ON A CONJECTURE OF MURTHY

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ABSTRACT. This article concerns Murthy's conjecture on complete intersections, made in 1975. The sole breakthrough on this conjecture has still been the result proved by Mohan Kumar in 1978. The conjecture is open in general. In this article we improve *Mohan Kumar's bound* when the base field is $\overline{\mathbb{F}}_p$. As an application, we prove that any local complete intersection surface in the affine space $\mathbb{A}^d_{\overline{\mathbb{F}}_p}$ is a set-theoretic complete intersection, generalizing a result of Bloch-Murthy-Szpiro.

1. INTRODUCTION

Let R be a commutative Noetherian ring. An ideal $I \subset R$ is called *efficiently generated* if $\mu(I) = \mu(I/I^2)$, where $\mu(-)$ stands for the minimal number of generators. If there is a projective R-module P of rank $= \mu(I/I^2)$ mapping onto I, then I is called *projectively generated*. Even when $\text{ht}(I) = \mu(I/I^2)$, there are examples when I is not even projectively generated. Any real maximal ideal of the real three sphere is one such.

A conjecture of M. P. Murthy asserts that if I is an ideal of a polynomial ring over a field, such examples cannot exist. Actually, Murthy did not pose this as a conjecture. From a question that Murthy asked in [19], the following conjecture has been formulated and has been colloquially known as *Murthy's complete intersection conjecture*.

Conjecture 1.1. *(Murthy) Let k be a field and let $A = k[T_1, \cdots, T_d]$ be the polynomial ring in d variables. Let $n \in \mathbb{N}$ and $I \subset A$ be an ideal such that $ht(I) = n = \mu(I/I^2)$. Then $\mu(I) = n$. In other words, I is efficiently generated.*

The conjecture is still open in general. The best known result on this conjecture is due to Mohan Kumar [17, Theorem 5], stated below. It was proved in 1978.

Theorem 1.2. [17] *Let k be a field and let $A = k[T_1, \cdots, T_d]$. Let $I \subset A$ be an ideal such that $\mu(I/I^2) = n \geq dim(A/I) + 2$. Then $\mu(I) = n$.*

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In an earlier version of this article, Theorem 1.5 (1) (the case $n = 2$) was proved under the hypothesis that $p \neq 2$. I am grateful to the referee for showing me how to remove that condition.

Briefly, Mohan Kumar's method goes as follows. It is well-known that applying a change of variables one can assume that I contains a monic polynomial in T_d with coefficients from $R := k[T_1, \cdots, T_{d-1}]$. With this observation, Mohan Kumar actually proves the following more general result:

Theorem 1.3. [17] *Let R be a commutative Noetherian ring and $I \subset R[T]$ be an ideal containing a monic polynomial. Let $\mu(I/I^2) = n \geq dim(R[T]/I) + 2$. Then I is projectively generated.*

Since projective $k[T_1, \cdots, T_d]$-modules are free by the Quillen-Suslin Theorem, Mohan Kumar obtains (1.2) as a corollary of (1.3). Later, Mandal improves (1.3) in [14], by showing that the ideal in (1.3) is in fact efficiently generated. A closer inspection of Mandal's proof shows that he essentially proves the following:

Theorem 1.4. [14] *Let R be a commutative Noetherian ring and $I \subset R[T]$ be an ideal containing a monic polynomial. Let $I = (f_1, \cdots, f_n) + I^2$, where $n \geq dim(R[T]/I) + 2$. Then there exist $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $g_i - f_i \in I^2$. In other words, the n generators of I/I^2 can be lifted to a set of n generators of I.*

Let us now highlight the impact of the above set of results in the area of projective modules and complete intersections. As an immediate application of (1.2), Mohan Kumar proved Förster's conjecture in [17]. Then he proved that any local complete intersection curve in the affine space $\mathbb{A}^d_k$ is a set-theoretic complete intersection [17], thus extending a result of Ferrand and Szpiro (who did it for $d = 3$, independently, in [7, 28]). Mandal's result from 1984 had a profound impact on the development of the theory of Euler class groups in the late 1990's.

In this article we improve Mohan Kumar's result (1.2) when $k = \overline{\mathbb{F}}_p$, in the following form (see (3.3) in the text). Recall that $\overline{\mathbb{F}}_p$ is the algebraic closure of the field of p elements, p a prime.

Theorem 1.5. *Let I be an ideal in $A = \overline{\mathbb{F}}_p[X_1, \cdots, X_d]$ with $\mu(I/I^2) = n \geq 2$. Assume that $n \geq dim(A/I) + 1$. Then $\mu(I) = n$. Moreover, any set of n generators of I/I^2 can be lifted to a set of n generators of I in the following cases:*

(1) *$n = 2 = \mu(I/I^2) = ht(I)$ and $d = 3$;*
(2) *$n \geq 3$.*

Further, for any algebraically closed field k of characteristic unequal to 2, in (3.7) we give an example of an ideal I of height 2 in $k[X_1, X_2, X_3, X_4]$ and a set of two generators of I/I^2 such that they cannot be lifted to a set of two generators of I, although $\mu(I) = 2$. This example is based on an example of V. Srinivas involving non-triviality of SK_1 for an affine surface over k. Therefore, as far as lifting of generators is concerned, the first condition stated above cannot be relaxed.

In tune with Mandal's result quoted above, we actually prove the following theorem first and then obtain (1.5) as an immediate application. See (3.2) below.

Theorem 1.6. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ and $n \geq 2$ be an integer. Let $I \subset R[T]$ be an ideal containing a monic polynomial such that $\mu(I/I^2) = n \geq dim(R[T]/I) + 1$. Then I is generated by n elements. Moreover, any set of n generators of I/I^2 can be lifted to a set of n generators of I in the following cases:*

(1) $n = 2 = \mu(I/I^2) = ht(I) = dim(R)$;
(2) $n \geq 3$.

We now illustrate two applications of our results.

Surfaces in the affine space. As we remarked above, the results of Mohan Kumar, Ferrand and Szpiro settled that any local complete intersection curve in $\mathbb{A}^d_k$ is a set-theoretic complete intersection, where k is any field. An analogous question for local complete intersection surfaces in $\mathbb{A}^d_k$ is open (see [13, 8] for some results in this direction). When $k = \overline{\mathbb{F}}_p$, Bloch-Murthy-Szpiro proved that any local complete intersection surface in $\mathbb{A}^4_{\overline{\mathbb{F}}_p}$ is a set-theoretic complete intersection. As an application of our main theorem, we extend their result and show that any local complete intersection surface in $\mathbb{A}^d_{\overline{\mathbb{F}}_p}$ $(d \geq 4)$ is a set-theoretic complete intersection (see (4.2) in the text).

The kernel of epimorphism of polynomial algebras. Consider the following question:

Question 1.7. *Let k be a field and $\phi : k[X_1, \cdots, X_n] \twoheadrightarrow k[Y_1, \cdots, Y_m]$ be a surjective k-algebra morphism and let I be the kernel of ϕ. Then is I generated by $n - m$ elements?*

When k is an algebraically closed field of characteristic zero, this is a famous open problem in Affine Algebraic Geometry. Even the case when $n = 5$ and $m = 2$ is open. For $k = \overline{\mathbb{F}}_p$, we immediately obtain the following theorem (5.2).

Theorem 1.8. *Let $\phi : \overline{\mathbb{F}}_p[X_1, \cdots, X_n] \twoheadrightarrow \overline{\mathbb{F}}_p[Y_1, \cdots, Y_m]$ be a surjective $\overline{\mathbb{F}}_p$-algebra morphism and let I be the kernel of ϕ. Assume that $n \geq 2m + 1$. Then $\mu(I) = n - m$.*

In particular, if $k = \overline{\mathbb{F}}_p$ and $m = 2$, then (1.7) has an affirmative answer for all $n \geq 3$.

2. Some assorted results

The purpose of this section is to collect various results from the literature. Quite often we would tailor them or improve them a little bit to suit our requirements in subsequent sections. We start with a result from [11] which is crucial to this paper.

Theorem 2.1. *Let R be an affine algebra of dimension one over $\overline{\mathbb{F}}_p$. Then, $SK_1(R)$ is trivial.*

Proof. See the last two paragraphs of the proof of [11, Theorem 6.4.1, page 274]. $\square$

The next result is due to Swan.

Proposition 2.2. [27, 9.10] *Let A be a ring and I be an ideal. Let $\gamma \in Sp_{2t}(A/I)$, $t \geq 1$. If the class of γ is trivial in $K_1Sp(A/I)$ and if $2t \geq sr(A) - 1$ (where $sr(-)$ means stable range), then γ has a lift $\alpha \in Sp_{2t}(A)$.*

The following corollary (of (2.1) and (2.2)) must be well-known but we did not find any suitable reference.

Corollary 2.3. *Let A be an affine algebra over $\overline{\mathbb{F}}_p$, and let $I \subset A$ be an ideal such that $dim(A/I) \leq 1$. Then, we have the following assertions.*

(1) *The canonical map $SL_n(A) \longrightarrow SL_n(A/I)$ is surjective for $n \geq 3$.*
(2) *If $dim(A) = 3$, then the canonical map $SL_2(A) \longrightarrow SL_2(A/I)$ is surjective.*

Proof. If $\dim(A/I) = 0$, then $SL_n(A/I) = E_n(A/I)$, and we are done because $E_n(A) \longrightarrow E_n(A/I)$ is surjective for $n \geq 2$. Therefore, we assume that $\dim(A/I) = 1$.

Let $n \geq 3$. Then, applying (2.1) on A/I, together with Vaserstein's stability results [29], we have $SL_n(A/I) = E_n(A/I)$ for $n \geq 3$ and we have thus proved (1).

To prove (2), we need some additional arguments. Let $\dim(A) = 3$, $\dim(A/I) = 1$, and let $\gamma \in SL_2(A/I)$ be arbitrary. By the remark following [26, 16.2], we have $K_1Sp(A/I) = SK_1(A/I)$ and therefore, by (2.1), $K_1Sp(A/I)$ is trivial. Note that $\gamma \in SL_2(A/I) = Sp_2(A/I)$. By [26, 17.3], we have $sr(A) \leq \max\{2, \dim(A)\} = 3$. We can now apply (2.2) with $t = 1$ to obtain $\alpha \in Sp_2(A)(= SL_2(A))$ which is a lift of γ. This completes the proof. $\square$

The next theorem is essentially an accumulation of results of various authors.

Theorem 2.4. *Let R be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$ and $I \subset R$ be an ideal such that $\mu(I/I^2) = d$. Suppose it is given that $I = (a_1, \cdots, a_d) + I^2$. Then, there exist $f_1, \cdots, f_d \in I$ such that $I = (f_1, \cdots, f_d)$ with $f_i - a_i \in I^2$ for $i = 1, \cdots, d$.*

Proof. This result was proved in [15] when R is reduced and in addition, R is smooth if $d = 2$. Let us first assume that R is reduced. In this case, we need only remove the smoothness assumption when $d = 2$. But it has been proved in [11] that $F^2K_0(R)$ is trivial even when R is singular and therefore the same proof as in [15] works.

Following the proof of [3, Corollary 4.6] (or its detailed version from [10, Corollary 4.13]), it is not difficult to prove that we can take R to be reduced to start with. We give a sketch below for the convenience of the reader.

Let $\mathfrak{n}$ be the nilradical of R and let bar denote reduction modulo $\mathfrak{n}$. We have,

$$\overline{I} = (\overline{a}_1, \cdots, \overline{a}_d) + \overline{I}^2 \text{ in } \overline{R}.$$

By the first paragraph, there exist $c_1, \cdots, c_d \in I$ such that $\overline{I} = (\overline{c}_1, \cdots, \overline{c}_d)$ with $\overline{c}_i - \overline{a}_i \in \overline{I}^2$ for $i = 1, \cdots, d$. As $\overline{I}/\overline{I}^2$ can be identified with $I/(I^2 + I \cap \mathfrak{n})$, we observe that $c_i - a_i \in I^2 + (I \cap \mathfrak{n})$ for $i = 1, \cdots, d$. We now consider the following fiber product diagram:

$$\begin{tikzcd}
I/(I^2 \cap \mathfrak{n}) \arrow[r] \arrow[d] & I/I^2 \arrow[d] \\
\overline{I} = I/(I \cap \mathfrak{n}) \arrow[r] & I/(I^2 + I \cap \mathfrak{n}) = \overline{I}/\overline{I}^2
\end{tikzcd}$$

The generators of I/I^2 (namely, the images of $a_1, \cdots, a_d$ in I/I^2) and the generators of $\overline{I}$ (namely, the images of $c_1, \cdots, c_d$ in $\overline{I}$) will patch to give us a set of generators of $I/(I^2 \cap \mathfrak{n})$. In other words, we can find $f_1, \cdots, f_d \in I$ such that:

(1) $I = (f_1, \cdots, f_d) + (I^2 \cap \mathfrak{n})$;
(2) $f_i - a_i \in I^2$ for $i = 1, \cdots, d$;
(3) $f_i - c_i \in I \cap \mathfrak{n}$ for $i = 1, \cdots, d$.

Therefore, $I = (f_1, \cdots, f_d) + I^2$ and $I = (f_1, \cdots, f_d) + (I \cap \mathfrak{n})$. An easy local checking ensures that $I = (f_1, \cdots, f_d)$. $\square$

We shall also need the following more general form of the above theorem. Its proof is exactly the same as the one given above.

Theorem 2.5. *Let R be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$ and L be a projective R-module of rank one. Let $I \subset R$ be an ideal of height d such that there is a surjection $\phi : L \oplus R^{d-1} \twoheadrightarrow I/I^2$. Then there is a surjection $\theta : L \oplus R^{d-1} \twoheadrightarrow I$ such that $\theta \otimes_R R/I = \phi$ (in other words, θ lifts ϕ).*

Some immediate corollaries are in order. Here we invoke a little bit of Euler class theory. Let A be a commutative Noetherian ring of dimension $d \geq 2$ and $\mathcal{L}$ be a projective A-module of rank one. For the definition of the d^{th} Euler class group $E^d(A, \mathcal{L})$ relative to $\mathcal{L}$, see [3]. From (2.5) and the definition of the Euler class group, we have:

Corollary 2.6. *Let R be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$ and L be a projective R-module of rank one. Then Euler class group $E^d(R, L)$ is trivial.*

Before deriving the next corollary, we insert a remark here.

Remark 2.7. Let A be a commutative Noetherian ring of dimension 2 and $\mathcal{L}$ be a projective A-module of rank one. Let P be a projective A-module of rank 2 with determinant $\mathcal{L}$. Fix an isomorphism $\chi : \mathcal{L} \xrightarrow{\sim} \wedge^2(P)$. Choose any surjection $\alpha : P \twoheadrightarrow J$ where $J \subset R$ is an ideal of height 2. It induces $\overline{\alpha} : P/JP \twoheadrightarrow J/J^2$. Note that $P/JP \xrightarrow{\sim} \mathcal{L}/J\mathcal{L} \oplus A/J$. Choose an isomorphism $\overline{\gamma} : \mathcal{L}/J\mathcal{L} \oplus A/J \xrightarrow{\sim} P/JP$ such that $\wedge^2 \overline{\gamma} = \chi \otimes_A A/J$. In [3], the Euler class of (P,χ) is defined as $e(P,\chi) := (J,\overline{\alpha\gamma}) \in E^2(A,\mathcal{L})$. As the dimension of the ring is 2 here, it can be easily checked, following the proof of [3, 3.1], that $e(P,\chi)$ is well-defined (one needs the additional assumption in [3, 3.1] only for $d \geq 3$). It then follows from [3, 4.4] that $e(P,\chi) = 0$ if and only if $P \stackrel{\sim}{\to} L \oplus A$.

Corollary 2.8. *Let R be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$ and P be a projective R-module of rank d. Then $P \simeq Q \oplus R$ for some R-module Q.*

Proof. For $d \geq 3$, this is proved in [18]. Let $d = 2$. In this case, R is assumed to be smooth in [18]. We can remove this assumption, as follows.

Now we assume that $d = 2$. Let $\wedge^2(P) \simeq L$. But by (2.6), $E^2(R, L)$ is trivial and therefore, the Euler class $e(P,\chi) = 0$ (see the remark above on the Euler class being well-defined). The proof is complete by [3, 4.4]. □

Remark 2.9. For a different proof of (2.8), see [11]. We believe the proof given here may be of some independent interest.

In particular, we obtain:

Corollary 2.10. *Let R be a two-dimensional affine algebra over $\overline{\mathbb{F}}_p$, and P be a projective R-module of rank $n \geq 2$ with determinant L. Then $P \simeq L \oplus R^{n-1}$. In particular, any projective R-module with trivial determinant is free.*

Proof. By a classical result of Serre [24], $P \stackrel{\sim}{\to} P' \oplus R^{n-2}$ for some projective R-module P' of rank 2. Note that the determinant of P' is isomorphic to L. Now, $P' \stackrel{\sim}{\to} L \oplus R$ by (2.8). □

The following corollary is now obvious.

Corollary 2.11. *Let R be a two-dimensional affine algebra over $\overline{\mathbb{F}}_p$, and L_1, L_2 be two projective R-modules, each of rank one. Then $L_1 \oplus L_2 \simeq R \oplus (L_1 \otimes L_2)$.*

We shall also need the following corollary. The idea of this proof lies in [9, Lemma 10.3.1].

Corollary 2.12. *Let R be a two-dimensional affine algebra over $\overline{\mathbb{F}}_p$, and P be a projective R-module of rank $n \geq 2$. Let K be any projective R-module of rank one. Then K is a direct summand of P.*

Proof. Let $\det(P) = L$. Then, by (2.10), $P \simeq R^{n-1} \oplus L$. Applying (2.11), we have

$$(L \otimes K^*) \oplus K \simeq (L \otimes K^* \otimes K) \oplus R = L \oplus R,$$

and therefore, $P \simeq R^{n-2} \oplus (L \oplus R) \simeq R^{n-2} \oplus (L \otimes K^*) \oplus K$. □

3. The main theorem

We start with a lemma of Mohan Kumar [16], recast slightly to suit our needs.

Lemma 3.1. *Let A be commutative Noetherian ring and $J \subset A$ be an ideal. Assume that $J = K + L$, where K, L are ideals and $L \subset J^2$. Then, there exist $e \in L$ such that:*

(1) *$J = (K, e)$ and $e(1-e) \in K$,*
(2) *If $J' = (K, 1-e)$, then $J \cap J' = K$ (note also that $J' + L = A$),*
(3) *For any $a \in A$, we have $(J, a) = (K, e + (1-e)a)$.*

We first prove a result on ideals containing monic polynomials. Compare it with the results of Mohan Kumar and Mandal [17, 14]. I learned slightly simplified proofs of their results from Raja Sridharan during my graduate days, and I mimic those arguments here.

Theorem 3.2. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ and $n \geq 2$ be an integer. Let $I \subset R[T]$ be an ideal containing a monic polynomial such that $\mu(I/I^2) = n \geq dim(R[T]/I) + 1$. Then I is generated by n elements. Moreover, any set of n generators of I/I^2 can be lifted to a set of n generators of I in the following cases:*

(1) *$n = 2 = \mu(I/I^2) = ht(I) = dim(R)$;*
(2) *$n \geq 3$.*

Proof. Let $f \in I$ be a monic polynomial and assume that $I = (f_1, \cdots, f_n) + I^2$. We treat two cases separately.

Case 1. Let $n = 2$. We have $I = (f_1, f_2) + I^2$. By Lemma 3.1 there exists $e \in I^2$ such that $e(1-e) \in (f_1, f_2)$. Therefore, $I_{1-e} = (f_1, f_2)_{1-e}$. On the other hand, $I_e = R[T]_e = (1, 0)$. Note that the unimodular row $(f_1, f_2)_{e(1-e)}$ over $R[T]_{e(1-e)}$ can be completed to a 2×2 matrix with determinant 1. Using a standard patching argument we obtain a projective $R[T]$-module P of rank 2 with *trivial determinant* such that P maps onto I. Since I contains the monic polynomial f, the module P_f has a unimodular element and hence becomes free. The Affine Horrocks Theorem implies that P is free. Consequently, I is generated by 2 elements, say, $I = (g_1, g_2)$.

We now show that if $\dim(R) = 2 = \text{ht}(I)$, then we can actually find $f_1', f_2' \in I$ such that $I = (f_1', f_2')$ with $f_i' - f_i \in I^2$, $i = 1, 2$. We have, $I = (f_1, f_2) + I^2$ and $I = (g_1, g_2)$ thus far. By [1, 2.2], there exists a matrix $\overline{\sigma} \in GL_2(R[T]/I)$ such that $(\overline{f}_1, \overline{f}_2) = (\overline{g}_1, \overline{g}_2)\overline{\sigma}$. Let $\det(\overline{\sigma}) = \overline{u}$. Let $v \in R[T]$ be such that $uv - 1 \in I$. Then $(v, g_2, -g_1)$ is a unimodular row over $R[T]$. Since the ideal $I = (g_1, g_2)$ contains the monic polynomial f, the unimodular row $(v, g_2, -g_1)$ is completable over $R[T]_f$, and is therefore completable over $R[T]$ by the Affine Horrocks Theorem. By [23, 2.3], there are generators h_1, h_2 of I and $\overline{\delta} \in GL_2(R[T]/I)$ such that $\det(\overline{\delta}) = \overline{u}$ and $(\overline{h}_1, \overline{h}_2) =$

$(\overline{g}_1, \overline{g}_2)\overline{\delta}$. Consequently, $(\overline{f}_1, \overline{f}_2) = (\overline{h}_1, \overline{h}_2)\overline{\tau}$ for some $\overline{\tau} \in SL_2(R[T]/I)$. We can now apply (2.3) and lift $\overline{\tau}$ to some $\tau \in SL_2(R[T])$. Write $(h_1, h_2)\tau = (f_1', f_2')$. Then f_1', f_2' is the desired set of generators for I.

Case 2. Let $n \geq 3$. Adding f^t to f_1 for suitable t we can assume that f_1 is monic. Let $J = I \cap R$ and $B = R[T]/(J^2[T], f_1)$. Then,

$$\overline{I} = (\overline{f}_2, \cdots, \overline{f}_n) + \overline{I}^2,$$

where bar means modulo $(J^2[T], f_1)$. Note that B is an affine algebra over $\overline{\mathbb{F}}_p$ and as $B \hookrightarrow R[T]/I$ is integral, we have $\dim(B) = \dim(R[T]/I) \leq n - 1$. By (2.4), there exist $h_2, \cdots, h_n \in I$ such that $\overline{I} = (\overline{h}_2, \cdots, \overline{h}_n)$. Then,

$$I = (f_1, h_2, \cdots, h_n) + J^2[T].$$

Applying (3.1) we find an ideal $I' \subset R[T]$ such that $I \cap I' = (f_1, h_2, \cdots, h_n)$ with $I' + J^2[T] = R[T]$. As I' contains a monic (namely, f_1), by [12, Lemma 1.1, Chapter III], $I' \cap R + J^2 = R$. Let $s \in J$ be such that $1 + s \in I'$. Then,

$$I_{1+s} = (f_1, h_2, \cdots, h_n)_{1+s}.$$

Let $\phi : R_{1+s}[T]^n \twoheadrightarrow I_{1+s}$ be the corresponding surjection. On the other hand, we have $I_s = R_s[T]$ and we have the surjection $\psi : R_s[T]^n \twoheadrightarrow I_s = R_s[T]$ which sends the first basis vector to 1 and the rest to zero. Now, over $R_{s(1+s)}[T]$ we have the unimodular row $(f_1, h_2, \cdots, h_n)$ whose one entry is monic. Since this row is elementarily completable by [21], a standard patching argument completes the proof. $\square$

As an immediate application, we prove the following result on Murthy's conjecture stated in the introduction. In fact, we prove a finer assertion.

Theorem 3.3. *Let I be an ideal in $A = \overline{\mathbb{F}}_p[X_1, \cdots, X_d]$ with $\mu(I/I^2) = n \geq dim(A/I) + 1$. Then $\mu(I) = n$. Moreover, any set of n generators of I/I^2 can be lifted to a set of n generators of I in the following cases:*

(1) $n = 2 = \mu(I/I^2) = ht(I)$, *and* $d = 3$;
(2) $n \geq 3$.

Proof. After a change of variables, I contains a monic polynomial in one of the variables, say, X_d. Write $R = \overline{\mathbb{F}}_p[X_1, \cdots, X_{d-1}]$ and apply the above theorem. $\square$

Remark 3.4. In (3.3) we can replace $\overline{\mathbb{F}}_p$ by a PID which is of finite type over $\overline{\mathbb{F}}_p$.

Corollary 3.5. *Let I be a non-zero ideal in $\overline{\mathbb{F}}_p[X_1, \cdots, X_d]$ with $\mu(I/I^2) = n$. Then, $\mu(I) = n$ if $n \leq 2$ or $n \geq d - 1$.*

Remark 3.6. In the above theorem (3.3) we have seen that if $n = 2 = \mu(I/I^2) = \text{ht}(I)$ and $d = 3$, then we can lift generators of I/I^2 to generators of I. In contrast, Bhatwadekar-Sridharan showed in [2, 3.15] that for the complete intersection ideal $I = (X_1^3 + X_2^3 - 1, X_3 - 1) \subset \mathbb{C}[X_1, X_2, X_3]$, there is a set of 2 generators of I/I^2 which cannot be lifted to a set of two generators of I. The crux of the matter in their example is that $SK_1(\mathbb{C}[X_1, X_2]/(X_1^3 + X_2^3 - 1))$ is non-trivial by [20, 2.5].

Example 3.7. In this example we show that, as far as lifting a given set of generators of I/I^2 to a set of generators of I is concerned, condition (1) in (3.2, 3.3) for $n = 2$ cannot be relaxed. In this sense, these results cannot be improved. The idea of this example is the same as [2, 3.15]. Let k be any algebraically closed field of characteristic unequal to 2. Let
$$B = \frac{k[X, Y, Z]}{(Z^2 - XY)}.$$
Srinivas [25] proved that $SK_1(B) \neq 0$ (In fact, he gave example of an explicit matrix in $SL_2(B)$ which is not even in the image of $SL_2(k[X, Y, Z]) \longrightarrow SL_2(B)$. His example is related to Cohn's example [5]). Write $f = Z^2 - XY$, $R = k[X, Y, Z]$, and consider the ideal $I = (f, T - 1) \subset R[T]$, where T is an indeterminate. Let bar denote reduction modulo I. We have, $R[T]/I = B$. By the aforementioned result of Srinivas [25], there is a $\overline{\sigma} \in SL_2(R[T]/I)$ such that it is not stably elementary. Let $(\overline{f}, \overline{T-1})\overline{\sigma} = (\overline{G}, \overline{H})$. Then, (i) I is an ideal of height 2, (ii) $I = (G, H) + I^2$, and (iii) I contains a monic, namely, $T - 1$. Assume, if possible, that there exist $g, h \in I$ such that $I = (g, h)$ with $g - G \in I^2$, and $h - H \in I^2$. Then, by [2, 2.14 (iv)] there exists a matrix $\sigma \in SL_2(R[T])$ *which is a lift of* $\overline{\sigma}$, and $(f, T - 1)\sigma = (g, h)$. But $\sigma \in SL_2(R[T]) = SL_2(k[X, Y, Z, T])$. As $SK_1(R[T]) = 0$, the image $\overline{\sigma}$ of σ cannot be a non-trivial element of $SK_1(B)$. This is a contradiction. Therefore, the generators $\overline{G}, \overline{H}$ of I/I^2 cannot be lifted to a set of generators of I.

4. Application I: set-theoretic generation of surfaces

In this section we derive from our main theorem in the last section that local complete intersection surfaces in the affine space $\mathbb{A}^d_{\overline{\mathbb{F}}_p}$ ($d \geq 4$) are set-theoretic complete intersections. This generalizes a result of Bloch-Murthy-Szpiro [4], who proved it for $d = 4$. In general, it is not known whether a local complete intersection surface in $\mathbb{A}^d_k$ is a set-theoretic complete intersection, when k is an arbitrary field. For some relevant results, we refer to the works of Lyubeznik [13, Theorem 6.3], and Hauber [8, Corollary 2.2].

We shall need the following refinement of Ferrand's construction for affine algebras over $\overline{\mathbb{F}}_p$. Ferrand's original construction in [7] was for local complete intersection ideals of height $d-1$ in a d-dimensional Noetherian ring. We shall follow the proof of Ferrand's result as given in [9, Theorem 10.3.3, pp 246]. Instead of repeating the whole set of arguments, we shall only indicate the refinements. Any diligent reader can consult *op. cit.* and complete the details.

Theorem 4.1. *Let R be an affine algebra of dimension $d \geq 4$ over $\overline{\mathbb{F}}_p$ and $I \subset R$ be a local complete intersection ideal of height $d-2$. Then, there is an ideal $J \subset R$ such that:*

(1) $I^2 \subset J \subset I$,
(2) *J is a local complete intersection ideal of height $d-2$,*
(3) *J/J^2 is a free R/J-module of rank $d-2$.*

Proof. We break the proof in steps.
Step 1. As I is a local complete intersection of height $d-2$, it follows that I/I^2 is a projective R/I-module of rank $d-2$. Write $K = (\wedge^{d-2} I/I^2)^*$. As $d-2 \geq 2$, applying (2.12) we conclude that K is a direct summand of I/I^2. Let $\phi : I/I^2 \twoheadrightarrow K$ be the projection map. The kernel of ϕ is then of the form J/I^2 where J is an ideal of R contained in I and containing I^2. Thus (1) is proved and we have,

$$I/I^2 \simeq J/I^2 \oplus K$$

Step 2. To prove that J is a local complete intersection. This part of the proof is exactly the same as in [9, Theorem 10.3.3].

Step 3. From (2) it follows that J/J^2 is a projective R/J-module of rank $d-2$. To show that J/J^2 is free, we first note that

$$J/IJ = J \otimes_R R/I = J \otimes_R R/J \otimes_{R/J} R/I \ = \ J/J^2 \otimes_{R/J} R/I.$$

Therefore, J/IJ is a projective R/I-module of rank $d-2$. Now it is enough to show that J/IJ is R/I-free. In view of (2.10), it is enough to show that the determinant of J/IJ is trivial. In other words, we have to show that $\wedge^{d-2}(J/IJ) \simeq R/I$. This is precisely what is proved in [9, Theorem 10.3.3]. □

We are now ready to prove that any local complete intersection surface in the affine space $\mathbb{A}^d_{\overline{\mathbb{F}}_p}$ $(d \geq 4)$ is a set-theoretic complete intersection.

Theorem 4.2. *Let $d \geq 4$ and I be a local complete intersection ideal of height $d-2$ in $R = \overline{\mathbb{F}}_p[X_1, \cdots, X_d]$. Then I is a set-theoretic complete intersection.*

Proof. By (4.1) there exists an ideal $J \subset R$ such that $\sqrt{I} = \sqrt{J}$ and $\mu(J/J^2) = d-2$.

Case 1. Assume that $d-2 = 2$. In this case, the same proof as in Case 1 of (3.2) shows that J is surjective image of a projective R-module P of rank 2. As P is free by the Quillen-Suslin Theorem, we have $\mu(J) = 2$.

Case 2. Assume that $d-2 \geq 3$. In this case, $\mu(J/J)^2 = d-2 \geq 3 \geq \dim(R/J)+1$. Applying (3.3), we conclude that $\mu(J) = d-2$. $\square$

5. APPLICATION II: KERNEL OF EPIMORPHISM OF POLYNOMIAL ALGEBRAS

In this section we address the following question:

Question 5.1. *Let k be a field and $\phi : k[X_1, \cdots, X_n] \twoheadrightarrow k[Y_1, \cdots, Y_m]$ be a surjective k-algebra morphism and let I be the kernel of ϕ. Then is I generated by $n-m$ elements?*

When k is an algebraically closed field of characteristic zero, this is a famous open problem in Affine Algebraic Geometry (see [6, Question 2.1] and the discussion surrounding it for an excellent survey).

For *any* field k, there are affirmative answers when: (1) $n = m+2$, or (2) $n \geq 2m+2$. Let us quickly indicate the solutions below. Before proceeding, note that under the hypothesis of (5.1), $\mu(I/I^2) = n-m$.

Proof of (1). $n = m+2$. Then $\mu(I/I^2) = n-m = 2$. Exactly the same argument as in (3.2) (Case 1), shows that I is surjective image of a projective $k[X_1, \cdots, X_n]$-module P of rank 2. But P is free by the Quillen-Suslin Theorem. $\square$

Proof of (2). $n \geq 2m+2$. Note that $\mu(I/I^2) = n-m \geq m+2 = \dim(k[X_1, \cdots, X_n]/I)+2$. Apply Mohan Kumar's result ((1.2) above). $\square$

When $k = \overline{\mathbb{F}}_p$, we can do a little better, as the following theorem shows.

Theorem 5.2. *Let $\phi : \overline{\mathbb{F}}_p[X_1, \cdots, X_n] \twoheadrightarrow \overline{\mathbb{F}}_p[Y_1, \cdots, Y_m]$ be a surjective $\overline{\mathbb{F}}_p$-algebra morphism and let I be the kernel of ϕ. Assume that $n \geq 2m+1$. Then $\mu(I) = n-m$.*

Proof. $\mu(I/I^2) = n-m \geq m+1 = \dim(\overline{\mathbb{F}}_p[X_1, \cdots, X_n]/I)+1$. Apply (3.3). $\square$

Remark 5.3. Let $k = \overline{\mathbb{F}}_p$ and $m = 2$. Then (5.1) has an affirmative answer for all $n \geq 3$.

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