

REMARKS ON EULER CLASS GROUPS AND TWO CONJECTURES

MRINAL KANTI DAS

1. INTRODUCTION

Recently, Asok-Fasel have settled two fascinating open problems in [AF]. Let us recall their results. Let k be a field and $X = \text{Spec}(R)$ be a smooth affine k -scheme of dimension $d \geq 2$. Let $E^d(R)$ be the d^{th} Euler class group of Nori-Bhatwadekar-Sridharan [BRS 1] and $\widetilde{CH}^d(X)$ be the Chow-Witt group defined by Barge-Morel [B-M] (and studied by Fasel [F] in detail). Also, consider the Chow group $CH^d(X)$ of zero cycles, and the weak Euler class group $E_0^d(R)$ of Bhatwadekar-Sridharan (introduced in [BRS 2]). Asok-Fasel prove the following comparison theorems.

Theorem 1.1. [AF, Theorems 3.2.1, 3.2.7] *Let k be a field and $X = \text{Spec}(R)$ be a smooth affine k -scheme of dimension $d \geq 2$. Assume that: (a) k has characteristic zero if $d = 3$, or (b) $\text{char}(k) \neq 2$, k is infinite and perfect if $d = 2$ or $d \geq 4$. Then,*

- (1) *there is a functorial isomorphism $E^d(R) \xrightarrow{\sim} \widetilde{CH}^d(X)$;*
- (2) *the canonical morphism $E_0^d(R) \longrightarrow CH^d(X)$ is an isomorphism.*

We shall remark that the above results can easily be extended to the following form.

Theorem 1.2. *Let k be an infinite perfect field and $X = \text{Spec}(R)$ be a smooth affine k -scheme of dimension $d \geq 2$. Then,*

- (1) *if $\text{char}(k) \neq 2$, there is a functorial isomorphism $E^d(R) \xrightarrow{\sim} \widetilde{CH}^d(X)$;*
- (2) *if $d = 2$, or $d \geq 3$ and $\text{char}(k) \neq 2$, the canonical morphism $E_0^d(R) \longrightarrow CH^d(X)$ is an isomorphism.*

Remark 1.3. The proof of $E_0^d(R) \xrightarrow{\sim} CH^d(X)$ (for $d \geq 3$) relies on the isomorphism $E^d(R) \xrightarrow{\sim} \widetilde{CH}^d(X)$ and the assumption $\text{char}(k) \neq 2$ stems from there. Therefore, the following remains open.

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Question 1.4. Can one remove the characteristic assumption from Theorem 1.2 (2) for $d \geq 3$?

The improvements of the results of Asok-Fasel in Theorem 1.2, despite appearing significant, did not warrant much work. The ingredients are all implicit in the existing theory of the Euler class groups as developed in [BRS 1, BRS 2, BRS 3, D1]. We have only highlighted them properly and made them explicit.

Let us explain to the reader why the odd assumption “ k has characteristic zero if $d = 3$ ” has landed in the hypothesis of Theorem 1.1. In [AF, Lemma 3.1.11] Asok-Fasel have used the *homotopy invariance* of the Euler class groups ($E^d(R) \xrightarrow{\sim} E^d(R[T])$). For $d \geq 4$ they refer to [MaY], while for $d = 3$ they have to use [D1]. In [D1], the theory of the Euler class group $E^d(R[T])$ ($d \geq 3$) was developed with the blanket assumption that $\mathbb{Q} \subset R$, and for this reason Asok-Fasel assume that $\text{char}(k) = 0$ in Theorem 1.1. To rectify this, here we define $E^d(R[T])$ for $d \geq 3$ where R is a regular ring containing a field k (R is of finite type over k if k is finite). Then we indicate how the homotopy invariance can be achieved when R is a smooth affine algebra of dimension $d \geq 3$ over an infinite field k . In our definition of $E^d(R[T])$, for an Euler cycle $(I, \omega_I) \in E^d(R[T])$ we take the *local orientation* ω_I as an equivalence class of surjections induced by the action of $SL_d(R[T]/I)$ on all surjections from $(R[T]/I)^d \rightarrow I/I^2$. On the other hand, the definition followed by [MaY] (from [BRS 4]) uses the action of the elementary group $E_d(R[T]/I)$ (which may be a proper subgroup of $SL_d(R[T]/I)$). Therefore, a priori the two definitions are different and it takes some amount of work to show their equivalence (one has to use Theorem 7.3 from below).

Our starting point of this article was what we just described above. But while carrying out the above task, we felt that we should also revamp some parts of the existing theory of Euler classes and present them in as much generality as possible so that others can use them if necessary. On the way, we have obtained some new results as well (see 3.12, 6.2, 6.3) which will perhaps attract attention of a general reader and will find some interesting applications. Again, it will be apparent to an expert that our methods involve hardly any substantial original idea.

2. THE EULER CLASS GROUP $E^d(R)$

The Euler class group $E^d(R)$ can be defined for any commutative Noetherian ring R of dimension $d \geq 2$. Let us first recall the two definitions of $E^d(R)$ from [BRS 1] and [BRS 3]. But before doing so, we insert a definition which will be frequently used.

Definition 2.1. (Local and global orientations:) Let R be a commutative Noetherian ring of dimension $d \geq 2$. Let $J \subset R$ be an ideal of height d such that J/J^2 is generated by d elements. Two surjections α, β from $(R/J)^d$ to J/J^2 are said to be related if there

exists $\sigma \in SL_d(R/J)$ such that $\alpha\sigma = \beta$. Clearly this is an equivalence relation on the set of surjections from $(R/J)^d$ to J/J^2 . Note that, if α can be lifted to a surjection $\theta : R^d \twoheadrightarrow J$, then β can also be lifted to some $\theta' : R^d \twoheadrightarrow J$. Let $[\alpha]$ denote the equivalence class of α . Such an equivalence class $[\alpha]$ is called a *local orientation* of J . By abuse of notation, we shall identify an equivalence class $[\alpha]$ with α . A local orientation α is called a *global orientation* if $\alpha : (R/J)^d \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : R^d \twoheadrightarrow J$.

Definition 2.2. (The Euler class group $E^d(R)$: R smooth affine) Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k . Let G_1 be the free abelian group on the set of pairs $(\mathfrak{m}, \omega_{\mathfrak{m}})$, where \mathfrak{m} is a maximal ideal of R and $\omega_{\mathfrak{m}}$ is a local orientation of \mathfrak{m} . Let $J = m_1 \cap \cdots \cap m_r$, where m_i are distinct maximal ideals of R . Any $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ induces surjections $\omega_i : (R/m_i)^d \twoheadrightarrow m_i/m_i^2$ for each i . We associate $(J, \omega_J) := \sum_1^r (m_i, \omega_i) \in G_1$. Let S_1 be the set of elements (J, ω_J) of G_1 for which ω_J is a global orientation and H_1 be the subgroup of G_1 generated by S_1 . The Euler class group $E^d(R)$ is defined as $E^d(R) \stackrel{\text{def}}{=} G_1/H_1$.

Definition 2.3. (The Euler class group $\tilde{E}^d(R)$: R Noetherian) Let R be a commutative Noetherian ring of dimension $d \geq 2$. Let G_2 be the free abelian group on the set of pairs $(\mathcal{N}, \omega_{\mathcal{N}})$ where \mathcal{N} is an \mathcal{M} -primary ideal for some maximal ideal \mathcal{M} of height d such that $\mathcal{N}/\mathcal{N}^2$ is generated by d elements and $\omega_{\mathcal{N}}$ is a local orientation of \mathcal{N} . Now let $J \subset R$ be an ideal of height d such that J/J^2 is generated by d elements and ω_J be a local orientation of J . Let $J = \cap_i \mathcal{N}_i$ be the (irredundant) primary decomposition of J . We associate to the pair (J, ω_J) , the element $\sum_i (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ of G_2 where $\omega_{\mathcal{N}_i}$ is the local orientation of \mathcal{N}_i induced by ω_J . By abuse of notation, we denote $\sum_i (\mathcal{N}_i, \omega_{\mathcal{N}_i})$ by (J, ω_J) . Let H_2 be the subgroup of G_2 generated by the set of pairs (J, ω_J) , where J is an ideal of height d and ω_J is a global orientation of J . The Euler class group of R is $\tilde{E}^d(R) \stackrel{\text{def}}{=} G_2/H_2$.

To explain that the above two notions coincide when R is a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k , we have the following proposition.

Proposition 2.4. *Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k . Then $E^d(R) \simeq \tilde{E}^d(R)$.*

Proof. The map which sends each $(\mathfrak{m}, \omega_{\mathfrak{m}})$ to itself gives rise to a morphism from G_1 to G_2 . Since this morphism takes H_1 inside H_2 , we get a canonical morphism $\phi : E^d(R) \rightarrow \tilde{E}^d(R)$. As remarked in [BRS 1, Remark 4.16], an element of $E^d(R)$ is represented by a single Euler cycle (J, ω_J) , where J is a reduced ideal of height d and ω_J is a local orientation. If $\phi((J, \omega_J)) = (J, \omega_J) = 0$ in $\tilde{E}^d(R)$, then by [BRS 3, Theorem 4.2] it follows that ω_J is a global orientation and therefore, $(J, \omega_J) = 0$ in $E^d(R)$. Therefore, ϕ is injective. Now let (I, ω_I) be an element of $\tilde{E}^d(R)$. As k is infinite and perfect, we

can apply Swan's Bertini theorem [BRS 2, Theorem 2.11] to find $(I', \omega_{I'}) \in \tilde{E}^d(R)$ such that $I + I' = R$, I' is reduced of height d , and $(I, \omega_I) + (I', \omega_{I'}) = 0$ in $\tilde{E}^d(R)$ (see [DRS, 2.7, 2.8] for a proof). As $(I', \omega_{I'})$ is in the image of ϕ , it follows that ϕ is surjective. \square

From now on, both the groups will be denoted by $E^d(R)$.

3. THE EULER CLASS OF A PROJECTIVE MODULE

Let R be a commutative Noetherian ring of dimension $d \geq 2$. Let P be a projective R -module of rank d such that $R \simeq \wedge^d(P)$ and let $\chi : R \xrightarrow{\sim} \wedge^d P$ be an isomorphism. Let $\varphi : P \twoheadrightarrow J$ be a surjection where J is an ideal of height d . Therefore we obtain an induced surjection $\bar{\varphi} : P/JP \twoheadrightarrow J/J^2$. As $\dim(R/J) = 0$, we see that P/JP is a free R/J -module of rank d . Let $\bar{\gamma} : (R/J)^d \simeq P/JP$ be an isomorphism such that $\wedge^d(\bar{\gamma}) = \bar{\chi}$. Let ω_J be the local orientation of J given by $\bar{\varphi} \bar{\gamma} : (R/J)^d \twoheadrightarrow J/J^2$. Let $e_d(P, \chi)$ be the image in $E^d(R)$ of the element (J, ω_J) . The Euler class of (P, χ) is defined to be $e_d(P, \chi)$, provided, the assignment sending the pair (P, χ) to the element $e_d(P, \chi)$ of $E^d(R)$ is independent of the choice of the surjection $\varphi : P \twoheadrightarrow J$.

Question 3.1. Is $e_d(P, \chi)$, as described above, well-defined?

We do not know the complete answer yet, when R is just commutative Noetherian. Before showing that $e_d(P, \chi)$ is well-defined for certain classes of rings, we first make the following definition.

Definition 3.2. A ring R is said to have property **E** if every projective $R[T]$ -module is extended from R .

In this context, we may recall the following conjecture.

Conjecture 3.3. (Bass-Quillen) Any regular ring has property **E**.

We shall comment more on this later. Let us also recall a very useful result by Roitman [Ro, Proposition 2].

Theorem 3.4. Let R be a ring and S be a multiplicative subset of R . If R has property **E** then so does R_S .

We shall need the following "moving lemma". The version given below can easily be proved following [BRS 4, 2.4], which in turn is essentially based on [BRS 3, 2.14].

Lemma 3.5. Let A be a Noetherian ring of dimension d and let $J \subset A$ be an ideal of height n such that $2n \geq d + 1$. Let P be a projective A -module of rank n and $\bar{\alpha} : P/JP \twoheadrightarrow J/J^2$ be a surjection. Then, there exists an ideal J' of A and a surjection $\beta : P \twoheadrightarrow J \cap J'$ such that:

- (1) $J + J' = A$,

- (2) $\beta \otimes A/J = \bar{\alpha}$,
- (3) $\text{ht}(J') \geq n$.

Given any ideal $K \subset A$ of height n , the map β can be chosen so that $J' + K = A$.

Remark 3.6. In the above lemma, if $n = d$ and if A is a geometrically reduced affine algebra over an infinite field k , then using Swan's Bertini Theorem one can ensure that the ideal J' (if proper) is reduced.

The proof of the following theorem is based on that of [BRS 4, 5.2].

Theorem 3.7. *Let A be a Noetherian ring of dimension $d \geq 2$ with the following assumptions:*

- (a) if $d = 2$: no restriction;
- (b) if $d = 3$: either A is a $\mathbb{Z}[\frac{1}{2}]$ -algebra, or A is regular;
- (c) $d \geq 4$: either A is a $\mathbb{Z}[\frac{1}{(d-1)!}]$ -algebra, or A has property **E**.

Let P be a projective A -module of rank d and let $\alpha(T) : P[T] \twoheadrightarrow I$ be a surjection where I is an ideal of $A[T]$ height d . Assume that $J = I(0)$ is a proper ideal, and further that P/NP is free, where $N = (I \cap A)^2$. Let $p_1, \dots, p_d \in P$ be such that their images in P/NP form a basis. Let $a_1, \dots, a_d \in J$ be such that $\alpha(0)(p_i) = a_i$. Then, there exists an ideal $K \subset A$ of height $\geq d$ such that $K + N = A$ and:

- (1) $I \cap K[T] = (F_1(T), \dots, F_d(T))$,
- (2) $F_i(0) - F_i(1) \in K^2, i = 1, \dots, d$,
- (3) $\alpha(T)(p_i) - F_i(T) \in I^2, i = 1, \dots, d$,
- (4) $F_i(0) - a_i \in J^2, i = 1, \dots, d$.

Proof. The proof follows [BRS 4] up to a certain part. Then, modifications are only needed in [BRS 4, page 151, second paragraph onward]. However, we write the whole proof for completeness.

Since $\alpha(0)(P/NP) = J/NJ$, it follows that $J = (a_1, \dots, a_d) + NJ$. Since $NJ \subset J^2$, there exists $c \in NJ$ such that $J = (a_1, \dots, a_d, c)$. Using a standard general position argument and Lemma 3.5, we can find $b_1, \dots, b_d \in A$ such that:

- (a) $(a_1 + cb_1, \dots, a_d + cb_d) = J \cap K$, where $\text{ht}(K) \geq d$;
- (b) $K + (c) = A$ and therefore, $K + N = A = K + J$.

As $c \in NJ$, we have, $c = \sum c_j d_j$, where $c_j \in N$ and $d_j \in J$. Let $q_j \in P$ be such that $\alpha(0)(q_j) = d_j$ (the map $\alpha(0) : P \rightarrow J$ is surjective). Let $\tilde{p}_i = p_i + b_i \sum c_j q_j$. Then $\tilde{p}_i - p_i \in NP$. Let $\alpha(T)(\tilde{p}_i) = G_i(T) \in I$. We record that $G_i(0) = a_i + cb_i$.

Let $I' = I \cap K[T]$. Then $I'/I'^2 = I/I^2 \oplus K[T]/K^2[T]$ (as $I + K[T] = A[T]$), and $I'(0) = J \cap K = (a_1 + cb_1, \dots, a_d + cb_d)$.

As P/NP is free, it follows that P_{1+N} is a free A_{1+N} -module with basis $\tilde{p}_1, \dots, \tilde{p}_d$. We have,

$$I'_{1+N} = I_{1+N} = (G_1(T), \dots, G_d(T)).$$

We can choose $a \in N$ such that $1 + a \in K$ and $I'_{1+a} = I_{1+a} = (G_1(T), \dots, G_d(T))$. On the other hand, $I'_a = K_a[T] = (a_1 + cb_1, \dots, a_d + cb_d)$ such that $G_i(0) = a_i + cb_i$ for $i = 1, \dots, d$.

We now split the cases.

Case 1. Assume that $d = 2$. Let $b = 1 + a$. The rows $(G_1(T), G_2(T))$ and $(a_1 + cb_1, a_2 + cb_2)$ are unimodular over the ring $A_{ab}[T]$, and they agree when T is set to zero. As any unimodular row of length two over any ring can be completed to a 2×2 with determinant one, we can find $\theta(T) \in SL_2(A_{ab}[T])$ such that $(G_1(T), G_2(T))\theta(T) = (1, 0)$. Then $(G_1(0), G_2(0))\theta(0) = (1, 0)$, implying that $(a_1 + cb_1, a_2 + cb_2)\theta(0) = (1, 0)$. Taking $\sigma(T) := \theta(T)\theta(0)^{-1}$, we observe that $\sigma(0) = id$, and $(G_1(T), G_2(T))\sigma(T) = (a_1 + cb_1, a_2 + cb_2) = (G_1(0), G_2(0))$. The rest of the arguments are the same as [BRS 4, 5.2].

Case 2. Assume that $d = 3$. Consider the two unimodular rows $(G_1(T), G_2(T), G_3(T))$ and $(a_1 + cb_1, a_2 + cb_2, a_3 + cb_3)$ over the ring $A_{a(1+aA)}[T]$. Note that $\dim(A_{a(1+aA)}) \leq 2$. If A is a $\mathbb{Z}[\frac{1}{2}]$ -algebra, then, by a result of Murthy [Ra1, Theorem 2.5], the unimodular row $(G_1(T), G_2(T), G_3(T))$ is locally completable, and therefore by Quillen's local-global principle, it is extended from $A_{a(1+aA)}$. On the other hand, if A is a regular ring, then by a result of Murthy [Mu1], the unimodular row $(G_1(T), G_2(T), G_3(T))$ is locally completable, and therefore by Quillen's local-global principle, it is extended from $A_{a(1+aA)}$. In other words, there exists $\theta(T) \in GL_3(A_{a(1+aA)}[T])$ such that

$$(G_1(T), G_2(T), G_3(T))\theta(T) = (G_1(0), G_2(0), G_3(0)) = (a_1 + cb_1, a_2 + cb_2, a_3 + cb_3)$$

Taking $\sigma'(T) := \theta(T)\theta(0)^{-1}$, we observe that $\sigma'(T) \in SL_3(A_{a(1+aA)}[T])$, $\sigma'(0) = id$, and $(G_1(T), G_2(T), G_3(T))\sigma'(T) = (a_1 + cb_1, a_2 + cb_2, a_3 + cb_3)$. We can find some b of the form $1 + \lambda a$ such that b is a multiple of $1 + a$, and some $\sigma(T) \in SL_3(A_{ab}[T])$ such that $\sigma(0) = id$ and $(G_1(T), G_2(T), G_3(T))\sigma(T) = (a_1 + cb_1, a_2 + cb_2, a_3 + cb_3)$ over the ring $A_{ab}[T]$. The rest is same as [BRS 4, 5.2].

Case 3. Assume that $d \geq 4$. Consider the unimodular rows $(G_1(T), \dots, G_d(T))$ and $(a_1 + cb_1, \dots, a_d + cb_d)$ over the ring $A_{a(1+aA)}[T]$. They agree on $T = 0$. Now, if A is a $\mathbb{Z}[\frac{1}{(d-1)!}]$ -algebra, then by [Ra2, Theorem 2.4], the row $(G_1(T), \dots, G_d(T))$ is locally completable, and therefore by Quillen's local-global principle, it is extended from $A_{a(1+aA)}$. On the other hand, if A has property **E**, then by Theorem 3.4, so does $A_{a(1+aA)}$. Therefore, $(G_1(T), \dots, G_d(T))$ is extended from $A_{a(1+aA)}$. Same line of arguments as in Case 2 will work. \square

We now prove:

Theorem 3.8. *Let R be a Noetherian ring of dimension $d \geq 2$ with the following assumptions:*

- (a) *if $d = 2$: no restriction;*
- (b) *if $d = 3$: either R is a $\mathbb{Z}[\frac{1}{2}]$ -algebra, or R is regular;*
- (c) *$d \geq 4$: either R is a $\mathbb{Z}[\frac{1}{(d-1)!}]$ -algebra, or R has property **E**.*

Let P be a projective R -module of rank d with trivial determinant. Fix an isomorphism $\chi : R \xrightarrow{\sim} \wedge^d P$. The Euler class $e_d(P, \chi)$ is well-defined.

Proof. Let $\beta : P \twoheadrightarrow J'$ be another surjection such that J' is an ideal of R of height d .

By [BRS 3, Lemma 3.0], there exists an ideal $I \subset R[T]$ of height d and a surjection $\phi(T) : P[T] \twoheadrightarrow I$ such that $I(0) = J$, $\phi(0) = \alpha$ and $I(1) = J'$, $\phi(1) = \beta$.

Let $N = (I \cap R)^2$. Then $\text{ht}(N) \geq d - 1$ and therefore, $\dim(R/N) \leq 1$. By Serre's splitting theorem, P/NP is a free R/N -module of rank d . On the other hand, by the same reasoning, $P[T]/IP[T]$ is a free $R[T]/I$ -module of rank d .

We can choose an isomorphism $\tau : (R/N)^d \xrightarrow{\sim} P/NP$ such that $\wedge^d \tau = \chi \otimes R/N$. This choice of τ gives us a basis of P/NP , which in turn induces a basis of the free module $P[T]/IP[T]$. Using this basis of $P[T]/IP[T]$ and the surjection $\phi(T) : P[T] \twoheadrightarrow I$, we obtain a surjection $\omega : (R[T]/I)^d \twoheadrightarrow I/I^2$. Note that, due to the choice of the basis, $\omega(0) = \omega_J : (R/J)^d \twoheadrightarrow J/J^2$, and $\omega(1) = \omega_{J'} : (R/J')^d \twoheadrightarrow J'/J'^2$.

Now using Theorem 3.7, we obtain an ideal K of height d with $K + J = R = K + J'$, and a surjection $\omega_K : (R/K)^d \twoheadrightarrow K/K^2$ such that

$$(J, \omega_J) + (K, \omega_K) = (J', \omega_{J'}) + (K, \omega_K) \text{ in } E^d(R).$$

This proves that $e_d(P, \chi)$ is well-defined. □

Remark 3.9. Note that, in [BRS 3] it has been proved that the Euler class $e(P, \chi)$ is well-defined if R is a commutative Noetherian ring containing \mathbb{Q} . But essentially they require that $(d - 1)!$ is invertible in R . We used that condition above. On the other hand, the Euler class is also well-defined when R is a smooth affine algebra over an infinite perfect field (the setup of [BRS 1]). We have stretched that to the condition “ R has property **E**” which will allow a bigger class of rings.

Now that the Euler class is proved to be well-defined, the following theorem can easily be established using the arguments in [BRS 1, BRS 3] verbatim.

Theorem 3.10. *Let R, P be as in the above theorem. Then, $P \simeq Q \oplus R$ for some R -module Q if and only if $e_d(P, \chi) = 0$ in $E^d(R)$.*

Remark 3.11. Let us now comment on property **E** and the Bass-Quillen conjecture. As mentioned above, if R is a regular ring of dimension 2, then by Murthy's result [Mu1], R has property **E**. Let R be a regular ring of dimension $d \geq 3$. Thanks to the works of Lindel [L] and Popescu [P], we know that R has property **E** in the following cases: (a) R contains a field, (b) The local rings of R are either unramified or excellent Henselian.

In particular, as indicated in [P], the regular rings in the following theorem have property **E** and therefore we can apply the Euler class theory to such rings.

Theorem 3.12. *Let R be a regular ring of dimension $d \geq 3$ which is either of the following:*

- (1) R is an A -algebra for some Dedekind domain A such that for every $\mathfrak{p} \in \text{Spec}(A)$ with $\mathfrak{p}R \neq R$, the ring $R/\mathfrak{p}R$ is regular and the quotient field of A/\mathfrak{p} is perfect;
- (2) In particular, for every prime integer p , either p is a unit in R , or R/pR is regular;
- (3) R is smooth and of finite type over \mathbb{Z}_p (the p -adic integers, p a prime).

Let P be a projective R -module of rank d with trivial determinant. Fix an isomorphism $\chi : R \xrightarrow{\sim} \wedge^d P$. Then, $P \simeq Q \oplus R$ for some R -module Q if and only if $e_d(P, \chi) = 0$ in $E^d(R)$.

Remark 3.13. If $d = 2$, then the above theorem also holds without any regularity assumption.

In Section 6, we shall give a much simpler criterion for a projective R module of rank d to have a free summand, where R is a d -dimensional affine \mathbb{Z} -algebra.

4. THE WEAK EULER CLASS GROUP

In this section we recall the definitions of the weak Euler class groups from [BRS 2, BRS 3]. The notation $\mu(-)$ stands for the minimal number of generators.

Definition 4.1. (The weak Euler class group $E_0^d(R)$: R smooth affine) Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k . Let G_1 be the free abelian group on the set B_1 of maximal ideals of R . Let $J = m_1 \cap \cdots \cap m_r$, where m_i are distinct maximal ideals of R . We associate $(J) := \sum_1^r (m_i) \in G_1$. Let S_1 be the set of elements (J) of G_1 such that $\mu(J) = d$. Let H_1 be the subgroup of G_1 generated by S_1 . The weak Euler class group $E_0^d(R)$ is defined as $E_0^d(R) := G_1/H_1$.

Definition 4.2. (The weak Euler class group $\tilde{E}_0^d(R)$: R Noetherian) Let R be a commutative Noetherian ring of dimension $d \geq 2$. Let G_2 be the free abelian group on the set of ideals \mathcal{N} where \mathcal{N} is an \mathcal{M} -primary ideal for some maximal ideal \mathcal{M} of height d such that $\mu(\mathcal{N}/\mathcal{N}^2) = d$. Now let $J \subset R$ be an ideal of height d such that $\mu(J/J^2) = d$. Let $J = \cap_i \mathcal{N}_i$ be the (irredundant) primary decomposition of J . We associate $(J) := \sum_i (\mathcal{N}_i)$ (in G_2) Let H_2 be the subgroup of G_2 generated by the set of elements (J) , where J is an ideal of height d and $\mu(J) = d$. The weak Euler class group of R is $\tilde{E}_0^d(R) \stackrel{\text{def}}{=} G_2/H_2$.

The same proof as in the case of the Euler class group will yield:

Proposition 4.3. *Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k . Then $E_0^d(R) \simeq \tilde{E}_0^d(R)$.*

From now on, the weak Euler class group will be denoted as $E_0^d(R)$. There is an obvious canonical surjective morphism $\psi : E^d(R) \rightarrow E_0^d(R)$ taking an Euler cycle (J, ω_J) to (J) .

Definition 4.4. (The weak Euler class of a projective R -module:) Let R be a commutative Noetherian ring of dimension $d \geq 2$. Let P be a projective R -module of rank d such that $R \simeq \wedge^d(P)$. Let $\varphi : P \rightarrow J$ be a surjection where J is an ideal of height d . The weak Euler class of P is defined as $e_d(P) := (J) \in E_0^d(R)$.

It is easy to see that $e_d(P)$ is well-defined whenever the Euler class $e_d(P, \chi)$ is so, for any fixed orientation $\chi : R \xrightarrow{\sim} \wedge^d(P)$. Therefore, for rings R as in 3.8, $e_d(P)$ is well-defined. For such rings, one can easily prove the relevant set of results as in [BRS 2, BRS 3]. We record one of them below.

Theorem 4.5. *Let R be a ring as in 3.8 and P be a projective R -module of rank d with trivial determinant. Assume that P is stably free (or, more generally, $[P] = [Q \oplus R]$ for some R -module Q of rank $d - 1$). Then, $e_d(P) = 0$.*

Proof. This is essentially contained in [BRS 2, BRS 3]. Still, let us give a sketch. Assume that P is stably isomorphic to $Q \oplus R$. By [BRS 3, 6.7], there exists an ideal J of height $\geq d$ and surjections $\alpha : P \rightarrow J$, $\beta : Q \oplus R \rightarrow J$. If $J = R$, then P has a free summand and $e_d(P, \chi) = 0$ in $E^d(R)$ for any $\chi : R \xrightarrow{\sim} \wedge^d(P)$. If J is proper, then, as $Q \oplus R$ has a free summand, it follows from [Mo, Theorem 1] that $\mu(J) = d$ and therefore, $(J) = 0$ in $E_0^d(R)$. As $e_d(P) = (J)$, the result follows. \square

5. SOME EXACT SEQUENCES

We first emphasize that, unless mentioned otherwise, in this section we take R to be commutative and Noetherian, *without any further assumption*.

Let $\dim(R) = d \geq 2$. The orbit space $Um_{d+1}(R)/E_{d+1}(R)$ has a group structure, thanks to the works of Vaserstein ($d = 2$) [SuVa] and van der Kallen ($d \geq 2$) [vdK]. The orbit space $Um_{d+1}(R)/SL_{d+1}(R)$ also has a group structure and this group is a natural quotient of $Um_{d+1}(R)/E_{d+1}(R)$. In this short section we recall the definitions of maps from these groups to the Euler class groups.

Notation. Let $(J, \omega_J) \in E^d(R)$ and let $\bar{u} \in (R/J)^*$. Choose any $\sigma \in GL_d(R/J)$ with determinant \bar{u} . The notation $(J, \bar{u}\omega_J)$ stands for the Euler cycle $(J, \omega_J\sigma)$. Note that this is well-defined as a local orientation of J is an equivalence class under the action of $SL_d(R/J)$.

5.1. Dimension two: Let $\dim(R) = 2$. Let $[a_1, a_2, a_3] \in Um_3(R)/SL_3(R)$. Using elementary transformations we may assume that the height of the ideal $J = (a_1, a_2)$ is 2. Let ω_J be the global orientation induced by (a_1, a_2) . Let $\theta : R^3 \rightarrow R$ be given by $e_i \mapsto a_i$, $1 \leq i \leq 3$, where e_1, e_2, e_3 are the standard basis vectors of R^3 . Let $P = \ker(\theta)$. Then we have a natural orientation $\chi : R \xrightarrow{\sim} \wedge^2(P)$ (see [BRS 3, p. 214]) and a straightforward computation (*op. cit.*) yields that $e(P, \chi) = (J, \overline{a_3}\omega_J)$. The association $\varphi : [a_1, a_2, a_3] \mapsto e_d(P, \chi) = (J, \overline{a_3}\omega_J)$ is proved to be a morphism of groups in [BRS 3] and the following sequence is exact:

$$(1) \quad 1 \longrightarrow Um_3(R)/SL_3(R) \xrightarrow{\varphi} E^2(R) \longrightarrow E_0^2(R) \longrightarrow 0$$

Similarly, we have a morphism $\phi' : Um_3(R)/E_3(R) \longrightarrow E^2(R)$ and an exact sequence:

$$(2) \quad Um_3(R)/E_3(R) \xrightarrow{\phi'} E^2(R) \longrightarrow E_0^2(R) \longrightarrow 0$$

Remark 5.1. The maps ϕ, ϕ' are well-defined because the Euler class $e_d(P, \chi)$ is so. Note that, as $d = 2$, we do not need any additional assumption on R for the Euler class to be well-defined.

5.2. Higher dimensions: Let $\dim(R) \geq 3$. Let $[a_1, \dots, a_{d+1}] \in Um_{d+1}(R)/E_{d+1}(R)$. As before, we may assume that the ideal $J = (a_1, \dots, a_d)$ has height d . Let ω_J be the global orientation of J induced by a_1, \dots, a_d . It has been proved in [DZ] that the association $\phi : [a_1, \dots, a_{d+1}] \mapsto (J, \overline{a_{d+1}}\omega_J)$ is well-defined, and is a morphism. Further, the following sequence of groups is exact:

$$(3) \quad Um_{d+1}(R)/E_{d+1}(R) \xrightarrow{\phi} E^d(R) \longrightarrow E_0^d(R) \longrightarrow 0$$

6. SOME RELATION WITH CANCELLATION

As mentioned above, in this section we intend to give a very straightforward criterion for a projective R module of rank d to have a free summand, where R is a d -dimensional affine \mathbb{Z} -algebra (without any smoothness assumption). This criterion is a result of the cancellative nature of the free module R^d (see [SuVa]). We shall provide some finer analysis of this phenomenon. Moreover, we think it is better to give a more general and unified treatment, in the form of following set of results.

In what follows we make the following assumption:

(*) *Let A be a commutative Noetherian ring such that for any finite type A -algebra S of dimension ≤ 2 , the free module S^2 is cancellative.*

Theorem 6.1. *Let R be a ring of $\dim(R) = d \geq 2$, which is of finite type over A , where A as in (*). Then, the canonical morphism $\psi : E^d(R) \longrightarrow E_0^d(R)$ is an isomorphism.*

Proof. We first assume that $\dim(R) = d = 2$. In this case, by hypothesis, R^2 is cancellative and therefore, $Um_{d+1}/SL_{d+1}(R)$ is trivial. The exact sequence 1 from Section 5 gives us the desired result.

We now assume that $d \geq 3$. As ψ is already surjective, we need to check its injectivity. Let $(J, \omega_J) \in E^d(R)$ be such that $\psi(J, \omega_J) = (J) = 0$ in $E_0^d(R)$. Then, from the exact sequence 3 in Section 5, it follows that there is a unimodular row $[a_1, \dots, a_{d+1}]$ such that:

- (1) $K := (a_1, \dots, a_d)$ has height d ;
- (2) $(K, \overline{a_{d+1}}\omega_K) = (J, \omega_J)$, where ω_K is the global orientation induced by a_1, \dots, a_d .

Therefore, it is enough to prove that $(K, \overline{a_{d+1}}\omega_K) = 0$. Note that we can also ensure that $\text{ht}(a_1, \dots, a_{d-2}) = d - 2$. Write $S = R/(a_1, \dots, a_{d-2})$.

Now $\overline{a_{d+1}}\omega_K$ is induced by $K = (a_1, \dots, a_{d+1}a_d) + K^2$. Let tilde denote reduction modulo (a_1, \dots, a_{d-2}) . Then, in S we have $\tilde{K} = (\tilde{a}_{d-1}, \tilde{a}_d)$ and

$$\tilde{K} = (\tilde{a}_{d-1}, \tilde{a}_{d+1}\tilde{a}_d) + \tilde{K}^2.$$

If $\dim(S) = 2$, by hypothesis we have S^2 is cancellative and arguing as in the first paragraph of this proof, we obtain that $\tilde{K} = (\tilde{b}, \tilde{c})$ such that $\tilde{b} - \tilde{a}_{d-1} \in \tilde{K}^2$, and $\tilde{c} - \tilde{a}_{d+1}\tilde{a}_d \in \tilde{K}^2$. We can then find suitable preimages b, c in K so that $K = (a_1, \dots, a_{d-2}, b, c)$ with $b - a_{d-1} \in K^2$ and $c - a_{d+1}a_d \in K^2$. This proves that $(K, \overline{a_{d+1}}\omega_K) = 0$. The case when $\dim(S) \leq 1$ follows from standard general position arguments. \square

For the following result, we cannot invoke the Euler class of a projective module. However, the underlying ideas are from the theory of Euler classes (as in [BRS 3, 3.3, 3.4]).

Theorem 6.2. *Let R be a ring of dimension $d \geq 2$, which is of finite type over A , where A as in (*). Let P be a projective R -module of rank d with $\wedge^d(P) \xrightarrow{\sim} R$. Assume that there is an R -linear surjection $\alpha : P \twoheadrightarrow J$ such that J is generated by d elements. Then $P \simeq Q \oplus R$ for some R -module Q .*

Proof. Let $\chi : R \xrightarrow{\sim} \wedge^d(P)$ be an isomorphism. Observe that P/JP is a free R/J -module of rank d . We choose an isomorphism $\sigma : (R/J)^d \xrightarrow{\sim} P/JP$ such that $\wedge^d(\sigma) = \chi \otimes_R R/J$. Write $\bar{\alpha} = \alpha \otimes_R R/J$. Let $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ be the composite:

$$(R/J)^d \xrightarrow{\sigma} P/JP \xrightarrow{\bar{\alpha}} J/J^2$$

Let ω_J correspond to $J = (b_1, \dots, b_d) + J^2$. On the other hand, it is given that J is generated by d elements. This implies that $(J) = 0$ in $E_0^d(R)$, and by the above theorem, $(J, \omega_J) = 0$ in $E^d(R)$.

Case 1. Let $d = 2$. In this case, the Euler class of a projective module is well-defined by Theorem 3.8 and we have $e_2(P, \chi) = (J, \omega_J) = 0$. Therefore, $P \simeq Q \oplus R$ for some R -module Q .

Case 2. Let $d \geq 3$. We have, $(J, \omega_J) = 0$ in $E^d(R)$. Therefore, there exist $c_1, \dots, c_d \in J$ such that $J = (c_1, \dots, c_d)$ and $b_i \equiv c_i \pmod{J^2}$ for $i = 1, \dots, d$. As the height of J is d , we can find $\lambda_1, \dots, \lambda_{d-1} \in R$ such that $\dim(R/(c'_1, \dots, c'_{d-1})) \leq 1$, where $c'_i = c_i + \lambda_i c_d$ for $i = 1, \dots, d-1$. Set $b'_i = b_i + \lambda_i b_d$ for $i = 1, \dots, d-1$. Therefore, we have, $J = (c'_1, \dots, c'_d)$ and $b'_i \equiv c'_i \pmod{J^2}$ for $i = 1, \dots, d$. Observe that the operations just performed correspond to an elementary matrix. We can then alter σ by this elementary matrix to obtain σ' so that the composite ω'_J

$$(R/J)^d \xrightarrow{\sigma'} P/J^2 \xrightarrow{\bar{\alpha}} J/J^2$$

gives rise to $J = (b'_1, \dots, b'_d) + J^2$. The upshot of this series of arguments is that, without loss of generality, we may assume that $\dim(R/(c_1, \dots, c_{d-1})) \leq 1$ to start with. We assume this and proceed.

Consider the polynomial algebra $R[T]$ and the following ideal in $R[T]$:

$$I = (c_1, \dots, c_{d-1}, T + c_d).$$

As $\dim(R[T]/I) = \dim(R/(c_1, \dots, c_{d-1})) \leq 1$, the projective $R[T]/I$ -module $P[T]/IP[T]$ is free of rank d . We choose an isomorphism $\theta : P[T]/IP[T] \xrightarrow{\sim} (R[T]/I)^d$ such that $\wedge^d \theta = \chi(T)^{-1} \otimes_{R[T]} R[T]/I$. Substituting $T = 0$ we observe that $\theta(0)^{-1}$ and σ differ by an automorphism $\bar{\delta} \in SL_d(R/J)$. But $SL_d(R/J) = E_d(R/J)$ and we can lift $\bar{\delta}$ to $\delta \in E_d(R)$. Taking $\theta' := \delta(T)\theta : P[T]/IP[T] \xrightarrow{\sim} (R[T]/I)^d$ we ensure that $\theta'(0)^{-1} = \sigma$.

Let $\beta : R[T]^d \twoheadrightarrow I$ be the surjection which sends e_i to c_i for $i = 1, \dots, d-1$, and e_d to $T + c_d$. We then have the induced surjection

$$\gamma := \bar{\beta}\theta : P[T]/IP[T] \twoheadrightarrow I/I^2.$$

Observe that $\gamma(0) = \bar{\alpha}$. We can now apply a result of Mandal [Ma, Theorem 2.1] to obtain a surjection $\eta : P[T] \twoheadrightarrow I$. Substituting $T = 1 - c_d$, we are done. \square

Corollary 6.3. *Let R be a ring of dimension $d \geq 2$ and P be a projective R -module of rank d with trivial determinant. Assume that there is an R -linear surjection $\alpha : P \twoheadrightarrow J$ such that J is an ideal of height d which is generated by d elements. Then $P \xrightarrow{\sim} Q \oplus R$ for some Q in the following cases:*

- (1) R is of finite type over \mathbb{Z} .
- (2) R is of finite type over a field k which satisfies one of the following:
 - (a) $p \neq \text{char}(k)$, $c.d._p(k) \leq 1$;
 - (b) $p = \text{char}(k)$ and k is perfect;

Proof. (1) Take $A = \mathbb{Z}$ and note that A satisfies the condition (*) due to a result of Suslin-Vaserstein [SuVa, Corollary 18.1, Theorem 18.2].

(2) Take $A = k$ as above and note that A satisfies the condition (*) due to a result of Suslin [Su, Theorem 2.4], modified by Parimala [Ra-vdK, Theorem 3.1]. \square

With the notations set up at the beginning of this section, one can easily prove the following “addition” and “subtraction” principles.

Theorem 6.4. *Let R be a ring of dimension $d \geq 2$, which is of finite type over A (where A as in (*)). Let I and J be two comaximal ideals of R , each of height d . If two of I , J , $I \cap J$ are generated by d elements, then so is the third.*

7. THE EULER CLASS GROUP $E^d(R[T])$

We may recall that in [D1], the d^{th} Euler class group $E^d(R[T])$ has been defined where R is a commutative Noetherian ring of dimension $d \geq 3$ containing the field of rationals. Recall also that for a Noetherian ring A of dimension δ , and an integer $n \geq \frac{1}{2}(\delta + 3)$, the n -th Euler class group $E^n(A)$ had already been defined in [BRS 4]. Therefore, if one takes a Noetherian ring R of dimension $d \geq 4$, the theory of $E^d(R[T])$ is available from [BRS 4] by taking $A = R[T]$, $\delta = d + 1$, and $n = d$. This was precisely done in [MaY]. However, this approach has a couple of disadvantages. First of all, it leaves out the case $d = 3$. Secondly, it does not allow one to define the Euler class of a projective $R[T]$ -module P of rank d (together with $\chi : R[T] \xrightarrow{\sim} \wedge^d(P)$). In order to define the Euler class of (P, χ) , one has to define an Euler cycle $(I, \omega_I) \in E^d(R[T])$ with the *local orientation* ω_I as an equivalence class of surjections induced by the action of $SL_d(R[T]/I)$ on all surjections from $(R[T]/I)^d \rightarrow I/I^2$. We did so in [D1]. In comparison, the definition in [BRS 4] uses the action of the elementary group $E_d(R[T]/I)$ (which may be a proper subgroup of $SL_d(R[T]/I)$).

In concurrence with [D1], we shall define $E^d(R[T])$ in this section where R is as described in the remark below.

Remark 7.1. In what follows, our hypothesis for the ring R will be as follows. We consider R to be a regular ring of dimension $d \geq 3$ with either of the following assumptions:

- (1) R is smooth and of finite type over a finite field k ; or
- (2) R contains an infinite field k .

Note that R has property **E** (see Definition 3.2).

Remark 7.2. Let $I \subset R[T]$ be an ideal of height d . If R contains an infinite field k , then by [BRS 1, Lemma 3.3], there exists $\lambda \in k$ such that $I(\lambda) = R$ or $\text{ht}(I(\lambda)) = d$. Changing T by $T - \lambda$, without loss of generality, one can assume that $I(0) = R$ or $\text{ht}(I(0)) = d$.

We used it heavily in [D1]. Here, when R is of finite type over a finite field, we have to use some different arguments than [D1]. We illustrate one such key situation below for the convenience of the reader.

Theorem 7.3. *Let R be as in 7.1. Let $I \subset R[T]$ be an ideal of height d with $\mu(I/I^2) = d$. Let α, β be two surjections from $(R[T]/I)^d$ to I/I^2 be such that there exists $\sigma \in SL_d(R[T]/I)$ with $\alpha\sigma = \beta$. Assume that α can be lifted to a surjection $\theta : R[T]^d \rightarrow I$. Then, β can also be lifted to a surjection $\phi : R[T]^d \rightarrow I$.*

Proof. We first show that we can lift β to a surjection $\gamma : R[T]^d \rightarrow I/(I^2T)$. We tackle two cases separately.

Case 1. We assume that R is smooth and of finite type over a finite field k .

If $I(0) = R$, then by [BRS 1, 3.9] we can lift β to a surjection $\gamma : R[T]^d \rightarrow I/(I^2T)$. We now show that we can do the same if $I(0)$ is proper. Note that $\text{ht}(I(0)) \geq d - 1$ and therefore, $\dim(R/I(0)) \leq 1$. We have $\alpha(0) : (R/I(0))^d \rightarrow I(0)/I(0)^2$, $\beta(0) : (R/I(0))^d \rightarrow I(0)/I(0)^2$ and $\alpha(0)\sigma(0) = \beta(0)$. Now, by [SuVa, Remark after 16.2], $K_1Sp(R/I(0)) = SK_1(R/I(0))$ whereas, by [Sw, Theorem 9.11] we have $K_1Sp(R/I(0)) = 0$. Therefore, $SK_1(R/I(0)) = 0$. Since by [Sw, Theorem 9.1], stable rank of $R/I(0) \leq 2$, and $d \geq 3$, it follows that $SL_d(R/I(0)) = E_d(R/I(0))$. As $E_d(R) \rightarrow E_d(R/I(0))$ is surjective, we can find $\tau \in E_d(R)$ which is a lift of $\sigma(0)$. Clearly, $\theta(0)\tau : R^d \rightarrow I(0)$ is a lift of $\beta(0)$. By [BRS 1, 3.9], β can be lifted to a surjection $\gamma : R[T]^d \rightarrow I/(I^2T)$.

Case 2. We assume that R contains an infinite field k . Using [BRS 1, Lemma 3.3], if necessary, we can replace T by $T - \lambda$ for a suitable $\lambda \in k$ so that $I(0) = R$ or $\text{ht}(I(0)) = d$. Then, $\dim(R/I(0)) = 0$ and therefore $SL_d(R/I(0)) = E_d(R/I(0))$. Rest is same as the last part of Case 1.

From both the cases, we have $\gamma : R[T]^d \rightarrow I/(I^2T)$ which is a lift of β . We now move to $R(T)$ and note that $\dim(R(T)/IR(T)) = 0$. Therefore, we have $SL_d(R(T)/IR(T)) = E_d(R(T)/IR(T))$ and the matrix $\sigma \otimes R(T)$ has a lift to $\sigma' \in E_d(R(T))$. Then, $(\theta \otimes R(T))\sigma' : R(T) \rightarrow IR(T)$ is a lift of $\beta \otimes R(T)$. We can now apply [BK, Proposition 4.9] to conclude that β has a lift to a surjection $\phi : R[T]^d \rightarrow I$. \square

Using [BK, Proposition 4.9] and the addition, subtraction principles from [DRS], it is easy to prove the following addition and subtraction principles. Basic line of arguments are the same as in [D1, Propositions 4.2, 4.3].

Theorem 7.4. *(Addition principle) Let R be as in 7.1. Let $I, J \subset R[T]$ be two comaximal ideals, each of height d . Let $I = (f_1, \dots, f_d)$ and $J = (g_1, \dots, g_d)$. Then $I \cap J = (h_1, \dots, h_d)$ where $h_i - f_i \in I^2$ and $h_i - g_i \in J^2$ for $i = 1, \dots, d$.*

Theorem 7.5. (*Subtraction principle*) Let R be as in 7.1. Let $I, J \subset R[T]$ be two comaximal ideals, each of height d . Let $I = (f_1, \dots, f_d)$ and $I \cap J = (h_1, \dots, h_d)$ where $h_i - f_i \in I^2$ for $i = 1, \dots, d$. Then $J = (g_1, \dots, g_d)$ with $h_i - g_i \in J^2$ for $i = 1, \dots, d$.

Let $I \subset R[T]$ be an ideal of height d such that $\mu(I/I^2) = d$. The group $SL_d(R[T]/I)$ acts on the set of all surjections $(R[T]/I)^d \rightarrow I/I^2$. By a *local orientation* of I we mean an equivalence class under that action. Theorem 7.3 tells us that if a surjection $\alpha : (R[T]/I)^d \rightarrow I/I^2$ has a lift to surjection $R[T]^d \rightarrow I$, then the same happens for any β in the equivalence class of α . A local orientation $[\alpha]$ is called *global* if α has a lift to a surjection $R[T]^d \rightarrow I$. As before, by abuse of notations, we shall identify a surjection $\alpha : (R[T]/I)^d \rightarrow I/I^2$ with its equivalence class.

Definition 7.6. (**The Euler class group** $E^d(R[T])$): Let R be as in 7.1. Let G be the free abelian group on the set B of pairs $(\mathcal{I}, \omega_{\mathcal{I}})$ where $\mathcal{I} \subset R[T]$ is an ideal of height d such that $\text{Spec}(R[T]/\mathcal{I})$ is connected, $\mathcal{I}/\mathcal{I}^2$ is generated by d elements and $\omega_{\mathcal{I}} : (R[T]/\mathcal{I})^d \rightarrow \mathcal{I}/\mathcal{I}^2$ is a local orientation of \mathcal{I} . Let $I \subset R[T]$ be an ideal of height d and ω_I a local orientation of I . Now I can be decomposed uniquely as $I = \mathcal{I}_1 \cap \dots \cap \mathcal{I}_r$, where the \mathcal{I}_i 's are ideals of $R[T]$ of height d , pairwise comaximal and $\text{Spec}(R[T]/\mathcal{I}_i)$ is connected for each i . Clearly ω_I induces local orientations $\omega_{\mathcal{I}_i}$ of \mathcal{I}_i for $1 \leq i \leq r$. By (I, ω_I) we mean the element $\Sigma(\mathcal{I}_i, \omega_{\mathcal{I}_i})$ of G . Let H be the subgroup of G generated by set S of pairs (I, ω_I) , where I is an ideal of $R[T]$ of height d generated by d elements and ω_I is a global orientation of I given by the set of generators of I . We define the d^{th} Euler class group of $R[T]$, denoted by $E^d(R[T])$, to be G/H .

We now state a lemma from [K, 4.1].

Lemma 7.7. Let G be a free abelian group with basis $B = (e_i)_{i \in \mathcal{I}}$. Let \sim be an equivalence relation on B . Define $x \in G$ to be "reduced" if $x = e_1 + \dots + e_r$ and $e_i \not\sim e_j$ for $i \neq j$. Define $x \in G$ to be "nicely reduced" if $x = e_1 + \dots + e_r$ is such that $e_i \not\sim e_j$ for $i \neq j$. Let $S \subset G$ be such that

- (1) Every element of S is nicely reduced.
- (2) Let $x, y \in G$ be such that each of $x, y, x + y$ is nicely reduced. If two of $x, y, x + y$ are in S , then so is the third.
- (3) Let $x \in G \setminus S$ be nicely reduced and let $\mathcal{J} \subset \mathcal{I}$ be finite. Then there exists $y \in G$ with the following properties : (i) y is nicely reduced; (ii) $x + y \in S$; (iii) $y + e_j$ is nicely reduced $\forall j \in \mathcal{J}$.

Let H be the subgroup of G generated by S . If $x \in H$ is nicely reduced, then $x \in S$.

We are now ready to prove:

Theorem 7.8. Let R be as in 7.1. Let I be any ideal of $R[T]$ of height d such that $\mu(I/I^2) = d$, and $\omega_I : (R[T]/I)^d \rightarrow I/I^2$ be a local orientation. Assume that the image of (I, ω_I) is

trivial in $E^d(R[T])$. Then ω_I is global. In other words, there is an $R[T]$ -linear surjection $\theta : (R[T]/I)^d \twoheadrightarrow I$ such that θ lifts ω_I .

Proof. We take G to be the free abelian group generated by B , as defined in (7.6). Define a relation \sim on B as: $(K, \omega_K) \sim (K', \omega_{K'})$ if $K = K'$. Then it is an equivalence relation.

Let $S \subset G$ be as in (7.6). In view of the above lemma, it is enough to show that the three conditions in (7.7) are satisfied. Condition (1) is clear, almost from the definition. The addition and subtraction principles (7.4, 7.5) will yield condition (2). Finally, applying the moving lemma [D1, Lemma 2.12], it is clear that (3) is also satisfied. \square

7.1. Homotopy invariance. Let F be a functor from the category of commutative Noetherian rings to the category of groups. Then, F is said to enjoy *homotopy invariance* if for a regular ring R , $F(R) \xrightarrow{\sim} F(R[T])$. Although the Euler class groups lack nice functorial properties, we can nevertheless ask whether the homotopy invariance holds. It is easy to see that the obvious canonical morphism $\varphi : E^d(R) \rightarrow E^d(R[T])$ is injective. The best result that we have is the following:

Theorem 7.9. *Let R be a regular ring of dimension $d \geq 3$ which is essentially of finite type over an infinite field k . Then the canonical morphism $\varphi : E^d(R) \rightarrow E^d(R[T])$ is an isomorphism.*

Proof. The same method as in [D1, Proposition 5.7] works. Instead of [BRS 1, Theorem 3.8], one has to use [D2, Theorem 4.12], as we did not assume k to be perfect here. \square

Question 7.10. Let R be smooth and of finite type over a finite field k . Is the canonical morphism $\varphi : E^d(R) \rightarrow E^d(R[T])$ an isomorphism?

8. TWO RESULTS OF ASOK-FASEL

Let $X = \text{Spec}(R)$ be a smooth affine k -algebra of dimension $d \geq 2$ where k is an infinite perfect field. As we have already collected the necessary results in the previous sections, here we shall just remark on the changes required in the arguments of [AF, Section 3].

We first rewrite a crucial result from [AF, Lemma 3.1.11]. We stick to the notations in their article and we do not explain any notation or terminology. This lemma is about the consistency of the definition of a map $\tilde{\psi}_d : Q_{2d}(R) \rightarrow E^d(R)$ where R is a smooth affine algebra of dimension $d \geq 3$ over an infinite perfect field k . The reader will notice that we confine ourselves to the ‘top’ case $d = n$ here, as the final results are about that setup. Also, we include the case $d = 2$ below which is not treated in [AF, Lemma 3.1.11].

Lemma 8.1. *Let $X = \text{Spec}(R)$ be a smooth affine k -scheme of dimension $d \geq 2$, where k is an infinite perfect field. Then map $\tilde{\psi}_d : Q_{2d}(R) \rightarrow E^d(R)$ is well-defined.*

Proof. There are two cases.

Case 1. Assume that $d \geq 3$. Follow the proof of [AF, Lemma 3.1.11] and at the end of the proof use Theorem 7.9.

Case 2. Assume that $d = 2$. (Following their notations) we have ω_0, ω_1 who are in the same homotopy orbit as v . Now, ω_i defines $(N_{\mu_i}, \omega_{N_{\mu_i}}) \in E^2(R)$ for $i = 0, 1$.

Following the proof, we finally have an ideal $I \subset R[T]$ of height 2 and a surjection $\omega_I : (R[T]/I)^2 \twoheadrightarrow I/I^2$ such that $I(0) = N_{\mu_0}$, $I(1) = N_{\mu_1}$ and ω_I restricted to $T = i$ gives $\omega_{N_{\mu_i}}$ for $i = 0, 1$. By [BRS 1, 4.3, 4.6], the elements $(N_{\mu_0}, \omega_{N_{\mu_0}})$ and $(N_{\mu_1}, \omega_{N_{\mu_1}})$ are equal in $E^2(R)$. \square

Remark 8.2. Ideas similar to the proof of Case 2 were used in [DTZ, Theorem 3.10].

Using the above lemma, one can extend [AF, Proposition 3.1.12] and eventually obtain the following theorem (cf. [AF, Theorem 3.1.13]).

Theorem 8.3. *Let $X = \text{Spec}(R)$ be a smooth affine k -scheme of dimension $d \geq 2$, where k is an infinite perfect field. The Segre class morphism $s : E^d(R) \rightarrow [X, Q_{2d}]_{\mathbb{A}^1}$ is an isomorphism.*

As a consequence, the following improvement of [AF, Theorem 3.2.1] is obtained, the proof being the same as in [AF]. Also note that, since we have included the case $d = 2$ in the above results, one does not need to treat this case separately as has been done in [AF].

Theorem 8.4. *Let k be an infinite perfect field of characteristic unequal to 2. Let $X = \text{Spec}(R)$ be a smooth affine k -scheme of dimension $d \geq 2$. Then the canonical morphism $E^d(R) \rightarrow \widetilde{CH}^d(X)$ is an isomorphism.*

Now we come to the comparison between the weak Euler class group $E_0^d(R)$ and the Chow group (of zero cycles) $CH^d(X)$.

Theorem 8.5. *Let k be an infinite perfect field and Let $X = \text{Spec}(R)$ be a smooth affine k -scheme of dimension $d \geq 2$. Then the canonical morphism $E_0^d(R) \rightarrow CH^d(X)$ is an isomorphism in the following cases:*

- (1) $d = 2$;
- (2) $d \geq 3$ and $\text{Char}(k) \neq 2$.

Proof. If we make the blanket assumption that $\text{Char}(k) \neq 2$, then the same proof as in [AF] works as we can use the exact sequence (2) from Section 5 for the case $d = 2$ as well.

However, for $d = 2$ one can give a direct proof without invoking the Chow-Witt group (where the characteristic assumption is required). Quite a long time ago, Bhatwadekar had told me that it is not difficult to prove that $E_0^2(R) \xrightarrow{\sim} CH^2(X)$. Somehow we never discussed his ideas. Let me try to give some arguments here.

Each of the following morphisms is surjective:

$$E_0^2(R) \twoheadrightarrow CH^2(X) \twoheadrightarrow F^2K_0(R)$$

Let $(J) \in E_0^2(R)$ be such that J is reduced of height d and its image in $CH^2(X)$ is trivial. We have $[R/J] = 0$ in $F^2K_0(R)$. Now, by [Mu2, 2.2], there is a projective R -module of rank 2 and a surjection $\alpha : P \twoheadrightarrow J$ such that

$$[P] - [R^2] = -[R/J] \text{ in } F^2K_0(R).$$

As $[R/J] = 0$, it follows that P is stably free. But we have the weak Euler class of P , $e_0(P) = (J) \in E_0^2(R)$. By Theorem 4.5, it follows that $(J) = 0$ in $E_0^2(R)$. Therefore, $E_0^2(R) \xrightarrow{\sim} CH^2(X) \xrightarrow{\sim} F^2K_0(R)$. \square

REFERENCES

- [AF] Aravind Asok and Jean Fasel, Euler class groups and motivic stable cohomotopy, preprint available at <https://arxiv.org/abs/1601.05723>.
- [B-M] J. Barge and F. Morel, Groupe de cycles orientés et classe d'Euler des fibrés vectoriels, *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000), 287-290.
- [BK] S. M. Bhatwadekar and M. K. Keshari, A question of Nori: Projective generation of ideals, *K-Theory* **28** (2003), 329-351.
- [BRS 1] S. M. Bhatwadekar and Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain and a question of Nori, *Invent. Math.* **133** (1998), 161-192.
- [BRS 2] S. M. Bhatwadekar and Raja Sridharan, Zero cycles and the Euler class groups of smooth real affine varieties, *Invent. Math.* **136** (1999), 287-322.
- [BRS 3] S. M. Bhatwadekar and Raja Sridharan, Euler class group of a Noetherian ring, *Compositio Math.* **122** (2000), 183-222.
- [BRS 4] S. M. Bhatwadekar and Raja Sridharan, On Euler classes and stably free projective modules, in: Algebra, arithmetic and geometry, Part I, II (Mumbai, 2000), 139-158, Tata Inst. Fund. Res. Stud. Math., 16, Tata Inst. Fund. Res., Bombay, 2002.
- [D1] Mrinal K. Das, The Euler class group of a polynomial algebra, *J. Algebra* **264** (2003), 582-612.
- [D2] Mrinal K. Das, On triviality of the Euler class group of a deleted neighbourhood of a smooth local scheme, *Transactions of the American Mathematical Society* **365** (2013), 3397-3411.
- [DRS] M. K. Das and Raja Sridharan, Good invariants for bad ideals, *J. Algebra* **323** (2010), 3216-3229.
- [DTZ] Mrinal K. Das, Soumi Tikader and Md. Ali Zinna, " \mathbb{P}^1 -gluing" for local complete intersections, preprint, available at: <https://arxiv.org/abs/1709.08627>.
- [DZ] Mrinal K. Das and Md. Ali Zinna, "Strong" Euler class of a stably free module of odd rank, *J. Algebra* **432** (2015), 185-204.
- [F] J. Fasel, Groupes de Chow-Witt, *Mém. Soc. Math. Fr. (N.S.)* **113** (2008): viii+197.
- [K] M. K. Keshari, Euler Class group of a Noetherian ring, M.Phil. thesis, available at : <http://www.math.iitb.ac.in/~keshari/acad.html>
- [L] H. Lindel, On the Bass-Quillen conjecture concerning projective modules over polynomial rings, *Invent. Math.* **65** (1981), 319-323.
- [Ma] S. Mandal, Homotopy of sections of projective modules, *J. Algebraic Geometry* **1** (1992), 639-646.

- [MaY] S. Mandal and Yong Yang, Excision in algebraic obstruction theory, *J. Pure Appl. Algebra* **216** (2012), 2159-2159.
- [Mo] N. Mohan Kumar, Some theorems on generation of ideals in affine algebras, *Comment.Math.Helv.* **59** (1984), 243-252.
- [Mu1] M. P. Murthy, Projective $A[X]$ -modules, *J. London Math. Soc.* **41** (1966), 453-456.
- [Mu2] M. P. Murthy, Zero cycles and projective modules, *Ann. Math.* **140** (1994), 405-434.
- [P] D. Popescu, Polynomial rings and their projective modules, *Nagoya Math. J.* **113** (1989), 121-128.
- [Ra1] R. A. Rao, Two examples of Bass-Quillen-Suslin conjectures, *Math. Ann.* **279** (1987), 227-238.
- [Ra2] R. A. Rao, The Bass-Quillen conjecture in dimension three but characteristic $\neq 2,3$ via a question of A. Suslin, *Invent. Math.* **93** (1988), 609-618.
- [Ra-vdK] R. A. Rao and van der Kallen, Improved stability for SK_1 and WMS_d of a non-singular affine algebra, K -theory (Strasbourg, 1992). Astérisque No. 226, 11, (1994), 411-420.
- [Ro] M. Roitman, On projective modules over polynomial rings, *J. Algebra* **58** (1979), 51-63.
- [Su] A. A. Suslin, Cancellation over affine varieties, *Journal of Soviet Math.* **27** (1984), 2974-2980.
- [SuVa] A. A. Suslin and N. Vaserstein, Serre's problem on projective modules over polynomial rings and Algebraic K-theory, *Math. of the USSR Izvestija* **10** (1976), 937-1001.
- [Sw] Richard G. Swan, Serre's problem. *Conference on Commutative Algebra 1975* (Queen's Univ., Kingston, Ont., 1975), pp. 10. Queen's Papers on Pure and Applied Math., **42**, Queen's Univ., Kingston, Ont., 1975.
- [vdK] Wilberd van der Kallen, A group structure on certain orbit sets of unimodular rows, *J. Algebra* **82** (1983), 363-397.

STAT-MATH UNIT, INDIAN STATISTICAL INSTITUTE, 203 B. T. ROAD, KOLKATA 700108 INDIA
 Email address: mrinal@isical.ac.in