that $I = (\lambda_1, \cdots, \lambda_n) + (I^2x)$ with $f_i - \lambda_i \in I^2$.

Now we go modulo y, i.e., to the ring $R[X]$. Consider the equation

$$\overline{I} = (\overline{\lambda}_1, \cdots, \overline{\lambda}_n) + (\overline{I}^2 X)$$

We know that $\overline{I} = (u_1, \cdots, u_n)$ such that $u_i - \overline{f_i} \in \overline{I}^2$. Now we move to the ring $R(X)$, where $R(X)$ denotes the ring obtained from $R[X]$ by inverting all monic polynomials. Then we have $\overline{\lambda}_i - u_i \in \overline{I}^2 R(X)$. Using Theorem 2.7, it follows that there exist $\delta_1, \cdots, \delta_n$ such that $\overline{I} = (\delta_1, \cdots, \delta_n)$ with $\overline{\lambda}_i - \delta_i \in (\overline{I}^2 X)$.

Again applying Lemma 2.9, we can find $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n) + (I^2xy)$ with $\lambda_i - g_i \in (I^2x)$. As $xy = 0$ in D, we have $I = (g_1, \cdots, g_n)$ with $f_i - g_i \in I^2$. This completes the proof. $\square$

Theorem 4.3. *Let R be a Noetherian ring of dimension d and $D = R[X,Y]/(XY)$. Let $I \subset D$ be an ideal and L be a projective D-module of rank one which is extended from $R[X]$. Write $P = L \oplus D^{n-1}$. Let $\alpha : P \twoheadrightarrow I/I^2$ be a surjection. Then there exists a surjection $\beta : P \twoheadrightarrow I$ that lifts α in each of the following cases:*

(1) *$ht(I) = n = d+1$.*
(2) *$\mathbb{Q} \subset R$, $ht(I) = n = d \geq 3$ and it is given that the induced maps $\overline{\alpha} : P/yP \twoheadrightarrow \overline{I}/\overline{I}^2$, $\widetilde{\alpha} : P/xP \twoheadrightarrow \widetilde{I}/\widetilde{I}^2$ can be lifted to surjections $\gamma : P/yP \twoheadrightarrow \overline{I}$ and $\delta : P/xP \twoheadrightarrow \widetilde{I}$, respectively.*

Proof. *Case 1:* Let $\text{ht}(I) = \text{rk}(P) = n = d+1$. Then note that $\text{ht}(\overline{I}) \geq n$.

We consider $\widetilde{\alpha} : P/xP \twoheadrightarrow \widetilde{I}/\widetilde{I}^2$ in $R[Y]$. Since $n = \dim(R) + 1$, by [D-Z, Proposition 2.13], there exists a surjection $\phi : P/xP \twoheadrightarrow \widetilde{I}$ which is a lift of $\widetilde{\alpha}$. Then by Lemma 2.9, we can lift α to a surjection $\psi : P \twoheadrightarrow I/(I^2x)$.

We go modulo y, i.e., to the ring $R[X]$ and consider $\overline{\psi} : P/yP \twoheadrightarrow \overline{I}/(\overline{I}^2X)$.

Now, as L is extended from $R[X]$, it follows from the proof of [B-R, Theorem 5.1] that L/yL and hence P/yP is extended from R. Therefore, using a theorem of Mandal [M 2, Theorem 2.1], we can find $\theta : P/yP \twoheadrightarrow \overline{I}$ which lifts $\overline{\psi}$. Therefore, it follows from Lemma 2.9 that we have a surjection $\beta : P \twoheadrightarrow I/(I^2xy)$ which is a lift of ψ. Since $xy = 0$ in D, therefore we have $\beta : P \twoheadrightarrow I$ such that $\beta \equiv \alpha$ modulo I^2.

Case 2: Let $\text{ht}(I) = \text{rk}(P) = d$ and assume that $\overline{\alpha}$, $\widetilde{\alpha}$ can be lifted to the surjections $\gamma : P/yP \twoheadrightarrow \overline{I}$ and $\delta : P/xP \twoheadrightarrow \widetilde{I}$ respectively. Since $\widetilde{\alpha}$ can be lifted to a surjection $\delta : P/xP \twoheadrightarrow \widetilde{I}$, applying Lemma 2.9, we can lift α to a surjection $\phi : P \twoheadrightarrow I/(I^2x)$. We go modulo y, i.e., to the ring $R[X]$. Consider $\overline{\phi} : P/yP \twoheadrightarrow \overline{I}/(\overline{I}^2X)$. We know that $\overline{\alpha}$ can be lifted to a surjection $\gamma : P/yP \twoheadrightarrow \overline{I}$. Now we move to the ring $R(X)$. Then $\overline{\phi} = \gamma$ modulo $\overline{I}^2R(X)$. Therefore, it follows from [D-Z, Theorem 4.1] that we can find a surjection $\psi : P/yP \twoheadrightarrow \overline{I}$ which lifts $\overline{\phi}$. Applying Lemma 2.9, we have a surjection $\beta : P \twoheadrightarrow I/(I^2xy)$ which lifts ϕ. Since $xy = 0$ in D, therefore $\beta : P \twoheadrightarrow I$ such that $\beta \equiv \alpha$ modulo I^2. This completes the proof. ☐

Remark 4.4. We can actually extend the above theorem for an arbitrary rank one projective D-module L (which is not necessarily extended from $R[X]$). We outline a strategy. Write $\mathbb{L} = L/yL$. Then $\mathbb{L}$ is a rank one projective $R[X]$-module. It is not difficult to believe that we can take R to be reduced. Then there exists an extension (see [B 1, Proposition 3.3] and [D-Z, Remark 5.6]) $R \hookrightarrow S$ such that:

(1) The projective $S[X]$-module $\mathbb{L} \otimes_{R[X]} S[X]$ is extended from S,
(2) S is module-finite over R,
(3) the canonical map $\text{Spec}(S) \to \text{Spec}(R)$ is bijective, and
(4) for every $\mathfrak{P} \in \text{Spec}(S)$, the inclusion $R/(\mathfrak{P} \cap R) \longrightarrow S/\mathfrak{P}$ is birational.

Write $E = S[X, Y]/(XY)$ and note that by [B-R, Theorem 5.1], the module $L \otimes E$ is extended from $S[X]$. Therefore Theorem 4.3 applies to the ring $E = S[X, Y]/(XY)$. One now needs some descent technique to conclude the result for D. This can be achieved by following the method of [D-Z, Section 5], which involves an induction argument on $\dim(R/C)$, where C is the conductor ideal of R in S. We do not repeat the details here. Keeping [D-Z] in hand, any diligent reader can work this out.

The following set of corollaries are now obvious from (4.3, 4.4) and we skip the proofs.

Corollary 4.5. *Let R be a ring of dimension $d \geq 1$ containing $\mathbb{Q}$ and $D = R[X,Y]/(XY)$. Let L be a rank one projective D-module. Then,*

1. *The $(d+1)$-th Euler class group $E^{d+1}(D, L)$ is trivial.*
2. *The weak Euler class group $E_0^{d+1}(D, L)$ is trivial.*

The result of Bhatwadekar and Roy from [B-R], can now be deduced (although we have the restriction that $\mathbb{Q} \subset D$.).

Corollary 4.6. *Let R be a ring of dimension $d \geq 1$ containing $\mathbb{Q}$ and D be as above. Let P be a projective D-module of rank $\geq d+1$. Then P contains a unimodular element.*

We now extend Theorem 4.2 to the polynomial extension of D.

Theorem 4.7. *Let R be a Noetherian ring of dimension $d \geq 2$ containing $\mathbb{Q}$ and let $D = R[X,Y]/(XY)$. Let $I \subset D[T]$ be an ideal of height n. Suppose that $I = (f_1, \cdots, f_n) + I^2$ where $n \geq d+1$. Then there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ and $f_i - g_i \in I^2$.*

Proof. The proof is similar to the proof of Theorem 3.15.

Note that $\dim(D) = d+1$. Now if $n \geq d+2 = \dim(D)+1$, there is nothing to prove. Therefore we assume that $n = d+1$.

Since R contains $\mathbb{Q}$, by [B-RS 1, Lemma 3.3], we can find some $\lambda \in \mathbb{Q}$ such that $I(\lambda) = D$ or $I(\lambda)$ is an ideal of D of height n. If necessary, we can replace T by $T - \lambda$ and assume that either $I(0) = D$ or $\text{ht}(I(0)) = n$.

If $I(0) = D$, by Lemma 2.8, we can lift $f_1, \cdots, f_n$ to a set of n generators of $I/(I^2T)$, say, $I = (l_1, \cdots, l_n) + (I^2T)$ with $f_i - l_i \in I^2$.

Now assume that $\text{ht}(I(0)) = n$, then we consider $I(0) = (f_1(0), \cdots, f_n(0)) + I(0)^2$. By Theorem 4.3, there exist $a_1, \cdots, a_n \in I(0)$ such that $I(0) = (a_1, \cdots, a_n)$ with $f_i(0) - a_i \in I(0)^2$. Therefore, again by Lemma 2.8, we can lift $f_1, \cdots, f_n$ to a set of n generators of $I/(I^2T)$, say, $I = (l_1, \cdots, l_n) + (I^2T)$ with $f_i - l_i \in I^2$.

Therefore, in any case, we can lift the given set of n generators of I/I^2 to a set of n generators of $I/(I^2T)$.

Let S be the set of all monic polynomials in $R[T]$ and $R(T) = S^{-1}R[T]$. We note that $\dim(R(T)) = \dim(R) = d$ and $S^{-1}D[T] = R(T)[X,Y]/(XY)$. Now we move to the ring $S^{-1}D[T]$. Again, applying Theorem 4.3, we get $IS^{-1}D[T] = (k_1, \cdots, k_n)$ such that $f_i - k_i \in I^2S^{-1}D[T]$.

Now we can apply Theorem 2.7 and obtain the desired set of generators for I. □

Note that modifying the proof of Theorem 4.7 and following Remark 4.4, we can also extend Theorem 4.7 in the following form.

Theorem 4.8. *Let R be a Noetherian ring of dimension $d \geq 3$ and $D = R[X,Y]/(XY)$. Let $I \subset D[T]$ be an ideal of height $n \geq 3$ and L be a projective $D[T]$-module of rank one. Suppose that $\alpha : L \oplus D[T]^{n-1} \twoheadrightarrow I/I^2$ where $n \geq d+1$. Then there exists $\beta : L \oplus D[T]^{n-1} \twoheadrightarrow I$ which lifts α.*

We then have

Corollary 4.9. *Let R be a ring of dimension $d \geq 3$ containing $\mathbb{Q}$ and $D = R[X,Y]/(XY)$. Then,*

(1) *The $(d+1)$-th Euler class group $E^{d+1}(D[T], L)$ is trivial.*
(2) *The weak Euler class group $E_0^{d+1}(D[T], L)$ is trivial.*

The following corollary is now obvious, which extends the result of Bhatwadekar-Roy [B-R, Theorem 5.3(ii)] to the the polynomial ring $D[T]$, with the assumption that $\mathbb{Q} \subset R$.

Corollary 4.10. *Let R be a ring of dimension $d \geq 3$, containing $\mathbb{Q}$ and D be as above. Let P be a projective $D[T]$-module of rank $\geq d+1$, then P contains a unimodular element.*

5. THE EULER CLASS GROUP OF D

In this section, R will denote a commutative Noetherian ring containing the field of rationals with dimension $d \geq 3$. Let $D = R[X,Y]/(XY) = R[x,y]$ and we retain the same notations as established at the beginning of the last section. Our aim is to define and study the d-th Euler class group of D. We retain the same notations established at the beginning of the last section.

5.1. **Addition and subtraction principles.** We first prove some 'addition' and 'subtraction' principles for ideals of the ring $D = R[X,Y]/(XY)$. These two principles form the technical heart of the Euler class theory and also have some intrinsic appeal.

Proposition 5.1. *(Addition principle) Let R be a Noetherian ring of dimension $d \geq 3$ and let $D = R[X,Y]/(XY)$. Let I, J be two comaximal ideals in D, each of height d. Suppose that $I = (f_1, \cdots, f_d)$ and $J = (g_1, \cdots, g_d)$. Then $I \cap J = (h_1, \cdots, h_d)$ where $h_i - f_i \in I^2$ and $h_i - g_i \in J^2$ for $1 \leq i \leq d$.*

Proof. Let $K = I \cap J$. Since I and J are comaximal, the generators of I and J induce a set of generators of K/K^2, say, $K = (k_1, \cdots, k_d) + K^2$ where $k_i - f_i \in I^2$ and $k_i - g_i \in J^2$ for $1 \leq i \leq d$.

Since $K \subset D$ is a proper ideal, it is easy to see that both $\overline{K}$ and $\widetilde{K}$ cannot be the whole ring. We consider the case when ht$(\overline{K}) = d$ and ht$(\widetilde{K}) = d$. Other cases can be tackled similarly.

Since I and J are comaximal ideals in D, it is easy to see that $\overline{K} = \overline{I} \cap \overline{J}$. Therefore, ht$(\overline{I}) \geq d$ and ht$(\overline{J}) \geq d$. Both of them cannot equal $R[X]$, as $\overline{K}$ is proper. If one of them, say $\overline{I} = R[X]$, then $\overline{K} = \overline{J} = (\overline{g}_1, \cdots, \overline{g}_d)$ whereas $\overline{g}_i - \overline{k}_i \in \overline{J}^2$. Since $\overline{I} = R[X]$, it follows that $\overline{g}_i - \overline{k}_i \in \overline{K}^2$.

Now assume that both $\overline{I}$ and $\overline{J}$ are proper ideals. In this case, by the addition principle as in [D 1, Proposition 4.2], $\overline{K} = (u_1, \cdots, u_d)$ such that $\overline{f}_i - u_i \in \overline{I}^2$ and $\overline{g}_i - u_i \in \overline{J}^2$.

Similarly, $\widetilde{K} = (v_1, \cdots, v_d)$ such that $\widetilde{f}_i - v_i \in \widetilde{I}^2$ and $\widetilde{g}_i - v_i \in \widetilde{J}^2$.

Therefore we have $\overline{K} = (u_1, \cdots, u_d)$ such that $\overline{k}_i - u_i \in \overline{K}^2$ and $\widetilde{K} = (v_1, \cdots, v_d)$ such that $\widetilde{k}_i - v_i \in \widetilde{K}^2$.

Finally applying Theorem 4.2, we can find $h_1, \cdots, h_d$ such that $K = I \cap J = (h_1, \cdots, h_d)$ where $h_i - f_i \in I^2$ and $h_i - g_i \in J^2$. □

Proposition 5.2. *(Subtraction principle) Let R be a Noetherian ring of dimension $d \geq 3$ and let $D = R[X,Y]/(XY)$. Let I, J be two comaximal ideals in D, each of height d. Suppose that $I = (f_1, \cdots, f_d)$ and $I \cap J = (h_1, \cdots, h_d)$ such that $h_i - f_i \in I^2$. Then, $J = (g_1, \cdots, g_d)$ where $h_i - g_i \in J^2$.*

Proof. Let $K = I \cap J$. First note that $J = (h_1, \cdots, h_d) + J^2$. We only consider the case when ht$(\overline{I}) =$ ht$(\widetilde{I}) = d$ and ht$(\overline{J}) =$ ht$(\widetilde{J}) = d$ (other cases can be handled similarly).

We now have $\overline{J} = (\overline{h}_1, \cdots, \overline{h}_d) + \overline{J}^2$ in $R[X]$ and $\widetilde{J} = (\widetilde{h}_1, \cdots, h_d) + \widetilde{J}^2$ in $R[Y]$. Then by [D 1, Proposition 4.3], $\overline{J} = (u_1, \cdots, u_d)$ such that $\overline{h}_i - u_i \in \overline{J}^2$.

Similarly, $\widetilde{J} = (v_1, \cdots, v_d)$ such that $\widetilde{h}_i - v_i \in \widetilde{J}^2$.

Finally applying Theorem 4.2, we can find $g_1, \cdots, g_d$ such that $J = (g_1, \cdots, g_d)$ where $g_i - h_i \in J^2$. □

5.2. **Definition of the group.** We proceed to define the d-th Euler class group of D. It is along the same line as the one given in Section 2. However the reader may note that here dim $(D) = d + 1$ and we are going to define the d-th Euler class group $E^d(D)$ and it involves some work. In fact, we closely follow [D 1] here.

Let $I \subset D$ be an ideal of height d such that I/I^2 is generated by d elements. Two surjections $\alpha, \beta : (D/I)^d \twoheadrightarrow I/I^2$ are said to be *related* if there exists an automorphism σ of $(D/I)^d$ of determinant 1 such that $\alpha\sigma = \beta$. This defines an equivalence relation on the set of surjections from $(D/I)^d$ to I/I^2. We call such an equivalence class a *local orientation* of I.

We now prove

Lemma 5.3. *Let $\alpha, \beta : (D/I)^d \twoheadrightarrow I/I^2$ be two surjections belonging to the same equivalence class. Suppose it is given that α can be lifted to a surjection $\phi : D^d \twoheadrightarrow I$. Then β can also be lifted to a surjection $\psi : D^d \twoheadrightarrow I$.*

Proof. Note that here α and β are represented by two sets of generators of I/I^2, say, $I = (f_1, \cdots, f_d) + I^2$ and $I = (g_1, \cdots, g_d) + I^2$ respectively. Suppose that ϕ is given by a set of generators of I, say, $h_1, \cdots, h_d$ such that $f_i - h_i \in I^2$.

Again we illustrate the proof in the case when ht$(\overline{I})$ = ht$(\widetilde{I}) = d$.

Consider the equations $\overline{I} = (\overline{f}_1, \cdots, \overline{f}_d) + \overline{I}^2$ and $\overline{I} = (\overline{g}_1, \cdots, \overline{g}_d) + \overline{I}^2$ in $R[X]$. By the assumption of the lemma, there exists an automorphism $\sigma \in SL_d(R[X]/\overline{I})$ such that $(\bar{f}_1, \cdots, \bar{f}_d)\sigma = (\bar{g}_1, \cdots, \bar{g}_d)$. Therefore, by [D 1, Proposition 4.4], $\overline{I} = (u_1, \cdots, u_d)$ such that $\overline{g}_i - u_i \in \overline{I}^2$.

Similarly, by [D 1, Proposition 4.4], $\widetilde{I} = (v_1, \cdots, v_d)$ such that $\widetilde{g}_i - v_i \in \widetilde{I}^2$.

Finally applying Theorem 4.2, we can find $k_1, \cdots, k_d$ such that $I = (k_1, \cdots, k_d)$ where $g_i - k_i \in I^2$.

The proof of the lemma is therefore complete. □

Definition 5.4. We call a local orientation $[\alpha]$ of I a *global orientation* of I if the surjection $\alpha : (D/I)^d \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : D^d \twoheadrightarrow I$.

Let G be the free abelian group on the set of pairs (I, ω_I) where $I \subset D$ is an ideal of height d with the property that Spec(D/I) is connected and I/I^2 is generated by d elements and $\omega_I : (D/I)^d \twoheadrightarrow I/I^2$ is a local orientation of I.

Let I be any ideal of D of height d such that I/I^2 is generated by d elements. Then there is a unique decomposition (see [D 1] for details), $I = I_1 \cap \cdots \cap I_k$, where Spec$(D/I_i)$ is connected and ht$(I_i) = d$ for each i, and $I_i + I_j = D$ for $i \neq j$. Now if ω_I is a local orientation of I then it naturally gives rise to $\omega_{I_i} : (D/I)^d \twoheadrightarrow I_i/{I_i}^2$ for $1 \leq i \leq k$. By (I, ω_I) we mean the element $\sum(I_i, \omega_{I_i}) \in G$.

Let H be the subgroup of G generated by the set of pairs (I, ω_I) in G such that ω_I is a global orientation.

Definition 5.5. We define the (d-th) Euler class group of D as $E^d(D) := G/H$

5.3. **Results on the Euler class group.** To establish most of the results on $E^d(D)$ we shall take advantage of the theory developed for $E^d(R[X])$ in [D 1]. In order to do so, we need to have a group morphism from $E^d(D)$ to $E^d(R[X])$. We prove that first.

Theorem 5.6. *Let R be a ring containing $\mathbb{Q}$ of dimension $d \geq 3$ and let $D = R[X,Y]/(XY) = R[x,y]$. There is a group homomorphism $\psi : E^d(D) \longrightarrow E(R[X])$ such that if $(I, \omega_I) \in E^d(D)$ has the property that $\overline{I}$ is an ideal of $R[X]$ of height d, then $\psi(I, \omega_I)) = (\overline{I}, \omega_{\overline{I}})$ in $E(R[X])$, where $\omega_{\overline{I}}$ is the local orientation of $\overline{I}$ induced by ω_I . If $\overline{I} = R[X]$, $\psi((I, \omega_I)) = 0$.*

Proof. We give a sketch of the proof. Let $I \subset D$ be an ideal such that Spec(D/I) is connected, ht$\,(I) = d = \mu(I/I^2)$. Let ω_I be a local orientation of I given by: $I = (f_1, \cdots, f_d) + I^2$. Note that either $\overline{I}$ is a proper ideal of $R[X]$ of height d or $\overline{I} = R[X]$. If $\overline{I}$ is proper, we naturally have the local orientation $\omega_{\overline{I}}$ given by: $\overline{I} = (\overline{f}_1, \cdots, \overline{f}_d) + \overline{I}^2$.

Recall that $E^d(D) = G/H$. The assignment sending (I, ω_I) to $(\overline{I}, \omega_{\overline{I}})$ will give rise to a morphism $\psi' : G \longrightarrow E^d(R[X])$ (if $\overline{I} = R[X]$, we map it to zero).

To see that ψ' extends to a morphism $\psi : E^d(D) \longrightarrow E^d(R[X])$ as described in the statement of this theorem, we are eventually reduced to checking the following: (1) If $(I, \omega_I), (J, \omega_J) \in E^d(D)$ such that $I + J = D$, then $\psi'((I, \omega_I) + (J, \omega_J)) = (\overline{I}, \omega_{\overline{I}}) + (\overline{J}, \omega_{\overline{J}})$, and (2) $H \subset \ker(\psi')$.

To prove the first assertion let $I = (f_1, \cdots, f_d) + I^2$ and $J = (g_1, \cdots, g_d) + J^2$ induce ω_I and ω_J respectively. Then, a simple application of the Chinese remainder theorem would yield, $I \cap J = (h_1, \cdots, h_d) + (I \cap J)^2$ such that $h_i - f_i \in I^2$ and $h_i - g_i \in J^2$. We thus get a local orientation $\omega_{I \cap J}$ induced by $h_1, \cdots, h_d$. Clearly, $\overline{I \cap J} = \overline{I} \cap \overline{J}$ and $\overline{I} + \overline{J} = R[X]$. Further, it is obvious from the relations that $\omega_{\overline{I}}$ and $\omega_{\overline{J}}$ together induce $\omega_{\overline{I \cap J}}$.

The second assertion is rather obvious because if $L = (k_1, \cdots, k_d) \subset D$ is an ideal of height d and ω_L is induced by these generators, then $\omega_{\overline{L}}$ is obviously induced by $\overline{k}_1, \cdots, \overline{k}_d$ and is therefore global. $\square$

We are now ready to prove the following result.

Theorem 5.7. *Let R be a ring of dimension $d \geq 3$ and $D = R[X, Y]/(XY)$. Let $I \subset D$ be an ideal of height d such that I/I^2 is generated by d elements and let $\omega_I : (D/I)^d \twoheadrightarrow I/I^2$ be a local orientation of I. Suppose that the image of (I, ω_I) is zero in $E^d(D)$. Then ω_I can be lifted to a surjection $\theta : D^d \twoheadrightarrow I$ (i.e., ω_I is a global orientation).*

Proof. Here ω_I is represented by a set of generators of I/I^2, say, $I = (f_1, \cdots, f_d) + I^2$.

Here we illustrate the proof in the case when $\text{ht}(\overline{I}) = \text{ht}(\widetilde{I}) = d$.

Now consider the equations $\overline{I} = (\overline{f}_1, \cdots, \overline{f}_d) + \overline{I}^2$ and $\widetilde{I} = (\widetilde{f}_1, \cdots, \widetilde{f}_d) + \widetilde{I}^2$ in $R[X]$ and $R[Y]$, respectively. Since $(I, \omega_I) = 0$ in $E^d(D)$, we have $(\overline{I}, \omega_{\overline{I}}) = 0$ in $E(R[X])$. Then by [D 1, Theorem 4.7], we have $\overline{I} = (u_1, \cdots, u_d)$ with $\overline{f}_i - u_i \in \overline{I}^2$. Again by [D 1, Theorem 4.7], we have $\widetilde{I} = (v_1, \cdots, v_d)$ with $\widetilde{f}_i - u_i \in \widetilde{I}^2$.

Now applying Theorem 4.2, we can find $g_1, \cdots, g_d$ such that $I = (g_1, \cdots, g_d)$ with $f_i = g_i \in I^2$. $\square$

Definition 5.8. (**Euler class of a projective module:** Let P be a projective D-module of rank n having trivial determinant. Let $\chi : D \xrightarrow{\sim} \wedge^d P$ be an isomorphism. To the pair (P, χ), we associate an element $e(P, \chi)$ of $E^d(D)$ as follows: Let $\lambda_0 : P \twoheadrightarrow I_0$ be a surjection, where I_0 is an ideal of D of height d. Let bar denote reduction modulo I_0. We obtain an induced surjection $\overline{\lambda}_0 : P/I_0P \twoheadrightarrow I_0/I_0^2$. Note that, since P has trivial determinant and $\dim(D/I_0) \leq 1$, by Lemma 2.12 we have $P/I_0P \simeq (D/I_0)^d$. We choose an isomorphism $\overline{\gamma} : (D/I_0)^d \xrightarrow{\sim} P/I_0P$, such that $\wedge^d \overline{\gamma} = \overline{\chi}$. Let ω_{I_0} be the composite surjection

$$(D/I_0)^d \overset{\overline{\gamma}}{\tilde{\to}} P/I_0P \overset{\overline{\lambda_0}}{\twoheadrightarrow} I_0/I_0^2.$$

Let $e(P,\chi)$ be the image in $E^d(D)$ of the element (I_0,ω_{I_0}). We say that (I_0,ω_{I_0}) is obtained from the pair (λ_0,χ). The *Euler class* of the pair (P,χ) is defined to be $e(P,\chi)$. In view of the following lemma, it is a valid definition.

Lemma 5.9. *The assignment sending the pair (P,χ) to the element $e(P,\chi)$, as described above, is well defined.*

Proof. Let $\lambda_i : P \twoheadrightarrow I_i$ $(i = 0,1)$ be two surjections so that (λ_i,χ) induce (I_i,ω_{I_i}). Applying the moving lemma (2.3) we can find an ideal $K \subset D$ and a local orientation ω_K of K such that $\text{ht}\,(K) \geq d$, $K + I_i = D$ for $i = 0,1$ and $(I_0,\omega_{I_0}) + (K,\omega_K) = 0$ in $E^d(D)$.

Now let $I = I_1 \cap K$ and ω_I be the local orientation of I induced by ω_{I_1} and ω_K. We want to show that $(I,\omega_I) = 0$ in $E^d(D)$. This proves the lemma because of $(I,\omega_I) = (I_1,\omega_{I_1}) + (K,\omega_K)$ in $E^d(D)$.

Now using the facts that the Euler class of a projective $R[X]$-module (resp., $R[Y]$-module) is well-defined, we can show that ω_I is a global orientation. This will prove $0 = (I,\omega_I) = (I_1,\omega_{I_1}) + (K,\omega_K)$ in $E^d(D)$ and therefore, $(I_0,\omega_{I_0}) = (I_1,\omega_{I_1})$. □

Theorem 5.10. *Let P be projective D-module of rank d having trivial determinant and $\chi : D \simeq \wedge^d P$. Let $e(P,\chi) = (I,\omega_I)$ in $E^d(D)$, where $I \subset D$ be an ideal of height d. Then there exists a surjection $\theta : P \twoheadrightarrow I$ such that (I,ω_I) is obtain from (Θ,χ).*

Proof. Since $\dim(D/I) \leq 1$ and P has trivial determinant, by Lemma 2.12, P/IP is a free D/I-module of rank d. Choose $\tau : (D/I)^d \simeq P/IP$ such that $\wedge^d\tau = \chi \otimes D/I$. Let $\omega_I\tau^{-1} : P/IP \twoheadrightarrow I/I^2$.

We may assume that $\text{ht}(\overline{I}) = \text{ht}(\widetilde{I}) = d$. Since $e(P/yP, \chi \otimes D/yD) = (\overline{I},\omega_{\overline{I}})$ in $E(R[X])$, by [D 1, Corollary 4.10], there exists a surjection $\eta_1 : P/yP \twoheadrightarrow \overline{I}$ such that $(\overline{I},\omega_{\overline{I}})$ is obtained from the pair $(\eta_1, \chi \otimes D/yD)$. In other words η_1 is a lift of $\overline{\omega_I\tau^{-1}}$.

Again applying [D 1, Theorem 4.10] there exists a surjection $\eta_2 : P/xP \twoheadrightarrow \widetilde{I}$ such that $(\widetilde{I},\omega_{\widetilde{I}})$ is obtained from the pair $(\eta_2, \chi \otimes D/xD)$. In other words η_2 is a lift of $\widetilde{\omega_I\tau^{-1}}$.

Now applying Theorem 4.3 one can find $\Theta : P \twoheadrightarrow I$ is a lift of $\omega_I\tau^{-1}$. Since $(\Theta \otimes D/I)\tau = \omega_I$ and $\wedge^d\tau = \chi \otimes D/I$, it follows that (I,ω_I) is obtained from (Θ,χ). □

We now prove that the Euler class is the precise obstruction for a projective D-module of rank d to have a unimodular element.

Theorem 5.11. *Let P be a projective D-module of rank d having trivial determinant and $\chi : D \simeq \wedge^d P$. Then $e(P,\chi) = 0$ in $E^d(D)$ if and only if P has a unimodular element.*

Proof. Let $e(P,\chi) = (I, \omega_I)$ in $E^d(D)$, where (I, ω_I) is obtained from the pair (α, χ). Now ω_I is represented by a set of generators of I/I^2, say, $I = (f_1, \cdots, f_d) + I^2$.

Assume that P has a unimodular element. Therefore, P/xP and P/yP also have unimodular elements.

Suppose both $\overline{I}$ and $\widetilde{I}$ are ideals of height d, then by [D 1, Corollary 4.11], we can lift the set of generators $\overline{f}_1, \cdots, \overline{f}_d$ of $\overline{I}/\overline{I}^2$ to a set of generators of $\overline{I}$, say $u_1, \cdots, u_d$. Again by [D 1, Corollary 4.11], there exist $v_1, \cdots, v_d$ such that $\widetilde{I} = (v_1, \cdots, v_d)$ such that $\widetilde{f_i} - v_i \in \widetilde{I}^2$.

Now applying Theorem 4.2, we can find $g_1, \cdots, g_d$ such that $I = (g_1, \cdots, g_d)$ with $f_i - g_i \in I^2$.

Conversely, assume that $e(P,\chi) = 0$ in $E^d(D)$. Which shows that P/xP and P/yP have unimodular elements. Let p_1 be a unimodular element of P/xP. Let tilde and bar denote reduction modulo X and Y respectively. Then $\bar{p}_1$ is a unimodular element of $P/(x,y)P$. As we have projective modules with *extended* determinants, it is easy to see from [B-L-Ra, Theorem 5.2, Remark 5.3], that the map $Um(P/yP) \twoheadrightarrow Um(P/(x,y)P)$ is surjective. Therefore there exists $p_2 \in Um(P/yP)$ such that $\widetilde{p}_2 = \bar{p}_1$.

Since the following square of rings

$$\begin{array}{ccc} D & \longrightarrow & R[X] \\ \downarrow & & \downarrow \\ R[Y] & \longrightarrow & R \end{array}$$

is Cartesian with vertical maps surjective, the unimodular elements p_1 and p_2 of P/xP and P/yP respectively will patch up together to give a unimodular element of P (the reader may have a look at [L, Lemma 1.14] for a general argument). $\square$

Remark 5.12. We can also extend the theory of $E^d(D)$ to $E^d(D, L)$, where L is a rank one projective D-module. We have already outlined such a strategy in Remark 4.4.

5.4. **Some maps between the Euler class groups.** In this subsection we study some maps between the Euler class groups.

Let R be a Noetherian ring containing $\mathbb{Q}$ with $\dim(R) = d \geq 3$. Let $D = R[X,Y]/(XY)$. Note that $R[X] \hookrightarrow D$ is not a flat extension. But it is easy to see that if $I \subset R[X]$ is an ideal of height d then $(I + (XY))/(XY)$ also has height d. Therefore, we have a map $\phi : E^d(R[X]) \longrightarrow E^d(D)$ which sends $(I, \omega_I) \in E^d(R[X])$ to $(\mathfrak{I}, \omega_{\mathfrak{I}}) \in E^d(D)$, where $\mathfrak{I} = (I + (XY))/(XY)$ and $\omega_{\mathfrak{I}}$ is the local orientation of $\mathfrak{I}$ induced by ω_I.

Now we have the following maps

$$E^d(R[X]) \xrightarrow{\phi} E^d(D) \xrightarrow{\psi} E^d(R[X]).$$

It is easy to see that the composition map $\psi\phi$ is identity on $E^d(R[X])$. Therefore ϕ is injective and ψ is surjective. The following question is natural.

Question 5.13. Is the map $\phi : E^d(R[X]) \longrightarrow E^d(D)$ surjective?

To answer the above question, we first prove the following theorem, which is an analogue of [B-R, Theorem 5.1]. Recall from [D 1] that there is a canonical map $\varphi : E^d(R) \longrightarrow E^d(R[X])$, which is injective but not necessarily surjective.

Theorem 5.14. *Let R be a ring containing $\mathbb{Q}$ of dimension $d \geq 3$ and let $D = R[X,Y]/(XY)$. Then the following statements are equivalent:*

1) *The canonical map $\varphi : E^d(R) \longrightarrow E^d(R[X])$ is surjective.*
2) *The map $\phi : E^d(R[X]) \longrightarrow E^d(D)$ is surjective.*
3) *The map $\phi\varphi : E^d(R) \longrightarrow E^d(D)$ is surjective.*

Proof. Clearly (3)⇒(2).

We now prove that (2)⇒(1). Let $(I, \omega_I) \in E^d(R[Y])$. By the hypothesis, $\phi((I, \omega_I)) = (\mathfrak{I}, \omega_{\mathfrak{I}}) \in E^d(D)$ has a preimage in $E^d(R[X])$, say (J, ω_J). But then we have $(I, \omega_I) = (J(0)R[Y], \omega_{J(0)} \otimes R[Y])$, showing that φ is surjective.

Finally we prove that (1)⇒(3). Let $(I, \omega_I) \in E^d(D)$. Let the local orientation ω_I be given by $I = (f_1, \cdots, f_d) + I^2$. Suppose $\overline{I}$ is a proper ideal of $R[X]$ of height d. Consider $(\overline{I}, \omega_{\overline{I}}) \in E^d(R[X])$, where $\omega_{\overline{I}}$ is the local orientation of $\overline{I}$ induced by ω_I. Now $\overline{\widetilde{\overline{I}}}$ is an ideal of R (not necessarily proper) with ht$(\overline{\widetilde{\overline{I}}}) \geq d-1$ and $\overline{\widetilde{\overline{I}}} = (\overline{\widetilde{f_1}}, \cdots, \overline{\widetilde{f_n}}) + \overline{\widetilde{\overline{I}}}^2$.

Now applying [D-RS, lemma 2.7], we can find an ideal $K \subset R$ of height d which is comaximal with $I \cap R$ and a local orientation ω_K of K such that $(\overline{\widetilde{\overline{I}}}, \omega_{\overline{\widetilde{\overline{I}}}}) + (K, \omega_K) = 0$ in $E^d(R)$. We denote the ideal $(KR[X,Y] + XY)/(XY)$ by KD.

Let $L = I \cap KD$. Since the ideals I and KD are comaximal, ω_I and ω_K induce $\omega_L : (D/L)^d \twoheadrightarrow L/L^2$ and we have the following equation in $E^d(D)$:

$$(L, \omega_L) = (I, \omega_I) + (KD, \omega_K \otimes D).$$

From hypothesis it is clear that $(\overline{L}, \omega_{\overline{L}}) = 0$ in $E^d(R[X])$ and $(\widetilde{L}, \omega_{\widetilde{L}}) = 0$ in $E^d(R[Y])$. Let the local orientation ω_L be given by $L = (g_1, \cdots, g_d) + L^2$. Here we consider the case when ht$(\overline{L}) = $ ht$(\widetilde{L}) = d$ (other cases can be handled similarly). Since $(\overline{L}, \omega_{\overline{L}}) = 0$ in $E(R[X])$, therefore there exist $u_1, \cdots, u_d$ such that $\widetilde{L} = (u_1, \cdots, u_d)$ with $\overline{g}_i - u_i \in \overline{L}^2$.

Again $(\widetilde{L}, \omega_{\widetilde{L}}) = 0$ in $E^d(R[Y])$, so we have $\widetilde{L} = (v_1, \cdots, v_d)$ such that $\widetilde{g}_i - v_i \in \widetilde{L}^2$.

Finally using Theorem 4.2, we can find $h_1, \cdots, h_d$ such that $L = (h_1, \cdots, h_d)$ with $g_i - h_i \in L^2$. This completes the proof. □

In general, the canonical map $\varphi : E^d(R) \longrightarrow E^d(R[X])$ is not surjective [B-RS 1, Example 6.4]. Therefore, from the above theorem it follows that the map $\phi : E^d(R[X]) \longrightarrow E^d(D)$ is not surjective in general.

Corollary 5.15. *Let R be a smooth affine domain containing $\mathbb{Q}$ with dim$(R) = d \geq 3$ and $D = R[X,Y]/(XY)$. Then the map $\phi : E^d(R[X]) \longrightarrow E^d(D)$ is an isomorphism.*

Proof. The proof follows from [D 1, Proposition 5.7] and the above theorem. □

Corollary 5.16. *Let R be a regular k-spot where k is a field of characteristic zero. Let dim$(R) = d \geq 3$ and $D = R[X,Y]/(XY)$. Then $E^d(D)$ is trivial.*

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References

[B 1] S. M. Bhatwadekar, Inversion of monic polynomials and existence of unimodular elements (II), *Math. Z.* **200** (1989), 233-238.

[B-L-Ra] S. M. Bhatwadekar, H. Lindel, Ravi A. Rao, The Bass-Murthy question: Serre dimension of Laurent polynomial extensions, *Invent. Math* **81** (1985), no. 1, 189203.

[B-R] S. M. Bhatwadekar, Amit Roy, Stability theorems for overrings of polynomial rings, *Invent.Math* **68** (1982), 117-127.

[B-RS 1] S. M. Bhatwadekar, Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain and a question of Nori, *Invent. Math* **133** (1998), 161-192.

[B-RS 2] S. M. Bhatwadekar, Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain (II), *K-Theory* **15** (1998), 293-300.

[B-RS 3] S. M. Bhatwadekar, Raja Sridharan, Euler class group of a Noetherian ring, *Compositio Math* **122** (2000), 183-222.

[D 1] M. K. Das, The Euler class group of a polynomial algebra, *J. Algebra* **264** (2003), 582-612.

[D 2] M. K. Das, The Euler class group of a polynomial algebra II, *J. Algebra* **299** (2006), 94-114.

[D-K] M. K. Das, M. K. Keshari, A question of Nori, Segre classes of ideals and other applications, *J. of Pure and Applied Algebra* **216** (2012) 21932203.

[D-RS] M. K. Das, Raja Sridharan, Good invariants for bad ideals, *J. Algebra* **323** (2010), 3216-3229.

[D-Z] M. K. Das, Md. Ali Zinna, The Euler class group of a polynomial algebra with coefficients in a line bundle, *Math. Z.* **276** (2014), 757-783.

[E-E] D. Eisenbud, E.G. Evans, Generating modules efficiently: Theorems from algebraic K -theory, *J. Algebra* **27** (1973) 278305.

[K] M. K. Keshari, Euler Class Group of a Noetherian Ring. M.Phil. thesis. available at : http://www.math.iitb.ac.in/˜keshari/acad.html

[Ka] D. Katz, Generating ideals up to projective equivalence, *Proc. Amer. Math. Soc*, **120** (1994), 401-414.

[L] H. Lindel, Unimodular elements in projective modules, *J. Algebra* **172** (1995) 301-319.

[M] S. Mandal, Basic elements and cancellation over laurent polynomial rings, *J. Algebra* **79** (1982) 251-257.

[M 1] S. Mandal, On efficient generation of ideals, *Invent. Math* **75** (1984), 59-67.

[M 2] S. Mandal, Homotopy of sections of projective modules, *J. Algebraic Geometry* **1** (1992), 639-646.

[M 4] S. Mandal, Projective Modules and Complete Intersections, Lecture Notes in Mathematics, vol. 1672, Springer, Berlin, 1997, pp. 1113.

[M 5] S. Mandal, On set theoretic intersection in Affine spaces, *J. of Pure and Applied Algebra* **51** (1988), 267-275.

[M-P] S. Mandal, K. Parker, Vanishing of Euler class groups, *J. Algebra* **308** (2007) 107-117.

[Mu] M. P. Murthy, Vector bundles over affine surfaces birationally equivalent to a ruled surface. *Ann. of Math* **89** (1969), 240-253.

[N] B. S. Nashier, Efficient generation of ideals in polynomial rings, *J. Algebra* **85** (1983), 287-302.

[P] B. R. Plumstead, The conjectures of Eisenbud and Evans, *Amer. J.Math.* **105** (1983), 1417-1433.

[Q] D. Quillen, Projective modules over polynomial rings, *Invent.Math.* **36** (1976), 167-171.

[R] Ravi A. Rao, Stability theorems for overrings of polynomial rings, II, *J. Algebra* **78** (1982), 437-444.

[Sa] Q. Samuel, Unique factorization domains, (Lecture Notes) Tata Institute, Bombay, 1964.

[Se] Serre, J-P, Sur les modules projectifs, *Semin. Dubreil-Pisot* **14** (1960-61).

[Su1] A. A. Suslin, Projective modules over a polynomial ring are free, *Soviet Math. Dokl.* **17** (1976), 1160-1164 (English translation).

[Su2] A. A. Suslin, On the structure of the special linear group over rings of polynomials, *Math. USSR-Izv* **11** (1977), 221-238.

[W] C. A. Weibel, K-theory and analytic isomorphisms, *Invent. Math* **61** (1980), 177-197.

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