A CRITERION FOR SPLITTING OF A PROJECTIVE MODULE IN TERMS OF ITS GENERIC SECTIONS

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ABSTRACT. Let R be a smooth affine domain of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$ with $p \neq 2$. Let P be a projective R-module of rank $d-1$ with trivial determinant. We prove that P splits off a free summand of rank one if and only if P surjects onto a complete intersection ideal of height $d-1$.

1. INTRODUCTION

Let R be a commutative Noetherian ring of (Krull) dimension d and P be a finitely generated projective R-module (of constant rank). A classical result of Serre [22] asserts that if rank$(P) \geq d+1$, then $P \simeq P' \oplus R$ for some R-module P'. Serre's result is best possible in the sense that there are examples of rings R of dimension d and projective R-modules of rank d which do not split a free summand of rank one. Since 1980's, a recurrent theme in this area has been to find the precise obstruction for a projective R-module of rank d to split a free factor. To this end, one would wonder, where should one look for such an obstruction? We digress a little here.

Let A be a commutative Noetherian ring and Q be a projective A-module. A remarkable result of Eisenbud-Evans [10] tells us that most of the A-linear maps $Q \longrightarrow A$ have the property that the image is an ideal of A of height equal to the rank of Q. We shall call an ideal $I \subset A$ a *generic section* of Q if there is a surjection $\alpha : Q \to I$ and ht$(I) =$ rank(Q).

Now let R be a commutative Noetherian ring of dimension $d \geq 2$ and P be a finitely generated projective R-module of rank d. For simplicity, let us assume that P has trivial determinant. The following result of Mohan Kumar [17, Theorem 1] gives us an indication of the necessary condition for P to split a free factor.

Theorem 1.1. *Let R be a commutative Noetherian ring of dimension d and P be a projective R-module of rank d (with trivial determinant). Let I be a generic section of P. Assume that $P \simeq P' \oplus R$. Then I is generated by d elements.*

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Let R be an affine algebra of dimension $d \geq 2$ over an algebraically closed field k and P be a projective R-module of rank d. Then Mohan Kumar [17] also proved:

Theorem 1.2. *$P \xrightarrow{\sim} Q \oplus R$ if and only if a generic section of P is generated by d elements.*

The above result of Mohan Kumar turned out to be a crucial step for the following seminal result of M. P. Murthy [19].

Theorem 1.3. *Let $X = Spec(R)$ be a smooth affine variety of dimension d over an algebraically closed field k and P be a projective R-module of rank d. Then, $P \simeq Q \oplus R$ if and only if its top Chern class $c_d(P) = 0$ in the Chow group $CH^d(X)$.*

If $P \simeq Q \oplus R$ then it easily follows that $c_d(P) = 0$. To prove the reverse implication, Murthy showed that if $c_d(P) = 0$ then P has a generic section I which is d-generated, and then he appealed to the result of Mohan Kumar stated above. Murthy's theorem establishes that the Chow group $CH^d(X)$ (of zero cycles) serves as the obstruction group and the top Chern class $c_d(P)$ of P as the precise obstruction for P to split a free summand. However, if the ground field is not algebraically closed, then this is no longer true, as the tangent bundle of real 2-sphere has trivial top Chern class but it does not have a free factor of rank one. This gave birth to the theory of the Euler class groups and the Euler classes, which were envisioned by M. V. Nori and studied in detail by Bhatwadekar-Sridharan. Given a smooth affine domain R of dimension d over an infinite perfect field k, the d^{th} Euler class group $E^d(R)$ was defined in [4]. For a projective R-module P of rank d together with an isomorphism $\chi : R \xrightarrow{\sim} \wedge^d(P)$, an element $e(P, \chi) \in E^d(R)$ was attached (called the *Euler class* of (P, χ)), and it was proved that $e(P, \chi) = 0$ if and only if $P \simeq P' \oplus R$ for some P'. Further, if $k = \overline{k}$, then the groups $E^d(R)$ and $CH^d(X)$ are isomorphic and $c_d(P)$ is the same as $e(P, \chi)$. With this development, the question of finding the obstruction for a projective module to split off a free summand was settled for modules of rank d.

However, since then, the progress for tackling projective modules of rank $\leq d - 1$ has been very slow. Bhatwadekar-Sridharan [7] extended the Euler class theory to accommodate projective modules of rank $n \geq (d+3)/2$ which are given by unimodular rows. Very recently, in pursuit of a conjecture of Murthy, Asok-Fasel [1, 2] proved that if $X = \text{Spec}(R)$ is a smooth affine variety of dimension $d (= 3, 4)$ over an algebraically closed field of characteristic unequal to 2, and if P is a projective R-module of rank $d - 1$, then $c_{d-1}(P)$ vanishes in $CH^{d-1}(X)$ if and only if $P \simeq P' \oplus R$. Their methods are quite involved and they have to employ some sophisticated machinery to prove these results, which gives an idea about the depth of the problem.

Keeping in tune with Mohan Kumar's results stated above, in this article we prove the following result when $k = \overline{\mathbb{F}}_p$ ($p \neq 2$). Recall that $\overline{\mathbb{F}}_p$ is the algebraic closure of the field of p elements, p a prime.

Theorem 1.4. *Let $p \neq 2$, and R be a smooth affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$, and P be a projective R-module of rank $d-1$ with trivial determinant. Then $P \xrightarrow{\sim} P' \oplus R$ for some R-module P' if and only if there is a generic section J of P such that J is a complete intersection.*

Let us now briefly illustrate our line of approach. Let R be a smooth affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$ (with no restriction on p). Let P be a projective R-module of rank $d-1$, together with an isomorphism $\chi : R \xrightarrow{\sim} \wedge^{d-1}(P)$. Following the footsteps of Bhatwadekar-Sridharan [4, 6, 7], we define a group $E^{d-1}(R)$ and associate to the pair (P, χ) an element $e_{d-1}(P, \chi)$ in $E^{d-1}(R)$ and prove the following result (see (4.12) for a detailed statement):

Theorem 1.5. *The element $e_{d-1}(P, \chi) = 0$ in $E^{d-1}(R)$ if and only if P splits a free summand of rank one.*

The group $E^{d-1}(R)$ is called the $(d-1)^{th}$ *Euler class group* of R and $e_{d-1}(P, \chi)$ the $(d-1)^{th}$ *Euler class* of (P, χ). Our definition of $E^{d-1}(R)$ is built on a similar definition given in [7], with a vital modification. As an application of the Euler class theory developed in Section 4, combined with the cancellation theorem of Fasel-Rao-Swan [11, 7.5], we are able to derive (1.4). In order to apply [11, 7.5] we have to assume that $p \neq 2$.

A question on threefolds: Let k be an algebraically closed field and R be an affine k-algebra of dimension 3. Let P be a projective R-module of rank 2 with trivial determinant. Obviously, any R-module in the isomorphism class of P has the same generic sections as P. In this context, we may pose the following question.

Question 1.6. *Does a generic section of P uniquely determine the isomorphism class of P? In other words, if Q is another projective R-module of rank two with trivial determinant such that P and Q have a common generic section J, then is P isomorphic to Q?*

We answer this question affirmatively when $k = \overline{\mathbb{F}}_p$, $p \neq 2$, and R is smooth. We would like to point out to the reader that apart from [11], we have not used any other cancellation results (e.g., those due to Asok-Fasel from [1]). On the other hand, as an application of our results we easily derive that the projective R-modules of rank 2 with trivial determinant are cancellative, where R is a smooth affine 3-fold over $\overline{\mathbb{F}}_p$ and $p \neq 2$.

2. SOME ASSORTED RESULTS

The purpose of this section is to collect various results from the literature. Quite often we would tailor them or improve them a little bit to suit our requirements in subsequent sections. We start with a lemma of Mohan Kumar [16], recast slightly to suit our needs.

Lemma 2.1. *Let A be commutative Noetherian ring and $J \subset A$ be an ideal. Assume that $J = K + L$, where K, L are ideals and $L \subset J^2$. Then, there exist $e \in L$ such that:*

(1) *$J = (K, e)$ and $e(1-e) \in K$,*
(2) *If $J' = (K, 1-e)$, then $J \cap J' = K$ (note also that $J' + L = A$),*
(3) *For any $a \in A$, we have $(J, a) = (K, e + (1-e)a)$.*

The following result from [13] is crucial to this paper.

Theorem 2.2. *Let R be an affine algebra of dimension one over $\overline{\mathbb{F}}_p$. Then, $SK_1(R)$ is trivial.*

Proof. See the last two paragraphs of the proof of [13, Theorem 6.4.1, page 274]. □

The next result is due to Swan.

Proposition 2.3. [27, 9.10] *Let A be a ring and I be an ideal. Let $\gamma \in Sp_{2t}(A/I)$, $t \geq 1$. If the class of γ is trivial in $K_1Sp(A/I)$ and if $2t \geq sr(A) - 1$ (where $sr(-)$ means stable range), then γ has a lift $\alpha \in Sp_{2t}(A)$.*

The following corollary (of (2.2) and (2.3)) must be well-known but we did not find any suitable reference.

Corollary 2.4. *Let A be an affine algebra over $\overline{\mathbb{F}}_p$, and let $I \subset A$ be an ideal such that dim$(A/I) \leq 1$. Then, we have the following assertions.*

(1) *The canonical map $SL_n(A) \longrightarrow SL_n(A/I)$ is surjective for $n \geq 3$.*
(2) *If dim$(A) = 3$, then the canonical map $SL_2(A) \longrightarrow SL_2(A/I)$ is surjective.*

Proof. If $\dim(A/I) = 0$, then $SL_n(A/I) = E_n(A/I)$, and we are done because $E_n(A) \longrightarrow E_n(A/I)$ is surjective for $n \geq 2$. Therefore, we assume that $\dim(A/I) = 1$.

Let $n \geq 3$. Then, applying (2.2) on A/I, together with Vaserstein's stability results [28], we have $SL_n(A/I) = E_n(A/I)$ for $n \geq 3$ and we have thus proved (1).

To prove (2), we need some additional arguments. Let $\dim(A) = 3$, $\dim(A/I) = 1$, and let $\gamma \in SL_2(A/I)$ be arbitrary. By the remark following [26, 16.2], we have $K_1Sp(A/I) = SK_1(A/I)$ and therefore, by (2.2), $K_1Sp(A/I)$ is trivial. Note that $\gamma \in SL_2(A/I) = Sp_2(A/I)$. By [26, 17.3], we have $sr(A) \leq \max\{2, \dim(A)\} = 3$. We can now apply (2.3) with $t = 1$ to obtain $\alpha \in Sp_2(A)(= SL_2(A))$ which is a lift of γ. This completes the proof. □

The next theorem is essentially an accumulation of results of various authors.

Theorem 2.5. *Let R be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$ and $I \subset R$ be an ideal such that $\mu(I/I^2) = d$. Suppose it is given that $I = (a_1, \cdots, a_d) + I^2$. Then, there exist $b_1, \cdots, b_d \in I$ such that $I = (b_1, \cdots, b_d)$ with $b_i - a_i \in I^2$ for $i = 1, \cdots, d$.*

Proof. This result was proved in [15] when R is reduced and in addition, R is smooth if $d = 2$. Let us first assume that R is reduced. In this case, we need only remove the smoothness assumption when $d = 2$. But it has been proved in [13] that $F^2K_0(R)$ is trivial even when R is singular and therefore we can follow the proof of [15].

It is not difficult to prove that we can take R to be reduced to start with. To see this, let $\mathfrak{n}$ be the nilradical of R and let bar denote reduction modulo $\mathfrak{n}$. Note that,

$$\overline{I} = (\overline{a}_1, \cdots, \overline{a}_d) + \overline{I}^2 \text{ in } \overline{R}.$$

By the above paragraph, there exist $\overline{c}_1, \cdots, \overline{c}_d$ such that $\overline{I} = (\overline{c}_1, \cdots, \overline{c}_d)$ with $\overline{c}_i - \overline{a}_i \in \overline{I}^2$. One can now follow the proof of [12, 4.13] (the 'injectivity' part) to see that there exist $b_1, \cdots, b_d \in I$ such that $I = (b_1, \cdots, b_d)$ with $b_i - a_i \in I^2$ for $i = 1, \cdots, d$. □

Some immediate corollaries are in order. For the definition of the 'top' Euler class group, see [6]. Here we abbreviate the d^{th} Euler class group $E^d(R, R)$ as $E^d(R)$. From (2.5) and the definition of $E^d(R)$, we have:

Corollary 2.6. *Let R be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$. The d^{th} Euler class group $E^d(R)$ is trivial.*

Corollary 2.7. *Let R be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$ and P be a projective R-module of rank d with trivial determinant. Then $P \xrightarrow{\sim} Q \oplus R$ for some R-module Q.*

Proof. For $d \geq 3$, this is proved in [18]. Let $d = 2$. In this case, R is assumed to be smooth in [18]. We can remove this assumption, as follows.

It is easy to see following the proof of [6, 3.1] that for *any* two-dimensional Noetherian ring A and a projective A-module M with trivialization $\chi : A \xrightarrow{\sim} \wedge^2(M)$, the Euler class $e(M, \chi)$ is a well-defined element of $E^2(A)$. Recall from [6, 4.4] that $e(M, \chi) = 0$ if and only if M splits a free summand.

By (2.6), $E^2(R)$ is trivial and therefore, the proof is complete by [6, 4.4]. □

Remark 2.8. For a different proof of (2.7), see [13]. We believe the proof given here may be of some independent interest.

We shall see repeated use of the following corollary in this article.

Corollary 2.9. *Let R be a two-dimensional affine algebra over $\overline{\mathbb{F}}_p$, and P be a projective R-module of rank 2 with trivial determinant. Then P is free. In fact, any projective R-module with trivial determinant is free.*

Proof. Let rank$(P) = n \geq 2$. By a classical result of Serre [22], $P \stackrel{\sim}{\to} P' \oplus R^{n-2}$ for some projective R-module P' of rank 2. Note that the determinant of P' is trivial. Now, P' is free by (2.7). $\square$

The following is a standard result. For a proof, the reader may see [12, 2.17].

Lemma 2.10. *Let A be a Noetherian ring and P a finitely generated projective A-module. Let $P[T]$ denote projective $A[T]$-module $P \otimes A[T]$. Let $\alpha(T) : P[T] \twoheadrightarrow A[T]$ and $\beta(T) : P[T] \twoheadrightarrow A[T]$ be two surjections such that $\alpha(0) = \beta(0)$. Suppose further that the projective $A[T]$-modules* ker $\alpha(T)$ *and* ker $\beta(T)$ *are extended from A. Then there exists an automorphism $\sigma(T)$ of $P[T]$ with $\sigma(0) = id$ such that $\beta(T)\sigma(T) = \alpha(T)$.*

The next lemma follows from the well known Quillen Splitting Lemma [20, Lemma 1], and its proof is essentially contained in [20, Theorem 1].

Lemma 2.11. *Let A be a Noetherian ring and P be a finitely generated projective A-module. Let $s, t \in A$ be such that $As + At = A$. Let $\sigma(T)$ be an $A_{st}[T]$-automorphism of $P_{st}[T]$ such that $\sigma(0) = id$. Then, $\sigma(T) = \alpha(T)_s \beta(T)_t$, where $\alpha(T)$ is an $A_t[T]$-automorphism of $P_t[T]$ such that $\alpha(T) = id$ modulo the ideal (sT) and $\beta(T)$ is an $A_s[T]$-automorphism of $P_s[T]$ such that $\beta(T) = id$ modulo the ideal (tT).*

We shall need the following *"moving lemma"*. The version given below can easily be proved following [7, 2.4], which in turn is essentially based on [6, 2.14].

Lemma 2.12. *Let A be a Noetherian ring of dimension d and let $J \subset A$ be an ideal of height n such that $2n \geq d + 1$. Let P be a projective A-module of rank n and $\overline{\alpha} : P/JP \twoheadrightarrow J/J^2$ be a surjection. Then, there exists an ideal J' of A and a surjection $\beta : P \twoheadrightarrow J \cap J'$ such that:*

(1) *$J + J' = A$,*
(2) *$\beta \otimes A/J = \overline{\alpha}$,*
(3) *$ht(J') \geq n$.*

Given any ideal $K \subset A$ of height n, the map β can be chosen so that $J' + K = A$.

We need the following result from [7].

Lemma 2.13. [7, 5.1] *Let A be a Noetherian ring and P a projective A-module of rank n. Let $\alpha : P \twoheadrightarrow J$ and $\beta : P \twoheadrightarrow J'$ be two surjections, where J, J' are ideals of A of height at least n. Then, there exists an ideal $I \subset A[T]$ of height $\geq n$ and a surjection $\phi(T) : P[T] \twoheadrightarrow I$ such that $I(0) = J$, $\phi(0) = \alpha$ and $I(1) = J'$, $\phi(1) = \beta$.*

Next we state a remarkable result of Rao [21, 3.1]. Rao proved it when A is local. The following version can be deduced from [21] by applying Quillen's local-global principle [20].

Theorem 2.14. *Let A be a Noetherian ring of dimension 3 such that $2A = A$. Take any unimodular row $(f_1(T), f_2(T), f_3(T))$ over $A[T]$. Then there is $\theta(T) \in GL_3(A[T])$ such that $(f_1(T), f_2(T), f_3(T))\theta(T) = (f_1(0), f_2(0), f_3(0))$. Replacing $\theta(T)$ by $\theta(T)\theta(0)^{-1}$ we can actually conclude that $\theta(T) \in SL_3(A[T])$ and $\theta(0) = id$.*

The above theorem will enable us to cover the case $d = 4$ in Sections 3 and 4 as it facilitates certain patching arguments. We illustrate one such instance in the following theorem, which is a variant of [7, 5.2].

Theorem 2.15. *Let A be a Noetherian ring of dimension $d \geq 3$ with the following additional assumptions: (i) $2A = A$ if $d = 4$, (ii) A is regular containing a field if $d \geq 5$. Let P be a projective A-module of rank $d-1$ and let $\alpha(T) : P[T] \twoheadrightarrow I$ be a surjection where I is an ideal of height $d-1$. Assume that $J = I(0)$ is a proper ideal, and further that P/NP is free, where $N = (I \cap A)^2$. Let $p_1, \cdots, p_{d-1} \in P$ be such that their images in P/NP form a basis. Let $a_1, \cdots, a_{d-1} \in J$ be such that $\alpha(0)(p_i) = a_i$. Then, there exists an ideal $K \subset A$ of height $\geq d-1$ such that $K + N = A$ and:*

(1) $I \cap K[T] = (F_1(T), \cdots, F_{d-1}(T))$,
(2) $F_i(0) - F_i(1) \in K^2$, $i = 1, \cdots, d-1$,
(3) $\alpha(T)(p_i) - F_i(T) \in I^2$,
(4) $F_i(0) - a_i \in J^2$.

Proof. This is essentially contained in [7], with A regular (containing a field). Therefore, for $d \geq 5$, we are done. For the remaining cases, retaining the notations from [7] we give a sketch of the necessary modification in [7, page 151, second paragraph].

We have (from the proof of [7, 5.2]), $I' = I \cap K[T]$. An element $a \in N$ is obtained so that $I'_{1+a} = I_{1+a} = (G_1(T), \cdots, G_{d-1}(T))$. On the other hand, $I'_a = KA_a[T] = (a_1 + cb_1, \cdots, a_{d-1} + cb_{d-1})$ such that $G_i(0) = a_i + cb_i$ for $i = 1, \cdots, d-1$.

We now split the cases.

Case 1. $d = 3$. Let $b = 1 + a$. The rows $(G_1(T), G_2(T))$ and $(a_1 + cb_1, a_2 + cb_2)$ are unimodular over the ring $A_{ab}[T]$, and they agree when T is set to zero. As any unimodular row of length two over any ring can be completed to a 2×2 with determinant one, we can find $\theta(T) \in SL_2(A_{ab}[T])$ such that $(G_1(T), G_2(T))\theta(T) = (1, 0)$. Then $(G_1(0), G_2(0))\theta(0) = (1, 0)$, implying that $(a_1 + cb_1, a_2 + cb_2)\theta(0) = (1, 0)$. Taking $\sigma(T) := \theta(T)\theta(0)^{-1}$, we observe that $\sigma(0) = id$, and $(G_1(T), G_2(T))\sigma(T) = (a_1 + cb_1, a_2 + cb_2) = (G_1(0), G_2(0))$. The rest of the arguments are the same as [7, 5.2].

Case 2. $d = 4$ and $2A = A$. Consider the unimodular rows $(G_1(T), G_2(T), G_3(T))$ and $(a_1 + cb_1, a_2 + cb_2, a_3 + cb_3)$ over the ring $A_{a(1+aA)}[T]$. Note that $\dim(A_{a(1+aA)}) \leq 3$. By (2.14) there exists $\sigma'(T) \in SL_3(A_{a(1+aA)}[T])$ such that $\sigma'(0) = id$ and

$$(G_1(T), G_2(T), G_3(T))\sigma'(T) = (a_1 + cb_1, a_2 + cb_2, a_3 + cb_3).$$

We can find some b of the form $1+\lambda a$ such that b is a multiple of $1+a$, and some $\sigma(T) \in SL_3(A_{ab}[T])$ such that $\sigma(0) = id$ and $(G_1(T), G_2(T), G_3(T))\sigma(T) = (a_1 + cb_1, a_2 + cb_2, a_3 + cb_3)$ over the ring $A_{ab}[T]$. The rest is same as [7, 5.2]. $\square$

Remark 2.16. The assumption $2A = A$ is not required for $d = 4$ if A is regular containing a field.

3. ADDITION AND SUBTRACTION PRINCIPLES

Adapting the arguments from [6, 3.2, 3.3], in this section we prove the following *"addition"* and *"subtraction"* principles. We shall need them in the next section.

Theorem 3.1. *(Addition principle) Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$. Let I, J be two comaximal ideals of R, each of height $d-1$. Assume that $I = (a_1, \cdots, a_{d-1})$ and $J = (b_1, \cdots, b_{d-1})$. Then, $I \cap J = (c_1, \cdots, c_{d-1})$ such that $c_i - a_i \in I^2$ and $c_i - b_i \in J^2$ for $i = 1, \cdots, d-1$.*

Proof. For $d \geq 5$, one may appeal to [7, 3.1]. The following proof works for $d \geq 3$.

Let 'bar' denote reduction modulo J. Note that $(\overline{a}_1, \cdots, \overline{a}_{d-1})$ is a unimodular row in R/J and can be completed to a matrix $\tau \in SL_{d-1}(R/J)$. Since $SL_{d-1}(R) \longrightarrow SL_{d-1}(R/J)$ is surjective by (2.4), we can lift τ to $\gamma \in SL_{d-1}(R)$, replace $(a_1, \cdots, a_{d-1})$ by $\gamma(a_1, \cdots, a_{d-1})$ and assume that $(a_1, \cdots, a_{d-2}) + J = R$ and $a_{d-1} \in J$. Adding suitable multiples of a_{d-1} to $a_1, \cdots, a_{d-2}$, we can further ensure that $\text{ht}(a_1, \cdots, a_{d-2}) = d-2$.

Let $K = (a_1, \cdots, a_{d-2})$ and let 'tilde' denote reduction modulo K. As $K + J = R$, the row $(\widetilde{b}_1, \cdots, \widetilde{b}_{d-1})$ is unimodular over R/K. As $\dim(R/K) \leq 2$, the stably free R/K-module defined by this unimodular row is free by (2.9). In other words, $(\widetilde{b}_1, \cdots, \widetilde{b}_{d-1})$ is completable over R/K. As a consequence, the unimodular row $(b_1, \cdots, b_{d-1})_{1+K}$ over the ring R_{1+K} is completable to a matrix in $SL_{d-1}(R_{1+K})$. We can clear denominators and find a suitable s of the form $1+t$, where $t \in K$, and some $\alpha \in SL_{d-1}(R_s)$ such that $(b_1, \cdots, b_{d-1})_s\alpha = (1, 0, \cdots, 0)$. Clearly, we can adjust so that $s \in J$.

We have $(I \cap J)_s = I_s = (a_1, \cdots, a_{d-1})_s$. Also, $(I \cap J)_t = J_t = (b_1, \cdots, b_{d-1})_t$.

As $t \in K = (a_1, \cdots, a_{d-2})$, the unimodular row $(a_1, \cdots, a_{d-1})_t$ contains a subrow of shorter length, namely, $(a_1, \cdots, a_{d-2})_t$. Therefore, $(a_1, \cdots, a_{d-1})_t$ is elementarily completable. In other words, there is $\sigma \in E_{d-1}(R_t)$ such that $(a_1, \cdots, a_{d-1})_t\sigma = (1, 0, \cdots, 0)$.

Over the ring R_{st} we have $(a_1, \cdots, a_{d-1})\sigma_s\alpha_t^{-1} = (b_1, \cdots, b_{d-1})$ and therefore,

$$(a_1, \cdots, a_{d-1})\alpha_t^{-1}(\alpha_t\sigma_s\alpha_t^{-1}) = (b_1, \cdots, b_{d-1}).$$

Write $\theta = \alpha_t\sigma_s\alpha_t^{-1}$. Note that σ_s is isotopic to identity and consequently so is θ. By (2.11), $\theta = \theta'_t\theta''_s$, where $\theta' \in SL_{d-1}(R_s)$ and $\theta'' \in SL_{d-1}(R_t)$. Let $(a_1, \cdots, a_{d-1})\alpha^{-1}\theta' =$

$(x_1, \cdots, x_{d-1})$ and $(b_1, \cdots, b_{d-1})\theta''^{-1} = (y_1, \cdots, y_{d-1})$. Then we have $(I \cap J)_s = (x_1, \cdots, x_{d-1})$ and $(I \cap J)_t = (y_1, \cdots, y_{d-1})$. As these generators agree over R_{st}, we can patch them to obtain a set of generators of $I \cap J$, say, $I \cap J = (z_1, \cdots, z_{d-1})$.

Recall that 'tilde' denotes reduction modulo I and 'bar' denotes that modulo J. Observe that by the above construction, $(\widetilde{a}_1, \cdots, \widetilde{a}_{d-1})$ and $(\widetilde{z}_1, \cdots, \widetilde{z}_{d-1})$ differ by an element of $SL_{d-1}(R/I)$. Similarly, $(\overline{b}_1, \cdots, \overline{b}_{d-1})$ and $(\overline{z}_1, \cdots, \overline{z}_{d-1})$ differ by an element of $SL_{d-1}(R/J)$. Now we use $SL_{d-1}(R) \twoheadrightarrow SL_{d-1}(R/I \cap J) \xrightarrow{\sim} SL_{d-1}(R/I) \times SL_{d-1}(R/J)$ to get a suitable $\beta \in SL_{d-1}(R)$ and observe that if $(z_1, \cdots, z_{d-1})\beta = (c_1, \cdots, c_{d-1})$, then these are the generators of $I \cap J$ we were looking for. □

Theorem 3.2. *(Subtraction principle) Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$ (with the additional assumption $p \neq 2$ if $d = 4$). Let I, J be two comaximal ideals of R such that ht$(I) = d-1$ and ht$(J) \geq d-1$. Let P be a projective R-module of rank $d-1$ with trivial determinant and $\chi : R \xrightarrow{\sim} \wedge^{d-1}P$ be an isomorphism. Let $\alpha : P \twoheadrightarrow I \cap J$ and $\beta : R^{d-1} \twoheadrightarrow I$ be surjections. Suppose that there exists an isomorphism $\delta : (R/I)^{d-1} \xrightarrow{\sim} P/IP$ such that:*

(1) $\wedge^{d-1}\delta = \chi \otimes R/I$,
(2) $(\alpha \otimes R/I)\delta = \beta \otimes R/I$.

Then, there exists a surjection $\gamma : P \twoheadrightarrow J$ such that $\gamma \otimes R/J = \alpha \otimes R/J$.

Proof. We first remark that since J is locally generated by $d-1$ elements, either it is proper of height $d-1$, or $J = R$.

Let β correspond to $I = (a_1, \cdots, a_{d-1})$. As before, we first note that we can always make changes up to $SL_{d-1}(R)$ in our arguments. We can make similar reductions as in the above proof and assume that: (1) $(a_1, \cdots, a_{d-2}) + J^2 = R$, (2) $a_{d-1} \in J^2$, and (3) ht$(a_1, \cdots, a_{d-2}) = d-2$. Once these are done, we pick some $\lambda \in (a_1, \cdots, a_{d-2})$ such that $\lambda \equiv 1$ modulo J^2, replace a_{d-1} by $\lambda + a_{d-1}$ and obtain $a_{d-1} \equiv 1$ modulo J^2.

Consider the following ideals in $R[T]$: $K' = (a_1, \cdots, a_{d-2}, T + a_{d-1})$, $K'' = J[T]$, and $K = K' \cap K''$. We aim to show that there is a surjection $\theta : P[T] \twoheadrightarrow K$ such that $\theta(0) = \alpha$. If we can achieve so then we can specialize at $1 - a_{d-1}$ to obtain $\gamma := \theta(1 - a_{d-1}) : P \twoheadrightarrow J$. As $a_{d-1} \equiv 1$ modulo J^2, we have $\gamma \otimes R/J = \theta(1 - a_{d-1}) \otimes R/J = \theta(0) \otimes R/J = \alpha \otimes R/J$, and we will be done. Rest of the proof is about finding such a θ, and we break it into two cases.

Case 1. Assume that $d \geq 5$. As $\dim(R[T]/K') \leq 2$, the module $P[T]/K'P[T]$ is free of rank $d-1$ by (2.9). We choose an isomorphism $\kappa(T) : (R[T]/K')^{d-1} \xrightarrow{\sim} P[T]/K'P[T]$ such that $\wedge^{d-1}\kappa(T) = \chi \otimes R[T]/K'$. Therefore, $\wedge^{d-1}\kappa(0) = \chi \otimes R/I = \wedge^{d-1}\delta$ and we conclude that $\kappa(0)$ and δ differ by an element $\mathfrak{a} \in SL_{d-1}(R/I)$. By (2.4), we can find a lift of $\mathfrak{a}$ in $SL_{d-1}(R)$ and use this to alter $\kappa(T)$ so that $\kappa(0) = \delta$.

Sending the canonical basis vectors to $a_1, \cdots, a_{d-2}, T + a_{d-1}$ respectively, we have a surjection from $R[T]^{d-1}$ to K', which will induce a surjection $\epsilon(T) : (R[T]/K')^{d-1} \twoheadrightarrow K'/K'^2$. Composing with $\kappa(T)^{-1}$ we have:

$$\pi(T) := \epsilon(T)\kappa(T)^{-1} : P[T]/K'P[T] \twoheadrightarrow K'/K'^2.$$

Observe that $\pi(0) = \epsilon(0)\kappa(0)^{-1} = (\beta \otimes R/I)\delta^{-1} = \alpha \otimes R/I$ (see hypothesis (2) above). We can now apply [14, 2.3] to obtain a surjection $\theta(T) : P[T] \twoheadrightarrow K$ so that $\theta(0) = \alpha$.

Case 2. Assume that $d = 3$ or 4. Write $L = (a_1, \cdots, a_{d-2})$. Note that $\dim(R/L) = 2$, and therefore, P/LP is free by (2.9). Consequently, P_{1+L} is a free R_{1+L}-module of rank $d-1$. We choose an isomorphism $\xi : R_{1+L}^{d-1} \xrightarrow{\sim} P_{1+L}$ such that $\wedge^{d-1}\xi = \chi \otimes R_{1+L}$. Note that ξ induces isomorphism $\xi(T) : R_{1+L}[T]^{d-1} \xrightarrow{\sim} P_{1+L}[T]$.

We have $K_{1+L} = K'_{1+L}$. Sending the canonical basis vectors to $a_1, a_2, \cdots, a_{d-2}, T + a_{d-1}$, respectively, we have a surjection $\pi(T) : R_{1+L}[T]^{d-1} \twoheadrightarrow K_{1+L}$. We therefore have

$$\gamma(T) := \pi(T)(\xi(T))^{-1} : P_{1+L}[T] \twoheadrightarrow K_{1+L}.$$

Recall that we have surjections $\alpha : P \twoheadrightarrow I \cap J$ and $\beta : R^{d-1} \twoheadrightarrow I$. We would like to compare $\gamma(0)$ and $\alpha_{1+L} : P_{1+L} \twoheadrightarrow I_{1+L}$.

Note that $\gamma(0) = \pi(0)\xi^{-1} = \beta_{1+L}\xi^{-1}$ and $I_{1+L}/I_{1+L}^2 = I/I^2$. We have induced surjections:

$$\overline{\beta\xi}^{-1} : P/IP \xrightarrow{\sim} (R/I)^{d-1} \twoheadrightarrow I/I^2$$
$$\overline{\beta}\delta^{-1} : P/IP \xrightarrow{\sim} (R/I)^{d-1} \twoheadrightarrow I/I^2$$

We have, $\wedge^{d-1}\delta = \chi \otimes R/I$ and $\wedge^{d-1}\xi = \chi \otimes R_{1+L}$. Further,

(1) $\gamma(0) \otimes R/I = (\beta \otimes R/I)(\xi^{-1} \otimes R/I)$,
(2) $\alpha_{1+L} \otimes R/I = (\beta \otimes R/I)\delta^{-1}$.

It follows that $\gamma(0) \otimes R/I$ and $\alpha_{1+L} \otimes R/I$ differ by an element $\mathfrak{a} \in SL(P/IP)$. As P/IP is free and $\dim(R/I) = 1$, by (2.4), $SL(P/IP) = E(P/IP)$ for $d = 4$. As $E(P_{1+L}) \longrightarrow E(P_{1+L}/(IP)_{1+L}) = E(P/IP)$ is surjective, we can find a lift of $\mathfrak{a}$ in $E(P_{1+L})$. On the other hand, if $d = 3$, applying [6, 2.3] we conclude that $\gamma(0)$ and α_{1+L} differ by an automorphism in $SL(P_{1+L})$. In either case, we can alter $\gamma(T)$ and assume that $\gamma(0) = \alpha_{1+L}$. We can find a suitable $t \in L$ such that: (a) $1 + t \in J$, (b) P_{1+tR} is free, and (c) there is a surjection $\gamma(T) : P_{1+tR}[T] \twoheadrightarrow I_{1+tR}$ such that $\gamma(0) = \alpha_{1+tR}$.

On the other hand, $\alpha_t : P_t \twoheadrightarrow J_t$ will induce $\alpha_t(T) : P_t[T] \twoheadrightarrow K_t$ (note that $K_t = K''_t = J_t[T]$). Now, the two surjections

$$\gamma(T)_t : P_{t(1+tR)}[T] \twoheadrightarrow K_{t(1+tR)}[T] = R_{t(1+tR)}[T]$$

$$\alpha_{t(1+tR)} : P_{t(1+tR)}[T] \twoheadrightarrow K_{t(1+tR)} = R_{t(1+tR)}[T]$$

are such that they agree when $T = 0$, and both their kernels are:

(1) free, if $d = 3$ (as any unimodular row of length 2 is completable),
(2) extended, if $d = 4$ and $2R = R$ (by (2.14)).

A standard patching argument using (2.11) yields a surjection $\theta : P[T] \twoheadrightarrow K$ such that $\theta(0) = \alpha$. $\square$

Taking $P = R^{d-1}$ in (3.2), we get the following corollary.

Corollary 3.3. *(Subtraction principle) Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$ (with the additional assumption $p \neq 2$ if $d = 4$). Let I, J be two comaximal ideals of R, each of height $d-1$. Assume that $I = (a_1, \cdots, a_{d-1})$ and $I \cap J = (c_1, \cdots, c_{d-1})$ such that $c_i - a_i \in I^2$ for $i = 1, \cdots, d-1$. Then, $J = (b_1, \cdots, b_{d-1})$ such that $c_i - b_i \in J^2$ for $i = 1, \cdots, d-1$.*

Taking $J = R$ in (3.2), we get the following corollary.

Corollary 3.4. *Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$ (with the additional assumption $p \neq 2$ if $d = 4$). Let I be an ideal of R such that $ht(I) = d-1$. Let P be a projective R-module of rank $d-1$ with trivial determinant and $\chi : R \overset{\sim}{\to} \wedge^{d-1}P$ be an isomorphism. Let $\alpha : P \twoheadrightarrow I$ and $\beta : R^{d-1} \twoheadrightarrow I$ be surjections. Suppose that there exists an isomorphism $\delta : (R/I)^{d-1} \overset{\sim}{\to} P/IP$ such that:*
(1) $\wedge^{d-1}\delta = \chi \otimes R/I$,
(2) $(\alpha \otimes R/I)\delta = \beta \otimes R/I$.
Then, there exists a surjection $\gamma : P \twoheadrightarrow R$.

Remark 3.5. When $p \neq 2$ and R is smooth, later in (5.4) we prove: If $I, J \subset R$ are comaximal ideals of height $d-1$, and if two of I, J, $I \cap J$ are complete intersections, then so is the third.

4. AN OBSTRUCTION GROUP AND AN OBSTRUCTION CLASS

Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$. Let $I \subset R$ be an ideal of height $d-1$ such that I/I^2 is generated by $d-1$ elements. Two surjections $\alpha, \beta : (R/I)^{d-1} \twoheadrightarrow I/I^2$ are said to be *related* if there exists $\sigma \in SL_{d-1}(R/I)$ such that $\alpha\sigma = \beta$. This defines an equivalence relation on the set of surjections from $(R/I)^{d-1}$ to I/I^2.

Lemma 4.1. *Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$. Let $I \subset R$ be an ideal of height $d-1$ such that $\mu(I/I^2) = d-1$. Let a surjection $\alpha : (R/I)^{d-1} \twoheadrightarrow I/I^2$ be such that it has a surjective lift $\theta : R^{d-1} \twoheadrightarrow I$. Then the same is true for any β related to α.*

Proof. Let $\overline{\sigma} \in SL_{d-1}(R/I)$ be such that $\alpha\overline{\sigma} = \beta$. By the hypothesis, there is a surjection $\theta : R^{d-1} \twoheadrightarrow I$ such that $\theta \otimes R/I = \alpha$. Applying Theorem 2.4 we can find a lift $\sigma \in SL_{d-1}(R)$ of $\overline{\sigma}$. Then $\theta\sigma : R^{d-1} \twoheadrightarrow I$ is a surjective lift of β. $\square$

Definition 4.2. An equivalence class of surjections from $(R/I)^{d-1}$ to I/I^2 will be called a *local orientation* of I. A local orientation ω_I of I will be called a *global orientation* if a surjection (hence all) in the class of ω_I can be lifted to a surjection from R^{d-1} to I.

Remark 4.3. Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$. We now define the $(d-1)^{\text{th}}$ Euler class group of R. Recall that for any commutative Noetherian ring A of dimension d and any integer n with $2n \geq d+3$, there is a notion of the n^{th} Euler class group $E^n(A)$ of A in [7]. Our definition is modeled on their definition but there is a difference: we consider the action of $SL_{d-1}(R/I)$ as defined above, whereas they consider the action of $E_{d-1}(R/I)$. This difference is crucial.

Definition 4.4. *The Euler class group $E^{d-1}(R)$:* Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$. Let G be the free abelian group on the set B of pairs $(\mathcal{I}, \omega_{\mathcal{I}})$ where $\mathcal{I} \subset R$ is an ideal of height $d-1$ with the property that Spec $(R/\mathcal{I})$ is connected, $\mu(\mathcal{I}/\mathcal{I}^2) = d-1$, and $\omega_{\mathcal{I}} : (R/\mathcal{I})^{d-1} \twoheadrightarrow \mathcal{I}/\mathcal{I}^2$ is a local orientation.

Let I be any ideal of R of height $d-1$ such that I/I^2 is generated by $d-1$ elements. Then I has a unique decomposition, $I = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_k$, where each of Spec$(R/\mathcal{I}_i)$ is connected, $\mathcal{I}_i$ are pairwise comaximal, and ht $\mathcal{I}_i = d-1$ (for a proof see [6] or [8]). Now if ω_I is a local orientation of I, then it naturally gives rise to $\omega_{\mathcal{I}_i} : (R/\mathcal{I}_i)^{d-1} \twoheadrightarrow \mathcal{I}_i/\mathcal{I}_i^2$ for $1 \leq i \leq k$. By (I, ω_I) we mean the element $\sum(\mathcal{I}_i, \omega_{\mathcal{I}_i}) \in G$.

Let H be the subgroup of G generated by the set S of pairs (I, ω_I) in G such that ω_I is a global orientation. We define the $(d-1)^{\text{th}}$ Euler class group of R as $E^{d-1}(R) := G/H$.

Remark 4.5. In view of (4.1), we would not distinguish between a surjection $\omega_I : (R/I)^{d-1} \twoheadrightarrow I/I^2$ and the equivalence class it represents.

We now state a lemma from [12, 4.1].

Lemma 4.6. *Let G be a free abelian group with basis $B = (e_i)_{i \in \mathcal{I}}$. Let $\sim$ be an equivalence relation on B. Define $x \in G$ to be "reduced" if $x = e_1 + \cdots + e_r$ and $e_i \neq e_j$ for $i \neq j$. Define $x \in G$ to be "nicely reduced" if $x = e_1 + \cdots + e_r$ is such that $e_i \not\sim e_j$ for $i \neq j$. Let $S \subset G$ be such that*

1. *Every element of S is nicely reduced.*
2. *Let $x, y \in G$ be such that each of $x, y, x+y$ is nicely reduced. If two of $x, y, x+y$ are in S, then so is the third.*
3. *Let $x \in G \setminus S$ be nicely reduced and let $\mathcal{J} \subset \mathcal{I}$ be finite. Then there exists $y \in G$ with the following properties : (i) y is nicely reduced; (ii) $x + y \in S$; (iii) $y + e_j$ is nicely reduced $\forall j \in \mathcal{J}$.*

Let H be the subgroup of G generated by S. If $x \in H$ is nicely reduced, then $x \in S$.

We are now ready to prove:

Theorem 4.7. *Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$ (with the additional assumption $p \neq 2$ if $d = 4$). Let I be any ideal of R of height $d-1$ such that $\mu(I/I^2) = d-1$, and $\omega_I : (R/I)^{d-1} \twoheadrightarrow I/I^2$ be a local orientation. Assume that the image of (I, ω_I) is trivial in $E^{d-1}(R)$. Then ω_I is global. In other words, there is an R-linear surjection $\theta : (R/I)^{d-1} \twoheadrightarrow I$ such that θ lifts ω_I.*

Proof. We take G to be the free abelian group generated by B, as defined in (4.4). Define a relation $\sim$ on B as: $(K, \omega_K) \sim (K', \omega_{K'})$ if $K = K'$. Then it is an equivalence relation.

Let $S \subset G$ be as in (4.4). In view of the above lemma, it is enough to show that the three conditions in (4.6) are satisfied. Condition (1) is clear, almost from the definition. The addition and subtraction principles (3.1, 3.3) will yield condition (2). Finally, applying the moving lemma (2.12), it is clear that (3) is also satisfied. □

Notation. Let $\sigma \in GL_{d-1}(R/I)$ and let $\det(\sigma) = \overline{u}$. As we are dealing with the action of $SL_{d-1}(R/I)$ on surjections from $(R/I)^{d-1}$ to I/I^2 in the definitions above, there will be no ambiguity if we write $\omega\sigma$ as $\overline{u}\omega$ and the corresponding element in $E^{d-1}(R)$ as $(I, \overline{u}\omega)$.

Remark 4.8. Let ω, ω' be two surjections from $(R/I)^{d-1}$ to I/I^2. With the above notations in mind, $(I, \omega') = (I, \overline{u}\omega)$ for some unit $\overline{u}$ in R/I by [3, 2.2].

We shall need the following result in Section 5.

Proposition 4.9. *Let R be as in (4.4) and $(I, \omega) \in E^{d-1}(R)$. Let $\overline{u}$ in R/I be a unit. The following assertions hold:*

(1) *$(I, \omega) = 0$ if and only if $(I, \overline{u}^2\omega) = 0$ in $E^{d-1}(R)$.*
(2) *$(I, \omega) = (I, \overline{u}^2\omega)$ in $E^{d-1}(R)$.*

Proof. Statement (1) is needed to prove (2). The proofs are the same as in [6, 5.3, 5.4] and we do not repeat them. One has to simply apply (2.4) at the appropriate places to lift automorphisms. □

Definition 4.10. *The $(d-1)^{th}$ Euler class of a projective module:* Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$, with the following additional assumptions: (i) $p \neq 2$ if $d = 4$, (ii) R is smooth if $d \geq 5$. Let P be a projective R-module of rank $d-1$ with trivial determinant. Fix an isomorphism $\chi : R \xrightarrow{\sim} \wedge^{d-1}(P)$. Let $\alpha : P \twoheadrightarrow J$ be a surjection where $J \subset R$ is an ideal of height $d-1$. Note that P/JP is a free R/J-module. Choose an isomorphism $\sigma : (R/J)^{d-1} \xrightarrow{\sim} P/JP$ such that $\wedge^{d-1}\sigma = \chi \otimes R/J$. Let ω_J be the composite:

$$(R/J)^{d-1} \overset{\sigma}{\underset{\sim}{\to}} P/JP \overset{\overline{\alpha}}{\twoheadrightarrow} J/J^2$$

We define the $(d-1)^{th}$ *Euler class* of the pair (P, χ) as $e_{d-1}(P, \chi) := (J, \omega_J)$ (we show below that this association does not depend on the chosen surjection $\alpha : P \twoheadrightarrow J$). We shall simply call it the Euler class here.

Theorem 4.11. *Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$, with the following additional assumptions: (i) $p \neq 2$ if $d = 4$, (ii) R is smooth if $d \geq 5$. Let P be a projective R-module of rank $d-1$ with trivial determinant. Fix an isomorphism $\chi : R \xrightarrow{\sim} \wedge^{d-1}P$. The Euler class $e_{d-1}(P, \chi)$ is well-defined.*

Proof. Let $\beta : P \twoheadrightarrow J'$ be another surjection such that J' is an ideal of R of height $d-1$.

By (2.13) there exists an ideal $I \subset R[T]$ of height $d-1$ and a surjection $\phi(T) : P[T] \twoheadrightarrow I$ such that $I(0) = J$, $\phi(0) = \alpha$ and $I(1) = J'$, $\phi(1) = \beta$.

Let $N = (I \cap R)^2$. Then $\text{ht}(N) \geq d-2$ and therefore, $\dim(R/N) \leq 2$. By (2.9), P/NP is a free R/N-module of rank $d-1$. On the other hand, by the same reasoning, $P[T]/IP[T]$ is a free $R[T]/I$-module of rank $d-1$.

We can choose an isomorphism $\tau : (R/N)^{d-1} \xrightarrow{\sim} P/NP$ such that $\wedge^{d-1}\tau = \chi \otimes R/N$. This choice of τ gives us a basis of P/NP, which in turn induces a basis of the free module $P[T]/IP[T]$. Using this basis of $P[T]/IP[T]$ and the surjection $\phi(T) : P[T] \twoheadrightarrow I$, we obtain a surjection $\omega : (R[T]/I)^{d-1} \twoheadrightarrow I/I^2$. Note that, due to the choice of the basis, $\omega(0) = \omega_J : (R/J)^{d-1} \twoheadrightarrow J/J^2$, and $\omega(1) = \omega_{J'} : (R/J')^{d-1} \twoheadrightarrow J'/J'^2$.

Now using (2.15), we obtain an ideal K of height $d-1$ with $K + J = R = K + J'$, and a surjection $\omega_K : (R/K)^{d-1} \twoheadrightarrow K/K^2$ such that

$$(J, \omega_J) + (K, \omega_K) = (J', \omega_{J'}) + (K, \omega_K) \text{ in } E^{d-1}(R).$$

This proves that $e_{d-1}(P, \chi)$ is well-defined. □

We now prove that $e_{d-1}(P, \chi)$ is the precise obstruction for P to split off a free summand of rank one.

Theorem 4.12. *Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$, with the following additional assumptions: (i) $p \neq 2$ if $d = 4$, (ii) R is smooth if $d \geq 5$. Let P be a projective R-module of rank $d-1$ with trivial determinant. Fix an isomorphism $\chi : R \xrightarrow{\sim} \wedge^{d-1}P$. Then $P \xrightarrow{\sim} Q \oplus R$ for some R-module Q if and only if $e_{d-1}(P, \chi) = 0$ in $E^{d-1}(R)$.*

Proof. Let $\alpha : P \twoheadrightarrow J$ be a surjection such that $J \subset R$ is an ideal of height $d-1$. Choose an isomorphism $\sigma : (R/J)^{d-1} \xrightarrow{\sim} P/JP$ such that $\wedge^{d-1}\sigma = \chi \otimes R/J$. Let ω_J be the composite:

$$(R/J)^{d-1} \overset{\sigma}{\xrightarrow{\sim}} P/JP \overset{\overline{\alpha}}{\twoheadrightarrow} J/J^2$$

Then, by definition (4.10), $e_{d-1}(P, \chi) = (J, \omega_J)$ in $E^{d-1}(R)$.

Assume first that $P \overset{\sim}{\to} Q \oplus R$ for some R-module Q. Write $\alpha = (\beta, a)$. Applying an elementary automorphism of P, we may assume that height of $K := \beta(Q)$ is at least $d-2$. Note that $J = (K, a)$.

As the determinant of Q is also trivial, we may assume that χ is induced by $\chi' : R \overset{\sim}{\to} \wedge^{d-2} Q$. As $\dim(R/K) \leq 2$, by (2.9) the R/K-module Q/KQ is free of rank $d-2$. Choose an isomorphism $\gamma' : (R/K)^{d-2} \overset{\sim}{\to} Q/KQ$ such that $\wedge^{d-2}\gamma' = \chi' \otimes R/K$. We have a surjection $(\beta \otimes R/K)\gamma' : (R/K)^{d-2} \twoheadrightarrow K/K^2$. Suppose that this induces $K = (\alpha_1, \cdots, \alpha_{d-2}) + K^2$. Then, by (2.1) there exists $e \in K^2$ such that $K = (\alpha_1, \cdots, \alpha_{d-2}, e)$ where $e(1-e) \in (\alpha_1, \cdots, \alpha_{d-2})$. As $J = (K, a)$, it follows from (2.1 (3)) that $J = (\alpha_1, \cdots, \alpha_{d-2}, b)$, where $b = e + (1-e)a$. Note that $b - a = e - ea \in K^2 \subset J^2$. It is now easy to see that ω_J is induced by $\alpha_1, \cdots, \alpha_{d-2}, b$. In other words, $(J, \omega_J) = 0$ in $E^{d-1}(R)$.

Conversely, assume that $e_{d-1}(P, \chi) = 0$. Then $(J, \omega_J) = 0$ in $E^{d-1}(R)$ and therefore by (4.7) there exists $\theta : R^{d-1} \twoheadrightarrow J$ which is a lift of ω_J. Let ω_J correspond to $J = (a_1, \cdots, a_{d-1}) + J^2$ and θ correspond to $J = (b_1, \cdots, b_{d-1})$. We have $b_i - a_i \in J^2$ for $1 \leq i \leq d-1$. Now apply (3.4). $\square$

Theorem 4.13. *Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$, with the following additional assumptions: (i) $p \neq 2$ if $d = 4$, (ii) R is smooth if $d \geq 5$. Let P be a projective R-module of rank $d-1$ with trivial determinant. Fix an isomorphism $\chi : R \overset{\sim}{\to} \wedge^{d-1} P$. Let $e_{d-1}(P, \chi) = (J, \omega_J)$ for some $(J, \omega_J) \in E^{d-1}(R)$. Then there is a surjection $\alpha : P \twoheadrightarrow J$ such that (J, ω_J) is induced by (α, χ).*

Proof. We choose an isomorphism $\sigma : (R/J)^{d-1} \overset{\sim}{\to} P/JP$ such that $\wedge^{d-1}\sigma = \chi \otimes R/J$. We have the composite:

$$P/JP \overset{\sigma^{-1}}{\overset{\sim}{\to}} (R/J)^{d-1} \overset{\omega_J}{\twoheadrightarrow} J/J^2,$$

let us call it $\overline{\theta}$. Applying (2.12), we can find an ideal $J' \subset R$ and a surjection $\eta : P \twoheadrightarrow J \cap J'$ such that: (i) $J + J' = R$, (ii) $\text{ht}(J') \geq d-1$, and (iii) $\eta \otimes R/J = \overline{\theta}$. If $J' = R$, we are done. Let $\text{ht}(J') = d-1$. It is easy to see that the pair (η, χ) induces $e_{d-1}(P, \chi) = (J, \omega_J) + (J', \omega_{J'})$ in $E^{d-1}(R)$. As $e_{d-1}(P, \chi) = (J, \omega_J)$, we have $(J', \omega_{J'}) = 0$ in $E^{d-1}(R)$. By (4.7), there is a surjection $\beta : R^{d-1} \twoheadrightarrow J'$ which is a lift of $\omega_{J'}$. Now apply (3.2). $\square$

Remark 4.14. We reiterate that for the definition and the results involving only $E^{d-1}(R)$, we do not need any smoothness assumption. We need smoothness whenever we talk about the Euler class of a projective module and $d \geq 5$. We could have imposed a blanket assumption that R is smooth throughout this section (or the article) but we decided against it as some subtle (and perhaps useful) points will be lost.

5. Applications I: splitting problem vis-à-vis complete intersections

In the preceding section we have established that the Euler class $e_{d-1}(P,\chi)$ is the precise obstruction for P to split off a free summand of rank one. In this section we refine the conclusion further and give it a much more simplified and tangible form.

Let R be an affine algebra of dimension d over $\overline{\mathbb{F}}_p$ and P be a projective R-module of rank $d-1$ with trivial determinant. Without getting into the Euler class theory, we (re)prove the following basic result, whose idea is essentially from [17, Theorem 1].

Theorem 5.1. *Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$ and P be a projective R-module of rank $d-1$ with trivial determinant. Let $\alpha : P \twoheadrightarrow J$ be an R-linear surjection such that $ht(J) = d-1$. Assume that $P \xrightarrow{\sim} Q \oplus R$ for some R-module Q. Then $\mu(J) = d-1$.*

Proof. Note that the case $d = 3$ is trivial. Therefore, assume that $d \geq 4$. Let $\alpha_{|_Q} = \beta$, and $\alpha(0,1) = a$. Using a standard general position argument we may assume that $\beta(Q) = K$ is such that $\dim(R/K) \leq 2$.

We have the induced surjection $Q/KQ \twoheadrightarrow K/K^2$. As the determinant of Q is trivial, by (2.9) Q/KQ is free of rank $d-2$. Therefore, $\mu(K/K^2) = d-2$, and by (2.1 (3)) we have $\mu(J) = d-1$. □

In this context, we can naturally ask the following question.

Question 5.2. *Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$ and P be a projective R-module of rank $d-1$ with trivial determinant. Let $\alpha : P \twoheadrightarrow J$ be an R-linear surjection such that $ht(J) = d-1$. Assume that $\mu(J) = d-1$. Then, is $P \xrightarrow{\sim} Q \oplus R$ for some R-module Q?*

To answer this question we first need the following result.

Theorem 5.3. *Let $p \neq 2$ and R be a smooth affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$. Let $J \subset R$ be an ideal of height $d-1$ such that $\mu(J) = d-1$. Suppose, it is given that $J = (b_1, \cdots, b_{d-1}) + J^2$. Then, there exist $c_1, \cdots, c_{d-1} \in J$ such that $J = (c_1, \cdots, c_{d-1})$ and $b_i \equiv c_i$ mod J^2 for $i = 1, \cdots, d-1$.*

Proof. Let $J = (a_1, \cdots, a_{d-1})$. By [3, 2.2], there is a matrix $\delta \in GL_{d-1}(R/J)$ such that $(\overline{a_1}, \cdots, \overline{a_{d-1}})\delta = (\overline{b_1}, \cdots, \overline{b_{d-1}})$. Let $u \in R$ be such that $\overline{u}$ =det$(\delta)^{-1}$. Using Swan's Bertini Theorem [5, 2.11], we may assume that $B = R/(a_3, \cdots, a_{d-1})$ is a smooth affine three-fold over $\overline{\mathbb{F}}_p$. By [11, 7.5], the unimodular row $(\widetilde{u}, \widetilde{a_1}, \widetilde{a_2}) \in Um_3(B)$ is completable. Applying [24, 2.4] we have a set of generators of J, say $J = (\beta_1, \cdots, \beta_{d-1})$, and a matrix $\overline{\theta} \in SL_{d-1}(R/J)$ such that $(\overline{\beta_1}, \cdots, \overline{\beta_{d-1}})\overline{\theta} = (\overline{b_1}, \cdots, \overline{b_{d-1}})$. We can now apply (2.4) and lift $\overline{\theta}$ to a matrix $\theta \in SL_{d-1}(R)$. Suppose that $(\beta_1, \cdots, \beta_{d-1})\theta = (c_1, \cdots, c_{d-1})$. Then $J = (c_1, \cdots, c_{d-1})$, as desired. □

The addition and subtraction principles (3.1, 3.3) proved earlier now take the following form.

Corollary 5.4. *Let $p \neq 2$, and let R be a smooth affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$. Let I, J be two comaximal ideals of R, each of height $d-1$. If two of $I, J, I \cap J$ are generated by $d-1$ elements, then so is the third.*

The following corollary will be used in the next section.

Corollary 5.5. *Let $p \neq 2$, and let R be a smooth affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$. Let $J \subset R$ be an ideal of height $d-1$ such that $\mu(J/J^2) = 2$. Let ω_1, ω_2 be two surjections from $(R/J)^{d-1}$ to J/J^2. Then $(J, \omega_1) = (J, \omega_2)$ in $E^{d-1}(R)$.*

Proof. If $(J, \omega_1) = 0$, then the conclusion follows from the above theorem. Assume that $(J, \omega_1) \neq 0$. Applying (2.12) we can find an ideal $J' \subset R$ and a surjection $\eta : R^{d-1} \twoheadrightarrow J \cap J'$ such that $J + J' = R$, $\text{ht}(J') \geq d-1$, and $\eta \otimes R/J = \omega_1$. We have $\text{ht}(J') = d-1$. Let $\omega_{J'} = \eta \otimes R/J'$. Then, $(J, \omega_1) + (J', \omega_{J'}) = 0$ in $E^{d-1}(R)$. By the Chinese Remainder Theorem, ω_2 and $\omega_{J'}$ will induce $\omega_{J \cap J'} : (R/J \cap J')^{d-1} \twoheadrightarrow J \cap J'/(J \cap J')^2$. Then, $(J \cap J', \omega_{J \cap J'}) = (J, \omega_2) + (J', \omega_{J'})$. As $\mu(J \cap J') = d-1$, by the above theorem, $(J \cap J', \omega_{J \cap J'}) = 0$, and we are done. $\square$

We answer Question 5.2 in the following form.

Theorem 5.6. *Let $p \neq 2$ and R be a smooth affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$ and P be a projective R-module of rank $d-1$ with trivial determinant. Then $P \xrightarrow{\sim} Q \oplus R$ for some R-module Q if and only if there is a surjection $\alpha : P \twoheadrightarrow J$ such that $J \subset R$ is an ideal of height $d-1$ and $\mu(J) = d-1$.*

Proof. Follows from (4.12), (5.3), and (4.7). $\square$

Remark 5.7. If $d \geq 5$, we can actually drop the smoothness assumption in the above theorem. But this will come at the expense of a restrictive hypothesis on p, as we will soon see below.

We need the following variant of (5.3). In this version we shall use the cancellation theorem of Dhorajia-Keshari [9, 3.5] instead of [11, Theorem 7.5], thus avoiding the restriction of smoothness. In the proof of (5.8) below we use some computations inside the Euler class group $E^{d-1}(R)$. We remind the reader that in Section 4 we only need smoothness to prove that the Euler class of a projective module is well-defined.

Theorem 5.8. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 5$ such that p and $(d-1)!$ are relatively prime. Let $J \subset R$ be an ideal of height $d-1$ such that $\mu(J) = d-1$. Suppose, it is given that $J = (b_1, \cdots, b_{d-1}) + J^2$. Then, there exist $c_1, \cdots, c_{d-1} \in J$ such that $J = (c_1, \cdots, c_{d-1})$ and $b_i \equiv c_i$ mod J^2 for $i = 1, \cdots, d-1$.*

Proof. Let $J = (a_1, \cdots, a_{d-1})$. These generators will induce $\omega : (R/J)^{d-1} \twoheadrightarrow J/J^2$. Note that $(J, \omega) = 0$ in $E^{d-1}(R)$ by definition.

Let the data $J = (b_1, \cdots, b_{d-1}) + J^2$ define $\omega' : (R/J)^{d-1} \twoheadrightarrow J/J^2$. By (4.8), $(J, \omega) = (J, \overline{u}\omega)$ for some $u \in R$ which is a unit modulo J. Let $v \in R$ be such that $uv \equiv 1$ modulo J. Note that $(v, a_1, \cdots, a_{d-1})$ is a unimodular row over R and by [9, 3.5] it is completable.

We can follow the arguments as in [23, Proposition, page 956, second proof] to conclude that there is a matrix $\sigma \in M_{d-1}(R)$ with determinant u^{d-2} modulo J such that, if $(a_1, \cdots, a_{d-1})\sigma = (f_1, \cdots, f_{d-1})$, then $J = (f_1, \cdots, f_{d-1})$. Let the generators $f_1, \cdots, f_{d-1}$ induce $\omega'' : (R/J)^{d-1} \twoheadrightarrow J/J^2$. Clearly, $0 = (J, \omega'') = (J, \overline{u}^{d-2}\omega)$.

Case 1. Let d be odd. Then, $(J, \omega') = (J, \overline{u}\omega) = (J, \overline{u}^{d-2}\omega) = 0$ by (4.9). Therefore, we are done by (4.7).

Case 2. Let d be even. We are required to prove that $(J, \overline{u}\omega) = 0$ and for that purpose, we may assume that $\overline{u}\omega$ is induced by $J = (ua_1, a_2, \cdots, a_{d-1}) + J^2$. We may further assume by some standard arguments that a_{d-1} is not a zero-divisor. Let $A = R/(a_{d-1})$ and 'tilde' denote reduction mod (a_{d-1}). It is easy to check that there is a group homomorphism $E^{d-2}(A) \longrightarrow E^{d-1}(R)$ (for an idea of proof, see [6, 7.4]). Then, in A we have $\widetilde{J} = (\widetilde{a_1}, \cdots, \widetilde{a_{d-2}})$. Also, $\widetilde{J} = (\widetilde{ua_1}, \widetilde{a_2}, \cdots, \widetilde{a_{d-2}}) + \widetilde{J^2}$, and it is enough to lift these to generators of $\widetilde{J}$. We now apply Case 1 and the morphism $E^{d-2}(A) \longrightarrow E^{d-1}(R)$ to conclude. □

It is now easy to prove the following result.

Theorem 5.9. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 5$ such that p and $(d-1)!$ are relatively prime. Let P be a projective R-module of rank $d-1$ with trivial determinant. Then $P \xrightarrow{\sim} Q \oplus R$ for some R-module Q if and only if there is a surjection $\alpha : P \twoheadrightarrow J$ such that $J \subset R$ is an ideal of height $d-1$ and $\mu(J) = d-1$.*

Proof. By a theorem of Eisenbud-Evans [10], there is a surjection $\beta : P \twoheadrightarrow J$ such that $J \subset R$ is an ideal of height $d-1$.

Let $P \xrightarrow{\sim} Q \oplus R$ for some R-module Q. We have proved in (5.1) that $\mu(J) = d-1$.

Conversely, assume that there is a surjection $\alpha : P \twoheadrightarrow I$ such that $I \subset R$ is an ideal of height $d-1$ and $\mu(I) = d-1$. Fix $\chi : R \xrightarrow{\sim} \wedge^{d-1}P$. The R/I-module P/IP is free. Choose $\sigma : (R/I)^{d-1} \xrightarrow{\sim} P/IP$ such that $\wedge^{d-1}\sigma = \chi \otimes R/I$. Let ω_I be the composite:

$$(R/I)^{d-1} \overset{\sigma}{\xrightarrow{\sim}} P/IP \overset{\overline{\alpha}}{\twoheadrightarrow} I/I^2.$$

Let ω_I correspond to $I = (f_1, \cdots, f_{d-1}) + I^2$.

By the hypothesis, $\mu(I) = d-1$. We can apply (5.8) and obtain $a_1, \cdots, a_{d-1} \in I$ such that $I = (a_1, \cdots, a_{d-1})$ where $a_i - f_i \in I^2$ for $1 \leq i \leq d-1$. Now we can apply (3.4). □

6. Applications II: Projective modules over threefolds

In this section we apply the Fasel-Rao-Swan cancellation theorem [11, 7.5] to the results obtained in the previous sections to answer Question 1.6 raised in the introduction. Throughout this section we assume that $p \neq 2$ and R is smooth.

Theorem 6.1. *Let $p \neq 2$, and let R be a smooth affine algebra of dimension 3 over $\overline{\mathbb{F}}_p$ and P, Q be projective R-modules of rank 2, each with trivial determinant. Fix isomorphisms $\chi_P : R \xrightarrow{\sim} \wedge^2 P$ and $\chi_Q : R \xrightarrow{\sim} \wedge^2 Q$. Then, $e_2(P, \chi_P) = e_2(Q, \chi_Q)$ in $E^2(R)$ if and only if P is isomorphic to Q.*

Proof. Let us first assume that P is isomorphic to Q. Let $\gamma : P \xrightarrow{\sim} Q$ be an isomorphism. Let $\alpha : Q \twoheadrightarrow J$ be a surjection, where J is an ideal of height 2. Suppose that (α, χ_Q) induces $e_2(Q, \chi_Q) = (J, \omega)$. Note that $\wedge^2(\gamma) : \wedge^2 P \xrightarrow{\sim} \wedge^2 Q$ is basically multiplication by a unit $\overline{u} \in (R/J)^*$. It then easily follows that $(\alpha\gamma, \chi_P)$ will induce $e(P, \chi_P) = (J, \overline{u}\omega)$. But $(J, \omega) = (J, \overline{u}\omega)$ by (5.5).

Conversely, assume that $e_2(P, \chi_P) = e_2(Q, \chi_Q)$ in $E^2(R)$. Let $e_2(P, \chi_P) = (J, \omega_J) = e_2(Q, \chi_Q)$. Then, by (4.13), there exist surjections $\alpha : P \twoheadrightarrow J$, $\beta : Q \twoheadrightarrow J$ such that (α, χ_P) and (β, χ_Q) both induce ω_J. In other words, we have a commutative diagram:

$$\begin{array}{ccccc} (R/J)^2 & \xrightarrow{\delta_1} & P/JP & \xrightarrow{\overline{\alpha}} & J/J^2 \\ \downarrow{\scriptstyle\sigma} & & & & \downarrow \\ (R/J)^2 & \xrightarrow{\delta_2} & Q/JQ & \xrightarrow{\overline{\beta}} & J/J^2 \end{array}$$

where $\sigma \in SL_2(R/J)$, $\wedge^2\delta_1 = \chi_P \otimes R/J$, $\wedge^2\delta_2 = \chi_Q \otimes R/J$, and the right vertical map is the identity. Let $\theta = \delta_2\sigma\delta_1^{-1}$. Then we have $\theta : P/JP \xrightarrow{\sim} Q/JQ$, $\overline{\alpha} = \overline{\beta}\theta$, and $\wedge^2\theta = (\wedge^2\delta_2)(\wedge^2\delta_1)^{-1}$ (as $\sigma \in SL_2(R/J)$). We can now apply [3, 3.5] to complete the proof. □

We can now answer Question 1.6 raised in the introduction.

Corollary 6.2. *Let $p \neq 2$, and let R be a smooth affine algebra of dimension 3 over $\overline{\mathbb{F}}_p$, and P, Q be projective R-modules of rank 2, each with trivial determinant. Then, $P \xrightarrow{\sim} Q$ if and only if there is an ideal $J \subset R$ of height 2 such that both P and Q map onto J.*

Proof. One way it is trivial. So let there exist an ideal J of height 2 such that there are surjections $\alpha : P \twoheadrightarrow J$ and $\beta : Q \twoheadrightarrow J$. Fix $\chi_P : R \xrightarrow{\sim} \wedge^2 P$ and $\chi_Q : R \xrightarrow{\sim} \wedge^2 Q$. Let (α, χ_P) induce $e_2(P, \chi_P) = (J, \omega_1)$ and (β, χ_Q) induce $e_2(Q, \chi_Q) = (J, \omega_2)$. The proof is complete by applying (5.5) and (6.1). □

Corollary 6.3. *Let $p \neq 2$, and let R be a smooth affine algebra of dimension 3 over $\overline{\mathbb{F}}_p$ and P be a projective R-module of rank 2 with trivial determinant. Then, P is cancellative.*

Proof. Let Q be an R-module of rank two such that $P \oplus R^n \xrightarrow{\sim} Q \oplus R^n$ for some n. As $P \oplus R$ is cancellative by [25], we have $P \oplus R \xrightarrow{\sim} Q \oplus R$. By [6, 6.7], there is an ideal $J \subset R$ of height at least two such that both P and Q map onto J. Applying the above corollary, we have $P \xrightarrow{\sim} Q$. □

REFERENCES

[1] Aravind Asok and Jean Fasel, A cohomological classification of vector bundles on smooth affine threefolds, *Duke Math. Journal* **163** (2014), 2561-2601.

[2] Aravind Asok and Jean Fasel, Splitting vector bundles outside the stable range and $\mathbb{A}^1$-homotopy sheaves of punctured affine spaces, *J. American Math. Soc.* **28** (2015), 1031-1062.

[3] S. M. Bhatwadekar, Cancellation theorems for projective modules over a two dimensional ring and its polynomial extensions, *Compositio Math.* **128** (2001), 339-359.

[4] S. M. Bhatwadekar and Raja Sridharan, Projective generation of curves in polynomial extensions of an affine domain and a question of Nori, *Invent. Math.* **133** (1998), 161-192.

[5] S. M. Bhatwadekar and Raja Sridharan, Zero cycles and the Euler class groups of smooth real affine varieties, *Invent. Math.* **136** (1999), 287-322.

[6] S. M. Bhatwadekar and Raja Sridharan, The Euler class group of a Noetherian ring, *Compositio Math.* **122** (2000), 183-222.

[7] S. M. Bhatwadekar and Raja Sridharan, On Euler classes and stably free projective modules, in: Algebra, arithmetic and geometry, Part I, II(Mumbai, 2000), 139–158, Tata Inst. Fund. Res. Stud. Math., 16, Tata Inst. Fund. Res., Bombay, 2002.

[8] M. K. Das, The Euler class group of a polynomial algebra, *J. Algebra* **264** (2003), 582-612.

[9] Alpesh M. Dhorajia and Manoj K. Keshari, note on cancellation of projective modules, *J. Pure and Appl. Algebra* **216** (2012), 126-129.

[10] D. Eisenbud and E. G. Evans, Generating modules efficiently: Theorems from algebraic K-Theory, *J. Algebra* **27** (1973), 278-305.

[11] Jean Fasel, Ravi A. Rao and R. G. Swan, On stably free modules over affine algebras, *Publ. Math. Inst. Hautes tudes Sci.* **116** (2012), 223243.

[12] Manoj K. Keshari, Euler Class group of a Noetherian ring, M.Phil. thesis, available at : https://arxiv.org/abs/1408.2645

[13] A. Krishna, V. Srinivas, Zero cycles on singular varieties, in Algebraic Cycles and Motives I, London Mathematical Society Lecture note ser., Cambridge University Press, 343, (2007), 264277.

[14] S. Mandal and Raja Sridharan, Euler classes and complete intersections, *J. Math. Kyoto Univ.* **36** (1996), 453-470.

[15] S. Mandal and M. P. Murthy, Ideals as sections of projective modules, *J. Ramanujan Math. Soc.* **13** (1998), 51–62.

[16] N. Mohan Kumar, Complete intersections, *J. Math. Kyoto Univ.* **17** (1977), 533-538.

[17] N. Mohan Kumar, Some theorems on generation of ideals in affine algebras, *Comment.Math.Helv.* **59** (1984), 243-252.

[18] N. Mohan Kumar, M. P. Murthy and Amit Roy, A cancellation theorem for projective modules over finitely generated rings, in : Algebraic geometry and commutative algebra, Vol. I, 281–287, Kinokuniya, Tokyo, 1988.

[19] M. P. Murthy, Zero cycles and projective modules, *Ann. Math.* **140** (1994), 405-434.

[20] D. Quillen, Projective modules over polynomial rings, *Invent. Math.* **36** (1976), 167-171.

[21] R. A. Rao, On completing unimodular polynomial vectors of length three, *Trans. Amer. Math. Soc.* **325** (1991), 231-239.

[22] J.-P. Serre, Modules projectifs et espaces fibres a fibre vectorielle, *Sem. Dubreil-Pisot* **23** (1957/58).

[23] Raja Sridharan, Non-vanishing sections of algebraic vector bundles,*J. Algebra* **176** (1995), 947-958.

[24] Raja Sridharan, Projective modules and complete intersections, *K-Theory* **13** (1998), 269-278.

[25] A. A. Suslin, A cancellation theorem for projective modules over algebras, *Soviet. Math. Dokl.* **18** (1977), 1281-1284.

[26] A. A. Suslin and L. N. Vaserstein, Serre's problem on projective modules over polynomial rings and algebraic K-theory, *Izv. Akad. Nauk. SSSR, Ser. Mat.* **40** (1976), 937-1001.

[27] Richard G. Swan, Serre's problem. *Conference on Commutative Algebra 1975* (Queen's Univ., Kingston, Ont., 1975), pp. 160. Queen's Papers on Pure and Applied Math., **42**, Queen's Univ., Kingston, Ont., 1975.

[28] L. N. Vaserstein, On the stabilization of the general linear group over a ring. *Mat. Sb. (N.S)* 79 (121) 405424 (Russian); translated as Math. USSR-Sb.8, 1969, 383400.

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