ORBIT SPACES OF UNIMODULAR ROWS OVER SMOOTH REAL AFFINE ALGEBRAS

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Dedicated to Professor S. M. Bhatwadekar on his seventieth birthday.

1. Introduction

Let R be a commutative Noetherian ring of (Krull) dimension d. It follows from a classical result of Bass [Ba] that the stably free R-modules of rank at least $d+1$ are all free. Let R be an affine algebra of dimension d over a field k. If k is algebraically closed, or more generally, if the cohomological dimension of k is at most one, Suslin proved that a stably free R-module of rank d is free (see [Su 1, Su 3]). These results of Suslin do not extend to arbitrary k. For example, if $d \neq 1, 3, 7$, the tangent bundle of a real d-sphere is stably free but not free. These examples also show that the aforementioned result of Bass is the best possible. Therefore, it is certainly of interest to understand the stably free R-modules of rank $d \geq 2$ when R is the coordinate ring of an affine variety over the field of real numbers. Among other results, we prove the following: *Let $X = Spec(R)$ be a smooth real affine variety of even dimension d, whose real points $X(\mathbb{R})$ constitute an orientable manifold. Then the set of isomorphism classes of (oriented) stably free R-modules of rank d is a free abelian group of rank equal to the number of compact connected components of $X(\mathbb{R})$.* In contrast, if $d \geq 3$ is odd, then the set of isomorphism classes of stably free R-modules of rank d is a $\mathbb{Z}/2\mathbb{Z}$-vector space (possibly trivial). We elaborate below.

The rings considered in this article are assumed to have (Krull) dimension at least two, unless mentioned otherwise. Recall that for any ring R of dimension d, a stably free R-module P of rank d corresponds to a *unimodular row* $(a_0, \cdots, a_d) \in R^{d+1}$ (meaning, there exist $b_0, \cdots, b_d \in R$ such that $\sum_0^d a_i b_i = 1$). The module P is free if and only if $(a_0, \cdots, a_d)$ is the first row of a matrix in $SL_{d+1}(R)$. Let $Um_{d+1}(R)$ be the set of unimodular rows of length $d+1$ over R. The preceding discussion inspires one to study the action of $SL_{d+1}(R)$ on $Um_{d+1}(R)$. The group $SL_{d+1}(R)$ and its elementary subgroup $E_{d+1}(R)$ act naturally on this set by multiplication from right. Thanks to the foundational works due to Vaserstein [SuVa, Section 5] (for $d = 2$) and

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van der Kallen [vdK 1] (for $d \geq 2$), the orbit space $Um_{d+1}(R)/E_{d+1}(R)$ carries the structure of an abelian group (inducing a group structure on $Um_{d+1}(R)/SL_{d+1}(R)$ as well [vdK 1]). Due to Jean Fasel's work [F 1], we now have a modern-day interpretation of $Um_{d+1}(R)/E_{d+1}(R)$ in terms of cohomology. In this article we compute this group and its quotient $Um_{d+1}(R)/SL_{d+1}(R)$, when R is a smooth affine domain over the reals of dimension $d \geq 2$. We now present our results one by one. But first, let us set up some notations.

Notation. Let $X = \text{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. We always assume that the set of real points $X(\mathbb{R})$ of X is non-empty, and therefore under the Euclidean topology, it is a smooth real manifold of dimension d. Let $\mathbb{R}(X)$ denote the ring obtained from R by inverting all the functions having no real zeros. Note that $\dim(R) = \dim(\mathbb{R}(X))$. Let $\mathcal{C}$ be the (finite) set of connected components of $X(\mathbb{R})$ which are compact. In this article we always assume that $X(\mathbb{R})$ is orientable.

The following is an accumulation of various results in the text (3.9, 4.5, 4.6, 4.9, 5.13).

Theorem 1.1. *Let $X = Spec(R)$ be as above. Then, we have the following assertions:*

1. $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \xrightarrow{\sim} \bigoplus_{C \in \mathcal{C}} \mathbb{Z}$.
2. *The canonical map $\beta : Um_{d+1}(R)/E_{d+1}(R) \to Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is a surjective morphism and $K = ker(\beta)$ is the unique maximal divisible subgroup of $Um_{d+1}(R)/E_{d+1}(R)$. Consequently, $Um_{d+1}(R)/E_{d+1}(R) \xrightarrow{\sim} K \oplus (\bigoplus_{C \in \mathcal{C}} \mathbb{Z})$.*
3. *Precisely, K consists of those elementary orbits which can be represented by a unimodular row whose one entry is a square. Further, for any $[(a_0, \cdots, a_d)] \in K$ and any $r \geq 1$, $[(a_0, \cdots, a_d)]^r = [(a_0, \cdots, a_d^r)]$.*
4. *If $d \geq 3$, then K is torsion-free.*

Let $[v] \in K$ be arbitrary. Using divisibility of K and taking $r = d!$ in (3) above, it immediately follows from a celebrated result of Suslin [Su 1] that v is the first row of a matrix in $SL_{d+1}(R)$. This observation makes the computation of $Um_{d+1}(R)/SL_{d+1}(R)$ quite easy. Note that the group $Um_{d+1}(R)/SL_{d+1}(R)$ is in bijection with the set of isomorphism classes of (oriented[1], if d is even) stably free R-modules of rank d. We prove the following result (Theorems 4.11 and 4.15 below):

Theorem 1.2. *Let $X = Spec(R)$ be as above. If the dimension d is <u>even</u>, then we have:*

$$\frac{Um_{d+1}(R)}{SL_{d+1}(R)} \xrightarrow{\sim} \frac{Um_{d+1}(\mathbb{R}(X))}{SL_{d+1}(\mathbb{R}(X))} \xrightarrow{\sim} \frac{Um_{d+1}(\mathbb{R}(X))}{E_{d+1}(\mathbb{R}(X))} \xrightarrow{\sim} \bigoplus_{C \in \mathcal{C}} \mathbb{Z}.$$

[1]For any ring R of dimension d, a unimodular row $(a_0, \cdots, a_d)$ gives rise to a stably free R-module P together with a canonical orientation $\chi : R \xrightarrow{\sim} \wedge^d(P)$. In this article, for d even, stably free modules are always chosen with an orientation. See Remark 2.8. We refer to [BRS 3, Page 214] for the details.

If d is odd, then $Um_{d+1}(R)/SL_{d+1}(R)$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space of rank $\leq |\mathcal{C}|$.

If d is odd, $Um_{d+1}(R)/SL_{d+1}(R)$ can be trivial. For example, by [F 1, Proposition 5.13], it is trivial when R is the coordinate ring of the real 3-sphere or the 7-sphere. We also touch on this in Subsection 4.3 using simpler arguments. For the other spheres of odd dimension, it follows from our results that this group is $\mathbb{Z}/2\mathbb{Z}$.

We now turn our attention to Mennicke symbols of Suslin. In [Su 2], Suslin used them to prove that the Milnor K-theory of a field injects into the Quillen K-theory modulo torsion. Our interest is in its connection with the group structure defined on $Um_{d+1}(A)/E_{d+1}(A)$, where A is a commutative Noetherian ring of dimension $d \geq 2$. In [vdK 2] van der Kallen introduced *weak Mennicke symbols* and showed that the universal weak Mennicke symbol $(wms, WMS_{d+1}(A))$ is in bijection with $Um_{d+1}(A)/E_{d+1}(A)$, thus giving the latter the structure of an abelian group. As a Mennicke symbol is also a weak Mennicke symbol, the universal Mennicke symbol $MS_{d+1}(A)$ is a quotient of $Um_{d+1}(A)/E_{d+1}(A)$. We prove the following results in Section 5.

Theorem 1.3. *Let $X = Spec(R)$ be as in Theorem 1.1. Then,*

1. $MS_{d+1}(\mathbb{R}(X)) \xrightarrow{\sim} \bigoplus_{C\in\mathcal{C}} \mathbb{Z}/2\mathbb{Z}$.
2. *The kernel L of the canonical surjection $\beta_0 : MS_{d+1}(R) \twoheadrightarrow MS_{d+1}(\mathbb{R}(X))$ is the unique maximal divisible subgroup of $MS_{d+1}(R)$. Consequently, $MS_{d+1}(R) \xrightarrow{\sim} L \oplus (\bigoplus_{C\in\mathcal{C}} \mathbb{Z}/2\mathbb{Z})$.*
3. *The kernel of the canonical surjection $Um_{d+1}(R)/E_{d+1}(R) \twoheadrightarrow MS_{d+1}(R)$ is a free abelian group of rank $|\mathcal{C}|$.*
4. *If $d \geq 3$, then L is torsion-free.*

We now spend a few words on our methods. As it turns out, the computation of $Um_{d+1}(R)/E_{d+1}(R)$, with explicit description of its maximal divisible subgroup K, is the key. Such computations become easier if there is another related group to compare with, whose structure is well-understood. Recall from [BRS 3, DZ, vdK 3] that if A is a Noetherian ring of dimension $d \geq 2$, there is an exact sequence

$$Um_{d+1}(A)/E_{d+1}(A) \xrightarrow{\phi_A} E^d(A) \longrightarrow E_0^d(A) \longrightarrow 0, \qquad (*)$$

where $E^d(A)$ is the d-th Euler class group of A and $E_0^d(A)$ is the d-th weak Euler class group of A. We recall the definition of $E^d(A)$ in Section 2. We do not use $E_0^d(A)$ in this article.

For smooth affine real varieties the following structure theorem was proved in [BRS 2].

Theorem 1.4. [BRS 2] *Let R be as in Theorem 1.1. Then, $E^d(R) \xrightarrow{\sim} E^d(\mathbb{C}) \oplus E^d(\mathbb{R}(X))$, where $E^d(\mathbb{C})$ is the subgroup generated by all those Euler cycles in $E^d(R)$, which are supported* *on complex maximal ideals of R. Further, $E^d(\mathbb{C})$ is uniquely divisible and $E^d(\mathbb{R}(X))$ is free abelian of rank $|\mathcal{C}|$.*

We compare the elementary orbit spaces with the Euler class groups. As mentioned in $(*)$ above, we have morphisms $\phi_{\mathbb{R}(X)} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \to E^d(\mathbb{R}(X))$, and $\phi_R : Um_{d+1}(R)/E_{d+1}(R) \to E^d(R)$. But we found these maps to be insufficient for our purposes. To have more leverage, we take a reverse path, as follows.

Let A be a regular domain of dimension $d \geq 2$, which is essentially of finite type over an infinite perfect field k such that $2A = A$. Based on the formalism developed in our earlier paper [DTZ], in Section 2 we introduce a map $\delta_A : E^d(A) \to Um_{d+1}(A)/E_{d+1}(A)$. When $k = \mathbb{R}$, this map gives us a lot of control.

Again let R be as in Theorem 1.1. In Section 3 we prove that $\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \to Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is an isomorphism. In Section 4 we prove that $\delta_R : E^d(R) \to Um_{d+1}(R)/E_{d+1}(R)$ is a morphism which is trivial on the divisible component $E^d(\mathbb{C})$. This enables us to analyze the kernel K of the natural map $\beta : Um_{d+1}(R)/E_{d+1}(R) \to Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ and deduce our main results. Composing with the canonical projection $\epsilon : Um_{d+1}(R)/E_{d+1}(R) \to Um_{d+1}(R)/SL_{d+1}(R)$, we also have a morphism $\delta'_R : E^d(R) \to Um_{d+1}(R)/SL_{d+1}(R)$, which turns out to be surjective (Theorem 4.12). Finally, the relation between the Euler class group and the elementary orbit space can be summed up in the form of the following exact sequence:

$$0 \to E^d(\mathbb{C}) \to E^d(R) \xrightarrow{\delta_R} Um_{d+1}(R)/E_{d+1}(R) \to K \to 1$$

We believe our methods are direct and basic and should be accessible to a wide class of audience. Evidently, the maps δ_R and $\delta_{\mathbb{R}(X)}$ make a lot of arguments remarkably easier, which can be seen in Sections 3 and 4.

When the real variety $X = \text{Spec}(R)$ is rational, Jean Fasel carried out some computation of the orbit spaces and Mennicke symbol using cohomological methods in [F 1, F 2]. Results in [F 1, Section 5] inspired us to take up this project.

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2. GENERALITIES: THE OBJECTS AND THE MAPS

In this section we recall some basic definitions and collect some useful results. We also establish a map δ_R from the Euler class group $E^d(R)$ to the group $Um_{d+1}(R)/E_{d+1}(R)$, when R is a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field of characteristic $\neq 2$. This definition involves homotopy orbits of certain objects. By *'homotopy'* we mean *'naive homotopy'*, as defined below.

Definition 2.1. Let F be a functor originating from the category of rings to the category of sets. For a given ring R, two elements $F(u_0), F(u_1) \in F(R)$ are said to be homotopic if there is an element $F(u(T)) \in F(R[T])$ such that $F(u(0)) = F(u_0)$ and $F(u(1)) = F(u_1)$.

Definition 2.2. Let F be a functor from the category of rings to the category of sets. Let R be a ring. Consider the equivalence relation on $F(R)$ *generated by* homotopies (the relation is easily seen to be reflexive and symmetric but is not transitive in general). The set of equivalence classes thus obtained will be denoted by $\pi_0(F(R))$ and an equivalence class will be called a *homotopy orbit*.

2.1. **Homotopy orbits of unimodular rows.**

For a ring R, consider the set

$$Um_{n+1}(R) := \{(a_1, \cdots, a_{n+1}) \in R^{n+1} \mid \sum_{i=1}^{n+1} a_i b_i = 1 \text{ for some } b_1, \cdots, b_{n+1} \in R\}$$

of *unimodular rows* of length $n+1$ in R. Then $F_{n+1}(R) := Um_{n+1}(R)$ is a functor. Two unimodular rows $(a_1, \cdots, a_{n+1})$ and $(a'_1, \cdots, a'_{n+1})$ are homotopic if there is $(f_1(T), \cdots, f_{n+1}(T)) \in Um_{n+1}(R[T])$ such that $f_i(0) = a_i$ and $f_i(1) = a'_i$ for $i = 1, \cdots, n+1$. The set $Um_{n+1}(R)$ has a base point, namely, $(0, \cdots, 0, 1)$.

We shall need the following theorem later. See also [F 1, Theorem 2.1] for a more general version.

Theorem 2.3. *Let R be a regular ring containing a field k. Then, for any $n \geq 2$ there is a bijection $\eta_R : \pi_0(Um_{n+1}(R)) \xrightarrow{\sim} Um_{n+1}(R)/E_{n+1}(R)$.*

Proof. Let $v \in Um_{n+1}(R)$. We define η_R by sending the homotopy orbit of v to the elementary orbit of v. But we have to ensure that η_R is well-defined. Let $u \in Um_{n+1}(R)$ be such that v is homotopic to u. Then, by definition, there exists $f(T) \in Um_{n+1}(R[T])$ such that $f(0) = v$ and $f(1) = u$. As R is a regular ring containing a field k, it follows from [Li, Po] that $f(T)$ is extended from R. In other words, there exists $\sigma(T) \in GL_{n+1}(R[T])$ such that $f(T)\sigma(T) = f(0)$. Therefore, $f(0)\sigma(0) = f(0)$. It then follows that $f(T)\sigma(T)\sigma(0)^{-1} = f(0)$. Writing $\tau = \sigma(T)\sigma(0)^{-1}$ we see that $\tau \in GL_{n+1}(R[T])$ and $\tau(0) = I_{n+1}$. By a result of Vorst [V, Theorem 3.3], we actually have $\tau \in E_{n+1}(R[T])$. As $u\tau(1) = f(1)\tau(1) = f(0) = v$, we are done proving that η_R is well-defined. Injectivity of η_R is clear because elementary matrices are homotopic to identity. Surjectivity is trivial. □

2.2. **The pointed set $Q'_{2n}(R)$ and its homotopy orbits:** Let R be any commutative Noetherian ring. Let $n \geq 2$ and consider the set

$$Q'_{2n}(R) = \{(x_1, \cdots x_n, y_1, \cdots, y_n, z) \in R^{2n+1} \mid \sum_{i=1}^{n} x_i y_i + z^2 = 1\}$$

with a base point $(0, \cdots, 0, 0, \cdots, 0, 1)$. Observe that there is an obvious map from $Q'_{2n}(R)$ to $Um_{n+1}(R)$ taking $(x_1, \cdots x_n, y_1, \cdots, y_n, z)$ to $(x_1, \cdots x_n, z)$, which will induce a set-theoretic map $\zeta_R : \pi_0(Q'_{2n}(R)) \to \pi_0(Um_{n+1}(R))$ between the homotopy orbit spaces.

2.3. **The Euler class group.** Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. Let B be the set of pairs (m, ω_m) where m is a maximal ideal of R and $\omega_m : (R/m)^d \twoheadrightarrow m/m^2$. Let G be the free abelian group generated by B. Let $J = m_1 \cap \cdots \cap m_r$, where m_i are distinct maximal ideals of R. Any $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ induces surjections $\omega_i : (R/m_i)^d \twoheadrightarrow m_i/m_i^2$ for each i. We associate $(J, \omega_J) := \sum_1^r (m_i, \omega_i) \in G$. Now, Let S be the set of elements (J, ω_J) of G for which ω_J has a lift to a surjection $\theta : R^d \twoheadrightarrow J$ and H be the subgroup of G generated by S . The Euler class group $E^d(R)$ is defined as $E^d(R) := G/H$.

Remark 2.4. The above definition appears to be slightly different from the one given in [BRS 1]. However, note that if $(J, \omega_J) \in S$ and if $\overline{\sigma} \in E_d(R/J)$, then the element $(J, \omega_J \overline{\sigma})$ is also in S. For details, see [DZ, Proposition 2.2].

Theorem 2.5. [BRS 1, 4.11] *Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. Let $J \subset R$ be a reduced ideal of height d and $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ be a surjection. Then, the following are equivalent:*

(1) *The image of $(J, \omega_J) = 0$ in $E^d(R)$*
(2) *ω_J can be lifted to a surjection $\theta : R^d \twoheadrightarrow J$.*

Remark 2.6. We shall refer to the elements of the Euler class group as *Euler cycles*. Let $I \subset R$ be an ideal of height d which is not necessarily reduced and let $\omega_I : (R/I)^d \twoheadrightarrow I/I^2$ be a surjection. Then also one can associate an element (I, ω_I) in $E^d(R)$ and prove the above theorem for (I, ω_I) (see [BRS 1, Remark 4.16]). Further, by [BRS 1, Remark 4.14], an arbitrary element of $E^d(R)$ can be represented by a single Euler cycle (J, ω_J), where J is a reduced ideal of height d and $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ is a surjection.

The following notation will be used in the rest of this article.

Notation. Let $\dim(R) = d$. Let $(J, \omega_J) \in E^d(R)$ and $u \in R$ be a unit modulo J. Let σ be any diagonal matrix in $GL_d(R/J)$ with determinant $\overline{u}$ (bar means modulo J). We shall denote the composite surjection

$$(R/J)^d \overset{\sigma}{\underset{\sim}{\to}} (R/J)^d \overset{\omega_J}{\twoheadrightarrow} J/J^2$$

by $\overline{u}\omega_J$. It is easy to check that the element $(J, \overline{u}\omega_J) \in E^d(R)$ is independent of σ (the key fact used here is that $SL_d(R/J) = E_d(R/J)$ as $\dim(R/J) = 0$).

2.4. **[The map $\phi_R : Um_{d+1}(R)/E_{d+1}(R) \to E^d(R)$].** Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. We now recall the definition of a group homomorphism $\phi_R : Um_{d+1}(R)/E_{d+1}(R) \to E^d(R)$. When d is even, ϕ_R has been defined in [BRS 3]. The extension to general d is available in [DZ, vdK 3]. We urge the reader to look at [DZ, Section 4] for the details.

Definition 2.7. Let $v = (a_1, \cdots, a_{d+1}) \in Um_{d+1}(R)$. Applying elementary transformations if necessary, we may assume that the height of the ideal $(a_1, \cdots, a_d)$ is d. Write $J = (a_1, \cdots, a_d)$ and let $\omega_J : R^d \twoheadrightarrow J$ be the surjection induced by $(a_1, \cdots, a_d)$. As a_{d+1} is a unit modulo J, we have $J = (a_1, \cdots, a_d a_{d+1}) + J^2$ and the corresponding element in $E^d(R)$ is $(J, \overline{a_{d+1}}\omega_J)$. Let $[v]$ denote the orbit of v in $Um_{d+1}(R)/E_{d+1}(R)$. Define $\phi_R([v]) = (J, \overline{a_{d+1}}\omega_J)$. It is proved in [DZ, vdK 3] that ϕ_R is a morphism.

Remark 2.8. When d is even, the above definition coincides with the one given in [BRS 3]. A short remark on the definition given in [BRS 3] is in order. Note that the unimodular row v gives rise to a stably free R-module, say, P of rank d together with a canonical orientation $\chi : R \xrightarrow{\sim} \wedge^d(P)$. Bhatwadekar-Sridharan defines $\phi_R([v])$ to be the *Euler class* of the pair (P, χ) which resides in $E^d(R)$. The computation of this Euler class in [BRS 3, Page 214] shows that it turns out to be exactly the one given above, namely, $(J, \overline{a_{d+1}}\omega_J)$.

2.5. **[The map $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$].** Let R be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k with Char$(k) \neq 2$. In [DTZ, Section 2 and Theorem 4.6] we established a set-theoretic bijection[2] from $E^d(R)$ to $\pi_0(Q'_{2d}(R))$ whose description goes as follows . Let $(J, \omega_J) \in E^d(R)$, where J is a reduced ideal of height d. Now $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ is given by $J = (a_1, \cdots, a_d) + J^2$, for some $a_1, \cdots, a_d \in J$. Applying the Nakayama Lemma one obtains $s \in J^2$ such that $J = (a_1, \cdots, a_d, s)$ with $s - s^2 = a_1 b_1 + \cdots + a_d b_d$ for some $b_1, \cdots, b_d \in R$ (see [Mo] for a proof). The assignment of (J, ω_J) to the homotopy orbit $[(2a_1, \cdots, 2a_d, 2b_1, \cdots, 2b_d, 1 - 2s)]$ in $\pi_0(Q'_{2d}(R))$ is a well-defined set-theoretic map

[2]Such a bijection has also been obtained in [AF, MaMi] using different arguments.

and is in fact, a bijection [DTZ, Section 2 and Theorem 4.6]. Let us call this map as ψ_R. Then the following composite

$$E^d(R) \xrightarrow{\psi_R} \pi_0(Q'_{2d}(R)) \xrightarrow{\zeta_R} \pi_0(Um_{d+1}(R)) \xrightarrow{\eta_R} Um_{d+1}(R)/E_{d+1}(R)$$

gives a set-theoretic map from $E^d(R)$ to $Um_{d+1}(R)/E_{d+1}(R)$. Let us call it δ_R.

Remark 2.9. Thus $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ takes (J, ω_J) (where ω_J is induced by $a_1, \cdots, a_d, s$, as above) to the orbit $[(2a_1, \cdots 2a_d, 1-2s)] \in Um_{d+1}(R)/E_{d+1}(R)$, where $(1-2s)^2 \equiv 1$ modulo the ideal $(2a_1, \cdots 2a_d)$. Conversely, let an orbit $[v] = [(x_1, \cdots, x_d, z)] \in Um_{d+1}(R)/E_{d+1}(R)$ be such that the ideal $(x_1, \cdots, x_d)$ is reduced of height d, and $z^2 \equiv 1$ modulo $(x_1, \cdots, x_d)$, then $[v]$ is in the image of δ_R.

Notation. An orbit $[(x_1, \cdots, x_d, z)] \in Um_{d+1}(R)/E_{d+1}(R)$ will be written as $[x_1, \cdots, x_d, z]$.

We now compute the composite map $\phi_R\delta_R : E^d(R) \longrightarrow E^d(R)$. The description of this composite will play a very important role in the next section.

Theorem 2.10. *Let R be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k with Char$(k) \neq 2$. For any $(J, \omega_J) \in E^d(R)$, we have*

$$\phi_R\delta_R((J, \omega_J)) = (J, 2^d\omega_J) - (J, -2^d\omega_J).$$

Consequently, if d is even or if $\sqrt{2} \in R$, then $\phi_R\delta_R((J, \omega_J)) = (J, \omega_J) - (J, -\omega_J)$.

Proof. Suppose that ω_J is given by $J = (a_1, \cdots, a_d) + J^2$. Using some standard arguments we may assume that ht$(a_1, \cdots, a_d) = d$. There is $s \in J^2$ with $s - s^2 \in (a_1, \cdots, a_d)$. Now, $s - s^2 = a_1b_1 + \cdots + a_db_d$, for some $b_1, \cdots, b_d \in R$. Then $\delta_R((J, \omega_J)) = [2a_1, \cdots, 2a_d, 1-2s] \in Um_{d+1}(R)/E_{d+1}(R)$. Write $K = (2a_1, \cdots, 2a_d) = (a_1, \cdots, a_d)$ (as $\frac{1}{2} \in R$).

If we write $J' = (a_1, \cdots, a_d, 1-s)$, then it is easy to see that $K = J \cap J'$, and $J' = (a_1, \cdots, a_d) + J'^2$. Therefore, we write $0 = (K, \omega_K) = (J, \omega_J) + (J', \omega_{J'})$ in $E^d(R)$, where ω_K is induced by the generators $a_1, \cdots, a_d$ of K and $\omega_{J'}$ is induced from the data $J' = (a_1, \cdots, a_d) + J'^2$.

Now, from the definition of ϕ_R it follows that $\phi_R\delta_R((J, \omega_J)) = (K, \overline{(1-2s)2^d}\omega_K)$. We write $u = (1-2s)$. Then, we have (here 'tilde' means modulo J^2, and so on),

$$\phi_R\delta_R((J, \omega_J)) = (K, \overline{u2^d}\omega_K) = (J, \widetilde{u2^d}\omega_J) + (J', \overline{\overline{u2^d}}\omega_{J'})$$

As $1 - 2s \equiv 1$ mod J and $1 - 2s \equiv -1$ mod J', we have $\phi_R\delta_R((J, \omega_J)) = (J, 2^d\omega_J) + (J', -2^d\omega_{J'})$. Further, note that $(J, -2^d\omega_J) + (J', -2^d\omega_{J'}) = 0$. Therefore, finally we have,

$$\phi_R\delta_R((J, \omega_J)) = (J, 2^d\omega_J) - (J, -2^d\omega_J).$$

If d is even or $\sqrt{2} \in R$, then 2^d is a square and it follows from [BRS 2, Lemma 3.4] that $\phi_R\delta_R((J, \omega_J)) = (J, \omega_J) - (J, -\omega_J)$. $\square$

The proof of the following corollary is routine and we omit the proof.

Corollary 2.11. *The composite $\phi_R\delta_R : E^d(R) \longrightarrow E^d(R)$ is a morphism of groups.*

Proposition 2.12. *Let R be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k with Char$(k) \neq 2$. Let $(J, \omega_J) \in E^d(R)$. Then $\delta_R((J, \omega_J) + (J, -\omega_J))$ is the trivial orbit in $Um_{d+1}(R)/E_{d+1}(R)$.*

Proof. Let ω_J be induced by $J = (a_1, \cdots, a_d) + J^2$. Take

$$K = (a_1, \cdots, a_{d-1}) + J^2.$$

Then $K = (a_1, \cdots, a_{d-1}, a_d^2) + K^2$ and by [BDM, Lemma 3.6], the corresponding ω_K will give $(K, \omega_K) = (J, \omega_J) + (J, -\omega_J)$ in $E^d(R)$.

Now there exists $t \in K^2$ such that $K = (a_1, \cdots, a_{d-1}, a_d^2, t)$ with

$$t - t^2 \in (a_1, \cdots, a_{d-1}, a_d^2).$$

We have, $\delta_R(K, \omega_K) = [2a_1, \cdots, 2a_{d-1}, 2a_d^2, 1 - 2t] = [2a_1, \cdots, 2a_{d-2}, 4a_{d-1}, a_d^2, 1 - 2t]$ (applying [vdK 2, Lemma 3.5 (iv)] here). But if we move the square, we remain in the same elementary orbit ([vdK 2, Lemma 3.5 (vi)]), implying that

$$[2a_1, \cdots, 2a_{d-2}, 4a_{d-1}, a_d^2, 1 - 2t] = [2a_1, \cdots, 2a_{d-2}, 4a_{d-1}, a_d, (1 - 2t)^2].$$

But $(1 - 2t)^2$ is 1 modulo $(a_1, \cdots, a_{d-1}, a_d^2)$ and therefore it is also 1 modulo the ideal $(2a_1, \cdots, 2a_{d-2}, 4a_{d-1}, a_d)$. As a consequence, this orbit is trivial. □

3. The "real" coordinate ring

We first prove a key lemma which is inspired by the proof of [OPS, Proposition 2.1].

Lemma 3.1. *Let R be a smooth affine domain over $\mathbb{R}$ of dimension $d \geq 2$. Let $v \in Um_{d+1}(R)$. Then there is some $t \in R$ and $(x_1, \cdots, x_d, z) \in Um_{d+1}(R)$ such that:*

1. *$[v] = [x_1, \cdots, x_d, z]$ in $Um_{d+1}(R)/E_{d+1}(R)$ (and hence in $Um_{d+1}(R)/SL_{d+1}(R)$);*
2. *$(zt^2)^2 \equiv 1$ modulo the (reduced) ideal $(x_1, \cdots, x_d)$;*
3. *$[x_1, \cdots, x_d, zt^2] = [x_1, \cdots, x_d, z][x_1, \cdots, x_d, t^2]$ in $Um_{d+1}(R)/E_{d+1}(R)$ (and hence in $Um_{d+1}(R)/SL_{d+1}(R)$);*
4. *The orbit $[x_1, \cdots, x_d, zt^2]$ is in the image of $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$.*

Proof. Let $v = (y_1, \cdots, y_d, w)$. We can use Swan's Bertini Theorem [BRS 2, Theorem 2.11] and find $\alpha_1, \cdots, \alpha_d \in R$ such that the ideal $I = (y_1 + \alpha_1 w, \cdots, y_d + \alpha_d w)$ is a reduced ideal of height d. We write $x_i = y_i + \alpha_i w$ for $i = 1, \cdots, d$, and we rename w as z. Then note that $[v] = [x_1, \cdots, x_d, z]$ in $Um_{d+1}(R)/E_{d+1}(R)$ (and hence in $Um_{d+1}(R)/SL_{d+1}(R)$).

As $(x_1, \cdots, x_d, z)$ is unimodular, there exist $b_1, \cdots, b_d, b \in R$ such that $x_1b_1 + \cdots + x_db_d + zb = 1$. Now, R/I is a finite direct product of $\mathbb{R}$ or $\mathbb{C}$. Therefore, the unit $\overline{b}^2 \in (R/I)^*$ is a fourth power, say, $\overline{b}^2 = \overline{t}^4$. Let $t \in R$ be a lift of $\overline{t}$. Then $z^2t^4 \equiv 1$ modulo I. Therefore, there exist $a_1, \cdots, a_d \in R$ such that $x_1a_1 + \cdots + x_da_d + (zt^2)^2 = 1$. It is then easy to see that the orbit $[x_1, \cdots, x_d, zt^2]$ is in the image of $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ (see Remark 2.9). Statement (3) is simply [vdK 2, Lemma 3.5 (v)]. □

We set up some notations. Let $X = \text{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Let $X(\mathbb{R})$ denote the set of real points of X. We assume that $X(\mathbb{R}) \neq \emptyset$. Therefore, under the Euclidean topology, $X(\mathbb{R})$ is a smooth real manifold of dimension d. Let $\mathbb{R}(X)$ denote the ring (informally dubbed as the "real" coordinate ring of the variety) obtained from R by inverting all functions which do not have any real zeroes. Since $\mathbb{R}(X)$ is a localization of R and $\dim(R) = \dim(\mathbb{R}(X))$, there is a canonical surjective group homomorphism $\Gamma : E^d(R) \twoheadrightarrow E^d(\mathbb{R}(X))$ (see [BRS 2, page 307].

Theorem 3.2. *The map $\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \longrightarrow Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is surjective.*

Proof. Take any orbit $[v] \in Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$. Note that Lemma 3.1 applies to $\mathbb{R}(X)$ as well. Therefore, we have

$$[x_1, \cdots, x_d, zt^2] = [x_1, \cdots, x_d, z][x_1, \cdots, x_d, t^2] \text{ in } Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)),$$

such that $[v] = [x_1, \cdots, x_d, z]$. The row $(x_1, \cdots, x_d, t^2)$ can be taken to $(x_1, \cdots, x_d, x_1^2 + \cdots + x_d^2 + t^2)$ using elementary transformations. Since $x_1^2 + \cdots + x_d^2 + t^2$ does not vanish at any real point, it is a unit in $\mathbb{R}(X)$. Consequently, $[x_1, \cdots, x_d, t^2]$ is trivial in $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ and the proof is complete by Lemma 3.1 (4). □

In this article we assume that $X(\mathbb{R})$ is orientable. In this case, the real line bundle on $X(\mathbb{R})$ induced by the canonical bundle $K_R := \wedge^d(\Omega_{R/\mathbb{R}})$ is trivial and therefore, by [BDM, Theorem 4.21], $E^d(\mathbb{R}(X))$ is torsion-free. We use only this piece of (nontrivial) information to prove that $\delta_{\mathbb{R}(X)}$ is bijective in Theorem (3.5) below. But before that we collect a crucial result from [BRS 2] in the form of the following proposition and a corollary (as they are implicit in [BRS 2]).

Proposition 3.3. *Let $\mathfrak{m}$ be a real maximal ideal of R. Assume that the real point corresponding to $\mathfrak{m}$ belongs to a compact connected component of $X(\mathbb{R})$. Then, for any $\omega_{\mathfrak{m}} : (R/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$, one has $(\mathfrak{m}, \omega_{\mathfrak{m}}) + (\mathfrak{m}, -\omega_{\mathfrak{m}}) = 0$ in $E^d(\mathbb{R}(X))$.*

Proof. See toward the end of the proof of [BRS 2, Theorem 4.13]. □

Corollary 3.4. *Let $J \subset \mathbb{R}(X)$ be a reduced ideal and $\omega_J : (\mathbb{R}(X)/J)^d \twoheadrightarrow J/J^2$ be a surjection. Then $(J, \omega_J) + (J, -\omega_J) = 0$ in $E^d(\mathbb{R}(X))$.*

Proof. Let $J = \mathfrak{m}_1 \cap \cdots \mathfrak{m}_r \cap \mathfrak{m}_{r+1} \cap \cdots \mathfrak{m}_s$. Assume that the real points corresponding to the maximal ideals $\mathfrak{m}_{r+1}, \cdots, \mathfrak{m}_s$ do not belong to any compact connected component of $X(\mathbb{R})$. Now ω_J will induce $\omega_i : (\mathbb{R}(X)/\mathfrak{m}_i)^d \twoheadrightarrow \mathfrak{m}_i/\mathfrak{m}_i^2$ for $i = 1, \cdots, s$ and we have: $(J, \omega_J) = \sum_{i=1}^{s} (\mathfrak{m}_i, \omega_i)$. By (the proof of) [BRS 2, Theorem 4.13], $(\mathfrak{m}_i, \omega_i) = 0$ for $i = r+1, \cdots, s$. The corollary now follows from the above proposition. □

Theorem 3.5. *The map $\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \to Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is a bijection.*

Proof. We proved above that $\delta_{\mathbb{R}(X)}$ is surjective. To prove that $\delta_{\mathbb{R}(X)}$ is injective, it is enough to prove that $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)}$ is injective. Since $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)}$ is a morphism of groups by Theorem 2.10, we pick $(J, \omega_J) \in E^d(\mathbb{R}(X))$ (with J reduced) such that $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)}((J, \omega_J)) = 0$ and prove that $(J, \omega_J) = 0$.

By the assumption, we have $(J, \omega_J) - (J, -\omega_J) = 0$. But as J is reduced, by Corollary 3.4 we also have $(J, \omega_J) + (J, -\omega_J) = 0$ in $E^d(\mathbb{R}(X))$. Therefore, $2(J, \omega_J) = 0$. But under the assumptions on $X(\mathbb{R})$, the group $E^d(\mathbb{R}(X))$ has no nontrivial torsion (see [BRS 2]). Therefore, $(J, \omega_J) = 0$. □

Corollary 3.6. *$\phi_{\mathbb{R}(X)} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \longrightarrow E^d(\mathbb{R}(X))$ is injective.*

Proof. As $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)}$ is injective and $\delta_{\mathbb{R}(X)}$ is a surjection, the result follows. □

The set-theoretic map $\delta_{\mathbb{R}(X)}$ turns out to be a group homomorphism.

Theorem 3.7. *The group homomorphism $\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \to Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is in fact an isomorphism of groups, where the group structure on $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is the one given in* [vdK 1].

Proof. Let us denote the group composition in $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ by $*$. In this proof the actual representation of elements would not matter. Therefore, let $\alpha, \beta \in E^d(\mathbb{R}(X))$. Our aim is to show that $\delta_{\mathbb{R}(X)}(\alpha + \beta) = \delta_{\mathbb{R}(X)}(\alpha) * \delta_{\mathbb{R}(X)}(\beta)$ (here $+$ is the group composition of the Euler class group). As $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)} : E^d(R) \longrightarrow E^d(R)$ is a morphism,

$$\begin{aligned}\phi_{\mathbb{R}(X)}(\delta_{\mathbb{R}(X)}(\alpha+\beta)) &= (\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)})(\alpha+\beta) = (\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)})(\alpha) + (\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)})(\beta) \\ &= \phi_{\mathbb{R}(X)}(\delta_{\mathbb{R}(X)}(\alpha)) + \phi_{\mathbb{R}(X)}(\delta_{\mathbb{R}(X)}(\beta)) = \phi_{\mathbb{R}(X)}(\delta_{\mathbb{R}(X)}(\alpha) * \delta_{\mathbb{R}(X)}(\beta))\end{aligned}$$

As $\phi_{\mathbb{R}(X)}$ is injective, we have $\delta_{\mathbb{R}(X)}(\alpha + \beta) = \delta_{\mathbb{R}(X)}(\alpha) * \delta_{\mathbb{R}(X)}(\beta)$. □

Let $X(\mathbb{R})$ be connected but not compact. Then we know from [BRS 2, Corollary 4.9] that the Euler class group $E^d(\mathbb{R}(X))$ is trivial. The same conclusion is now immediate for the group $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$.

Corollary 3.8. *Let $X(\mathbb{R})$ be connected but not compact. The group $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is then trivial.*

Proof. We have $\delta_{\mathbb{R}(X)}$ surjective and under the assumptions, $E^d(\mathbb{R}(X))$ is trivial by [BRS 2, Corollary 4.9]. $\square$

As a consequence of the results obtained in this section and the structure theorem for the Euler class groups as established in [BRS 2, BDM], we obtain the following structure theorem.

Theorem 3.9. *Let $X = Spec(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Assume that $X(\mathbb{R})$ is orientable. Let $\mathcal{C}$ be the set of compact connected components of $X(\mathbb{R})$. Then,*

$$Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \xrightarrow{\sim} \bigoplus_{C \in \mathcal{C}} \mathbb{Z}$$

Corollary 3.10. *The composite group homomorphism $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \longrightarrow E^d(\mathbb{R}(X))$ is multiplication by 2.*

Proof. It is clearly enough to consider the case when $X(\mathbb{R})$ is compact and connected. Then we know from [BRS 2] that $E^d(\mathbb{R}(X))$ is generated by $(\mathfrak{m}, \omega)$, where $\mathfrak{m}$ is any real maximal ideal and $\omega : (\mathbb{R}(X)/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ is any surjection.

Now, from Theorem 2.10, we have $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)}((\mathfrak{m}, \omega)) = (\mathfrak{m}, \omega) - (\mathfrak{m}, -\omega)$. But by Proposition 3.3, $(\mathfrak{m}, \omega) + (\mathfrak{m}, -\omega) = 0$. Therefore, it follows that $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)}((\mathfrak{m}, \omega)) = 2(\mathfrak{m}, \omega)$. $\square$

The following theorem will be useful in the next section.

Theorem 3.11. *Let $X = Spec(R)$ be a smooth affine variety of <u>even</u> dimension d over $\mathbb{R}$. Then,*

$$Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \xrightarrow{\sim} Um_{d+1}(\mathbb{R}(X))/SL_{d+1}(\mathbb{R}(X)).$$

Proof. It suffices to prove that the canonical projection $\epsilon : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \to Um_{d+1}(\mathbb{R}(X))/SL_{d+1}(\mathbb{R}(X))$ is injective. Recall from (2.7) and the subsequent remark (or [DZ, Section 4]), that the morphism $\phi_{\mathbb{R}(X)} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \longrightarrow E^d(\mathbb{R}(X))$ is such that when d is even, then $\phi_{\mathbb{R}(X)}([v])$ is precisely the Euler class of the stably free module associated to the unimodular row v in a canonical way.

Now let $v = (a_1, \cdots, a_{d+1}) \in Um_{d+1}(\mathbb{R}(X))$ be a completable matrix. It is enough to show that this unimodular row is elementarily completable. As v is completable, the Euler class of the stably free module associated to v is trivial, and therefore, $\phi_{\mathbb{R}(X)}([v]) = 0$ in $E^d(\mathbb{R}(X))$. As $\phi_{\mathbb{R}(X)}$ is injective, $[v]$ is trivial in $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$. $\square$

4. MAIN THEOREMS: ORBIT SPACES

Let $X = \text{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over reals. As before, we always assume that $X(\mathbb{R})$ is orientable. As in the previous sections, we are treating the orbit spaces of unimodular rows as multiplicative groups.

Recall from [BRS 2, page 307] that there is a canonical surjective morphism $\Gamma : E^d(R) \twoheadrightarrow E^d(\mathbb{R}(X))$. Bhatwadekar-Sridharan denotes the kernel of this map by $E^d(\mathbb{C})$. They prove that $E^d(\mathbb{C})$ is the subgroup of $E^d(R)$ generated by all $(\mathfrak{m}, \omega_{\mathfrak{m}})$, where $\mathfrak{m}$ runs over the complex maximal ideals of R, and $\omega_{\mathfrak{m}} : (R/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ is any surjection. Let $\beta : Um_{d+1}(R)/E_{d+1}(R) \longrightarrow Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ be the canonical map.

4.1. **The elementary orbit space.** We have the following commutative diagram with exact rows. As $\delta_{\mathbb{R}(X)}$ is an isomorphism and Γ is surjective, it follows that β is a surjective morphism. Write $K = \ker(\beta)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E^d(\mathbb{C}) & \longrightarrow & E^d(R) & \xrightarrow{\Gamma} & E^d(\mathbb{R}(X)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \delta_R} & & \downarrow{\scriptstyle \delta_{\mathbb{R}(X)}} & & \\
1 & \longrightarrow & K & \longrightarrow & \frac{Um_{d+1}(R)}{E_{d+1}(R)} & \xrightarrow{\beta} & \frac{Um_{d+1}(\mathbb{R}(X))}{E_{d+1}(\mathbb{R}(X))} & \longrightarrow & 1
\end{array}$$

Proposition 4.1. *The restriction of δ_R on the subgroup $E^d(\mathbb{C})$ is trivial.*

Proof. Let $(J, \omega) \in E^d(\mathbb{C})$, where J is a product of complex maximal ideals. It can be derived from [BDM, 4.25, 4.26] and the proof of [BDM, 4.29] that $E^d(\mathbb{C})$ is a torsion-free divisible group. As $E^d(\mathbb{C})$ is divisible, there is some $(I, \omega_I) \in E^d(\mathbb{C})$ such that $(J, \omega) = 2(I, \omega_I)$. Note that $(I, \omega_I) = (I, -\omega_I)$. Therefore, $(J, \omega) = (I, \omega_I) + (I, -\omega_I)$, and consequently, by Proposition 2.12, $\delta_R((J, \omega)) = 0$. $\square$

Proposition 4.2. *The morphism β is injective on the image of the map δ_R.*

Proof. Let $(J, \omega_J) \in E^d(R)$ be such that $\beta\delta_R((J, \omega_J)) = [0, \cdots, 0, 1]$. Then, from the diagram we have, $\delta_{\mathbb{R}(X)}\Gamma((J, \omega_J)) = [0, \cdots, 0, 1]$. Since $\delta_{\mathbb{R}(X)}$ is an isomorphism, we have $\Gamma((J, \omega_J)) = 0$ in $E^d(\mathbb{R}(X))$. By exactness of the top row, $(J, \omega_J) \in E^d(\mathbb{C})$. But then $\delta_R((J, \omega_J)) = [0, \cdots, 0, 1]$ by the above proposition. $\square$

Theorem 4.3. *The map δ_R is a group homomorphism.*

Proof. We have to prove that if (J, ω_J) and (I, ω_I) are two elements of $E^d(R)$ such that J, I are both reduced ideals and $J + I = R$, then

$$\delta_R((J, \omega_J) + (I, \omega_I)) = \delta_R((J, \omega_J))\delta_R((I, \omega_I)),$$

where the multiplication on the right is that of $Um_{d+1}(R)/E_{d+1}(R)$. There are three cases to consider.

Case 1. Both J and I are contained only in complex maximal ideals of R. Then, both (J, ω_J) and (I, ω_I) are from $E^d(\mathbb{C})$. This case follows trivially from Proposition 4.1.

Case 2. Both J and I are contained only in real maximal ideals of R.

Note that the exact sequence in the top row of the above diagram splits. There is a split morphism $\theta_1 : E^d(\mathbb{R}(X)) \to E^d(R)$. Define

$$\theta_2 := \delta_R\, \theta_1\, (\delta_{\mathbb{R}(X)})^{-1} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \to Um_{d+1}(R)/E_{d+1}(R).$$

It is easy to see that θ_2 is a split map (for the bottom row) and $\delta_R\theta_1 = \theta_2\delta_{\mathbb{R}(X)}$. We then have the following diagram, where $\overline{\delta_R}$ denotes the restriction of δ_R. The same for $\overline{\beta}$ and $\overline{\Gamma}$.

$$\begin{array}{ccc} \theta_1(E^d(\mathbb{R}(X)) & \xrightarrow[\overline{\Gamma}]{\sim} & E^d(\mathbb{R}(X)) \\ \downarrow \overline{\delta_R} & & \downarrow \delta_{\mathbb{R}(X)} \\ \theta_2\left(\frac{Um_{d+1}(\mathbb{R}(X))}{E_{d+1}(\mathbb{R}(X))}\right) & \xrightarrow[\sim]{\overline{\beta}} & \frac{Um_{d+1}(\mathbb{R}(X))}{E_{d+1}(\mathbb{R}(X))} \end{array}$$

We can treat the elements (J, ω_J) and (I, ω_I) as elements of $\theta_1(E^d(\mathbb{R}(X))$. It is therefore enough to prove that $\overline{\delta_R}$ is a morphism. This is clear from the diagram.

Case 3. In this case we assume that J is contained only in complex maximal ideals and I is contained only in real maximal ideals of R.

For convenience, we write $x = (J, \omega_J)$ and $y = (I, \omega_I)$. Note that $\delta_R(x)$ is trivial. It is therefore enough to show that $\delta_R(x+y) = \delta_R(y)$. We compute: $\beta(\delta_R(x+y)) = \delta_{\mathbb{R}(X)}\Gamma(x+y) = \delta_{\mathbb{R}(X)}\Gamma(y) = \beta(\delta_R(y))$. By Proposition 4.2, β is injective on the image of δ_R. We are done. □

Proposition 4.4. *The group K is 2-divisible.*

Proof. Let $[x_1, \cdots, x_d, z] \in K$. We make the same choice of t as in Lemma 3.1 and so that $(t^2z)^2 \equiv 1$ modulo $(x_1, \cdots, x_d)$. We have $[x_1, \cdots, x_d, t^2z] = [x_1, \cdots, x_d, z][x_1, \cdots, x_d, t^2]$. Clearly, $\beta([x_1, \cdots, x_d, t^2]) = [0, \cdots, 0, 1]$, and therefore it follows that $[x_1, \cdots, x_d, t^2z] \in K$. But $[x_1, \cdots, x_d, t^2z] = \delta_R((J, \omega_J))$ for some $(J, \omega_J) \in E^d(R)$. Then $\delta_{\mathbb{R}(X)}\Gamma((J, \omega_J)) = [0, \cdots, 0, 1]$. As $\delta_{\mathbb{R}(X)}$ is an isomorphism, we see that $\Gamma((J, \omega_J)) = 0$ and thus $(J, \omega_J) \in E^d(\mathbb{C})$. Therefore, $[x_1, \cdots, x_d, t^2z] = \delta_R((J, \omega_J)) = [0, \cdots, 0, 1]$ by Proposition 4.1.

So $[x_1, \cdots, x_d, z][x_1, \cdots, x_d, t^2] = [0, \cdots, 0, 1]$. Now recall the way we chose t^2 in the proof of Lemma 3.1. As t^2 is a unit in a product of fields which are either $\mathbb{R}$ or $\mathbb{C}$, it follows that $t^2 = c^4$ modulo $(x_1, \cdots, x_d)$, for some c and therefore, $[x_1, \cdots, x_d, t^2] = [x_1, \cdots, x_d, c^4] = [x_1, \cdots, x_d, c^2]^2$ by [vdK2, 3.5 (v)]. Note also that $[x_1, \cdots, x_d, c^2] \in K$. We therefore have $[x_1, \cdots, x_d, z] = ([x_1, \cdots, x_d, c^2]^{-1})^2$. Thus K is 2-divisible. □

Corollary 4.5. *K is the unique maximal divisible subgroup of $Um_{d+1}(R)/E_{d+1}(R)$.*

Proof. Let D be the unique maximal divisible subgroup of $Um_{d+1}(R)/E_{d+1}(R)$. Then $K \subseteq D$. Write $H = Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$. Now, we have $Um_{d+1}(R)/E_{d+1}(R) = K \oplus H$ and H is reduced. It follows that

$$D = (D \cap K) \oplus (D \cap H) = K \oplus (D \cap H).$$

But then $(D \cap H)$ is a direct summand of the divisible group D and is contained in the reduced group H, implying that $(D \cap H)$ is trivial. Therefore, $D = K$. $\square$

Combining Theorem 3.9 and the results proved above, we have the following:

Theorem 4.6. *Let $X = Spec(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Assume that $X(\mathbb{R})$ is orientable. Let $\mathcal{C}$ be the set of compact connected components of $X(\mathbb{R})$. Then,*

$$Um_{d+1}(R)/E_{d+1}(R) \xrightarrow{\sim} K \oplus (\bigoplus_{C \in \mathcal{C}} \mathbb{Z}),$$

where K is the unique maximal divisible subgroup of $Um_{d+1}(R)/E_{d+1}(R)$.

In Section 5, we shall prove that K is torsion-free if $d \geq 3$. A summary of our conclusions above fits in an exact sequence, as given below.

Theorem 4.7. *The sequence $0 \to E^d(\mathbb{C}) \to E^d(R) \xrightarrow{\delta_R} Um_{d+1}(R)/E_{d+1}(R) \to K \to 1$ is an exact sequence of abelian groups.*

We now analyze the subgroup K in intricate detail. This is in fact, a preparation for the next subsection.

Theorem 4.8. *Let $[x_1, \cdots, x_d, z] \in K$. Then $[x_1, \cdots, x_d, z] = [x_1, \cdots, x_d, -z]$ and as a consequence, $[x_1, \cdots, x_d, z]^n = [x_1, \cdots, x_d, z^n]$ for any $n \geq 1$.*

Proof. Let $\alpha_1 x_1 + \cdots + \alpha_d x_d + bz = 1$. Recall from Lemma 3.1 that t is chosen so that $b^2 \equiv t^4 \bmod (x_1, \cdots, x_d)$ and then $[x_1, \cdots, x_d, t^2 z] = [x_1, \cdots, x_d, z][x_1, \cdots, x_d, t^2]$. As $\alpha_1 x_1 + \cdots + \alpha_d x_d + (-b)(-z) = 1$, in a similar manner we have, $[x_1, \cdots, x_d, -t^2 z] = [x_1, \cdots, x_d, -z][x_1, \cdots, x_d, t^2]$. As $[x_1, \cdots, x_d, z] \in K$, the argument as in Proposition 4.4 shows that $[x_1, \cdots, x_d, t^2 z] = [0, \cdots, 0, 1]$. As $(t^2 z)^2 \equiv 1$ modulo $(x_1, \cdots, x_d)$, by [vdK 2, Lemma 3.5 (iii)], $[x_1, \cdots, x_d, -t^2 z]$ is nothing but the inverse of $[x_1, \cdots, x_d, t^2 z]$, and hence trivial. Therefore, $[x_1, \cdots, x_d, z] = [x_1, \cdots, x_d, -z]$ and by [Ra, Lemma 1.3.1], $[x_1, \cdots, x_d, z]^n = [x_1, \cdots, x_d, z^n]$ for any $n \geq 1$. $\square$

It is obvious that any unimodular row over R with one square entry is in K. The following easy corollary is the converse.

Corollary 4.9. *Any element in K is of the form $[x_1, \cdots, x_d, w^2]$.*

Proof. Let $[v] \in K$. As K is 2-divisible, $[v] = [x_1, \cdots, x_d, w]^2$ for some $[x_1, \cdots, x_d, w] \in K$. Then, $[v] = [x_1, \cdots, x_d, w]^2 = [x_1, \cdots, x_d, w^2]$ by the above theorem. $\square$

Corollary 4.10. *Let $v \in Um_{d+1}(R)$ be such that $[v] \in K$. Then the row v can be completed to a matrix in $SL_{d+1}(R)$.*

Proof. As K is divisible, $[v] = [x_1, \cdots, x_d, w]^{d!}$ for some $[x_1, \cdots, x_d, w] \in K$. Then, $[v] = [x_1, \cdots, x_d, w]^{d!} = [x_1, \cdots, x_d, w^{d!}]$ by the above theorem. Under the canonical morphism $Um_{d+1}(R)/E_{d+1}(R) \twoheadrightarrow Um_{d+1}(R)/SL_{d+1}(R)$, the image of $[v]$ is trivial by a celebrated theorem of Suslin [Su 1]. $\square$

4.2. **Stably free modules.** We now proceed to compute the group $Um_{d+1}(R)/SL_{d+1}(R)$. In order to do so, we consider the following composite morphisms. We shall call the first composite as δ'_R, and the second one as $\delta'_{\mathbb{R}(X)}$.

(1) $E^d(R) \overset{\delta_R}{\to} Um_{d+1}(R)/E_{d+1}(R) \overset{\epsilon_R}{\to} Um_{d+1}(R)/SL_{d+1}(R)$, and
(2) $E^d(\mathbb{R}(X)) \overset{\delta_{\mathbb{R}(X)}}{\to} Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \overset{\epsilon_{\mathbb{R}(X)}}{\to} Um_{d+1}(\mathbb{R}(X))/SL_{d+1}(\mathbb{R}(X))$.

Note that by results proved in the previous section, $\delta'_{\mathbb{R}(X)}$ is a morphism. We shall refer to the following commutative diagram with exact rows. Since $\epsilon_{\mathbb{R}(X)}$ and β are both surjective, it follows that γ is also a surjective morphism.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E^d(\mathbb{C}) & \longrightarrow & E^d(R) & \xrightarrow{\Gamma} & E^d(\mathbb{R}(X)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \delta_R & & \downarrow \delta_{\mathbb{R}(X)} & & \\
1 & \longrightarrow & K & \longrightarrow & \frac{Um_{d+1}(R)}{E_{d+1}(R)} & \xrightarrow{\beta} & \frac{Um_{d+1}(\mathbb{R}(X))}{E_{d+1}(\mathbb{R}(X))} & \longrightarrow & 1 \\
 & & \downarrow \overline{\epsilon} & & \downarrow \epsilon_R & & \downarrow \epsilon_{\mathbb{R}(X)} & & \\
1 & \longrightarrow & \ker(\gamma) & \longrightarrow & \frac{Um_{d+1}(R)}{SL_{d+1}(R)} & \xrightarrow{\gamma} & \frac{Um_{d+1}(\mathbb{R}(X))}{SL_{d+1}(\mathbb{R}(X))} & \longrightarrow & 1
\end{array}
$$

We now prove the following theorem.

Theorem 4.11. *Let d be <u>even</u>. Then, $Um_{d+1}(R)/SL_{d+1}(R) \overset{\sim}{\to} \bigoplus_{C \in \mathcal{C}} \mathbb{Z}$, where $\mathcal{C}$ is the set of all compact connected components of $X(\mathbb{R})$.*

Proof. We have observed that γ is surjective. As ϵ_R is surjective, and $\epsilon_{\mathbb{R}(X)}$ is an isomorphism by Theorem 3.11, it follows that the induced map $\overline{\epsilon} : K \to \ker(\gamma)$ is also surjective. It is immediate from Corollary 4.10 that $\overline{\epsilon}$ is the trivial morphism, implying that $\ker(\gamma)$ is trivial. Now apply Theorem 3.11. $\square$

Theorem 4.12. *The morphism $\delta'_R : E^d(R) \to Um_{d+1}(R)/SL_{d+1}(R)$ is surjective.*

Proof. From the above theorem, this is obvious when d is even. For general d we need additional arguments. Recall from the proof of Theorem 4.3 that there is a split morphism $\theta_1 : E^d(\mathbb{R}(X)) \to E^d(R)$ for the top row. Also, there is a split morphism $\theta_2 : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \to Um_{d+1}(R)/E_{d+1}(R)$. We checked that the restriction of δ_R on $\theta_1(E^d(\mathbb{R}(X)))$ is an isomorphism onto $\theta_2(Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)))$.

Note that ϵ_R is surjective and it is trivial on K. As $Um_{d+1}(R)/E_{d+1}(R) = K \oplus \theta_2(Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)))$, it follows that for any element in $Um_{d+1}(R)/SL_{d+1}(R)$, there is a preimage in $\theta_2(Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)))$, which further has a preimage in $\theta_1(E^d(\mathbb{R}(X))) \subset E^d(R)$ under δ_R. □

We record the following corollary which will be used soon.

Corollary 4.13. *Let $X(\mathbb{R})$ be orientable, compact and connected. Then $Um_{d+1}(R)/SL_{d+1}(R)$ is generated by $\delta'_R((\mathfrak{m}, \omega_\mathfrak{m}))$, where $\mathfrak{m}$ is any real maximal ideal of R and $\omega_\mathfrak{m} : (R/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ is any surjection.*

Proof. By [BRS 2, 4.12, 4.13], $E^d(\mathbb{R}(X)) = \mathbb{Z}$, and it is generated by any $(\mathfrak{m}, \omega_\mathfrak{m})$ as in the statement of this corollary. By the proof of the above theorem, $\delta'_R((m, \omega_\mathfrak{m}))$ generates $Um_{d+1}(R)/SL_{d+1}(R)$. □

Remark 4.14. Let d be even. If $E^d(\mathbb{C}) = 0$ (for example, when R is the coordinate ring of a real sphere, or when Spec(R) is a rational variety), then it follows that $E^d(R)$ is isomorphic to $Um_{d+1}(R)/SL_{d+1}(R)$. Consequently, under this assumption, a stably free R-module P of rank d is free if and only if it has a unimodular element (see also [F 1, Theorem 5.10]).

We are now ready to compute $Um_{d+1}(R)/SL_{d+1}(R)$ when d is odd.

Theorem 4.15. *Let $d \geq 3$ be odd. Then $Um_{d+1}(R)/SL_{d+1}(R)$ is an $\mathbb{F}_2$-vector space of rank $\leq |\mathcal{C}|$, where $\mathcal{C}$ is the set of all compact connected components of $X(\mathbb{R})$.*

Proof. By [vdK 1, 4.3], the group $Um_{d+1}(R)/SL_{d+1}(R)$ satisfies Mennicke relations. In particular, for any orbit $[x_1, \cdots, x_d, z]$, and for any $r \geq 1$, one has $[x_1, \cdots, x_d, z]^r = [x_1, \cdots, x_d, z^r]$. Let us keep this in mind.

Recall that we proved that δ'_R is a surjective morphism. We actually proved that for any $[v] \in Um_{d+1}(R)/SL_{d+1}(R)$, there is $(J, \omega_J) \in \theta_1(E^d(\mathbb{R}(X)))$ such that $\delta'_R((J, \omega_J)) = [v]$. Let ω_J be induced by $J = (a_1, \cdots, a_s, s)$ where $s - s^2 \in (a_1, \cdots, a_d)$. Then, by the definition of δ'_R, it follows that $[v] = [2a_1, \cdots, 2a_d, 1 - 2s]$. Since Mennicke relations hold in $Um_{d+1}(R)/SL_{d+1}(R)$, $[v]^2 = [2a_1, \cdots, 2a_d, 1 - 2s]^2 = [2a_1, \cdots, 2a_d, (1 - 2s)^2] = [0, \cdots, 0, 1]$. It shows that every element of $Um_{d+1}(R)/SL_{d+1}(R)$ is 2-torsion.

As $\theta_1(E^d(\mathbb{R}(X)))$ is isomorphic to $E^d(\mathbb{R}(X))$, and $E^d(\mathbb{R}(X)) = \bigoplus_{C \in \mathcal{C}} \mathbb{Z}$, the result follows. □

4.3. **Computations on spheres.** Let us now apply the above computations on real spheres. Consider the coordinate ring of the d-dimensional real sphere $S^d(\mathbb{R})$ for $d \geq 2$ (no further assumption on d):

$$R = \frac{\mathbb{R}[X_1, \cdots, X_{d+1}]}{(X_1^2 + \cdots + X_{d+1}^2 - 1)} = \mathbb{R}[x_1, \cdots, x_{d+1}]$$

We now have the following result (see also [F 1, Corollary 5.12]).

Theorem 4.16. *Let R be the coordinate ring of $S^d(\mathbb{R})$. Then $Um_{d+1}(R)/SL_{d+1}(R)$ is generated by the orbit of the tangent bundle.*

Proof. By Corollary 4.13, $Um_{d+1}(R)/SL_{d+1}(R)$ is generated by $\delta'_R((\mathfrak{m}, \omega_{\mathfrak{m}}))$ (see notations therein), where $\mathfrak{m}$ is any real maximal ideal of R and $\omega_{\mathfrak{m}} : (R/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ is any surjection.

We now concentrate on the orbit $[x_1, \cdots, x_{d+1}]$ of the tangent bundle. We have the following relations among the ideals involved:

$$(x_1, \cdots, x_d) = (x_1, \cdots, x_d, 1 - x_{d+1}) \cap (x_1, \cdots, x_d, 1 + x_{d+1}) = \mathfrak{m}_1 \cap \mathfrak{m}_2,$$

and $\mathfrak{m}_1$, $\mathfrak{m}_2$ are both real maximal ideals. Let $s = \frac{1}{2}(1 - x_{d+1})$. Then, $s - s^2 = \frac{1}{4}(1 - x_{d+1}^2) \in (x_1, \cdots, x_d)$. Therefore, $\mathfrak{m}_1 = (\frac{1}{2}x_1, \cdots, \frac{1}{2}x_d, \frac{1}{2}(1 - x_{d+1}))$ will induce $\omega_{\mathfrak{m}_1}$ and by definition, $\delta'_R((\mathfrak{m}_1, \omega_{\mathfrak{m}_1})) = [x_1, \cdots, x_d, x_{d+1}]$. This shows that $Um_{d+1}(R)/SL_{d+1}(R)$ is generated by the orbit of the tangent bundle. □

The following corollary is now obvious.

Corollary 4.17. *All stably free modules of top rank on $S^3(\mathbb{R})$ and $S^7(\mathbb{R})$ are free. For odd $d \neq 1, 3, 7$, the set of isomorphism classes of stably free modules of rank d over $S^d(\mathbb{R})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

Proof. For $S^3(\mathbb{R})$ and $S^7(\mathbb{R})$, the orbit of the tangent bundle in each case is trivial. For $d \neq 3, 7$, the orbit of the tangent bundle is non-trivial. □

Remark 4.18. The assertion on $S^3(\mathbb{R})$ and $S^7(\mathbb{R})$ was first proved in [F 1, Proposition 5.13].

Remark 4.19. Most of the results of this paper can be extended to smooth affine varieties over $\mathbf{R}$, where $\mathbf{R}$ is any real closed field. One needs to use the structure theorem for the Euler class group in this case which is available in [BS].

5. MENNICKE SYMBOLS

We briefly recall the definition of Mennicke symbols.

Definition 5.1. Let B be a ring. A Mennicke symbol of length $n+1 \geq 3$ is a pair (ψ, G), where G is a group and $\psi : Um_{n+1}(B) \to G$ is a map such that:

ms1. $\psi((0, \cdots, 0, 1)) = 1$ and $\psi(v) = \psi(v\sigma)$ for any $\sigma \in E_{n+1}(B)$;

ms2. $\psi((b_1, \cdots, b_n, x))\psi((b_1, \cdots, b_n, y)) = \psi((b_1, \cdots, b_n, xy))$ for any two unimodular rows $(b_1, \cdots, b_n, x)$ and $(b_1, \cdots, b_n, y)$.

Clearly, a universal Mennicke symbol $(ms, MS_{n+1}(B))$ exists.

W. van der Kallen introduced the weak Mennicke symbol in [vdK 2]. Now let $\dim(B) = n \geq 2$. It was proved in [vdK 2] that the universal weak Mennicke symbol $(wms, WMS_{n+1}(B))$ is in bijective correspondence with $Um_{n+1}(B)/E_{n+1}(B)$, giving the latter a group structure. A Mennicke symbol of length $n+1$ is also a weak Mennicke symbol of length $n+1$ and there is a unique surjective morphism $WMS_{n+1}(B) \to MS_{n+1}(B)$. Summing up, we have a surjective morphism $f_B : Um_{n+1}(B)/E_{n+1}(B) \to MS_{n+1}(B)$, whose kernel is generated by all elements of the following form:

$$[b_1, \cdots, b_n, x][b_1, \cdots, b_n, y][b_1, \cdots, b_n, xy]^{-1}.$$

Let $X = \mathrm{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Assume that $X(\mathbb{R})$ is orientable. To compute $MS_{d+1}(R)$, we first focus on $MS_{d+1}(\mathbb{R}(X))$. We shall consider the following diagram. Here L denotes the kernel of the natural map $\beta_0 : MS_{d+1}(R) \to MS_{d+1}(\mathbb{R}(X))$. As $f_{\mathbb{R}(X)}\beta = \beta_0 f_R$ is surjective, it follows that β_0 is surjective.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & \frac{Um_{d+1}(R)}{E_{d+1}(R)} & \xrightarrow{\beta} & \frac{Um_{d+1}(\mathbb{R}(X))}{E_{d+1}(\mathbb{R}(X))} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f_R} & & \downarrow{\scriptstyle f_{\mathbb{R}(X)}} & & \\
1 & \longrightarrow & L & \longrightarrow & MS_{d+1}(R) & \xrightarrow{\beta_0} & MS_{d+1}(\mathbb{R}(X)) & \longrightarrow & 1
\end{array}$$

Theorem 5.2. *$MS_{d+1}(\mathbb{R}(X))$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space.*

Proof. Take any element $ms((x_1, \cdots, x_d, z)) \in MS_{d+1}(\mathbb{R}(X))$, where $(x_1, \cdots, x_d, z)$ is a unimodular row over $\mathbb{R}(X)$. It is clear that

$$(ms((x_1, \cdots, x_d, z)))^2 = ms((x_1, \cdots, x_d, z^2)) = (f_{\mathbb{R}(X)}([x_1, \cdots, x_d, z^2])).$$

But $[x_1, \cdots, x_d, z^2] = [0, \cdots, 0, 1]$ in $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$. It follows that every element in $MS(\mathbb{R}(X))$ is 2-torsion and therefore it is a $\mathbb{Z}/2\mathbb{Z}$-vector space. $\square$

It follows from the above theorem and Theorem 3.9 that $MS_{d+1}(\mathbb{R}(X))$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space of dimension $\leq |\mathcal{C}|$. We now claim that it is actually of dimension $|\mathcal{C}|$. To see this, we first need the following easy lemma.

Lemma 5.3. *Let $(\mathfrak{m}, \omega_{\mathfrak{m}}) \in E^d(\mathbb{R}(X))$ and let $\lambda, \mu \in (\mathbb{R}(X)/\mathfrak{m})^* = \mathbb{R}^*$. Then, $(\mathfrak{m}, \lambda\omega_{\mathfrak{m}}) + (\mathfrak{m}, \mu\omega_{\mathfrak{m}}) - (\mathfrak{m}, \lambda\mu\omega_{\mathfrak{m}})$ is equal to:*

(1) $(\mathfrak{m}, \omega_{\mathfrak{m}})$ *if* $\lambda > 0$, $\mu > 0$;
(2) $-3(\mathfrak{m}, \omega_{\mathfrak{m}})$ *if* $\lambda < 0$, $\mu < 0$;
(3) $(\mathfrak{m}, \omega_{\mathfrak{m}})$ *if* λ *and* μ *have opposite signs.*

Proof. This is nothing but a straightforward computation using [BRS 2, Lemma 3.4] and Proposition 3.3. □

Corollary 5.4. *Let $J = (a_1, \cdots, a_d)$ be a reduced ideal of height d in $\mathbb{R}(X)$. Let $\omega_J : (\mathbb{R}(X)/J)^d \twoheadrightarrow J/J^2$ be the surjection induced by $a_1, \cdots, a_d$. Let λ, μ be units modulo J. Then $(J, \lambda\omega_J) + (J, \mu\omega_J) - (J, \lambda\mu\omega_J) \in 4E^d(\mathbb{R}(X))$.*

Proof. We have $(J, \omega_J) = 0$ in $E^d(\mathbb{R}(X))$. Let $J = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k$, where each $\mathfrak{m}_i$ is a maximal ideal. We then have,

$$0 = (J, \omega_J) = (\mathfrak{m}_1, \omega_{\mathfrak{m}_1}) + \cdots + (\mathfrak{m}_k, \omega_{\mathfrak{m}_k}), \qquad (*)$$

where $\omega_{\mathfrak{m}_i} : (\mathbb{R}(X)/\mathfrak{m}_i)^d \twoheadrightarrow \mathfrak{m}_i/\mathfrak{m}_i^2$ is the surjection induced by ω_J.

Let us write λ as the tuple $(\lambda_1, \cdots, \lambda_k)$, where λ_i is the image of λ in $\mathbb{R}(X)/\mathfrak{m}_i$. Similarly, $\mu = (\mu_1, \cdots, \mu_k)$. By renaming if necessary, we may assume that λ_i and μ_i are both negative for $i = 1, \cdots, r$, for some r ($0 \le r \le k$). Then an easy verification using the lemma above will show that

$$(J, \lambda\omega_J) + (J, \mu\omega_J) - (J, \lambda\mu\omega_J) = (\mathfrak{m}_{r+1}, \omega_{\mathfrak{m}_{r+1}}) + \cdots + (\mathfrak{m}_k, \omega_{\mathfrak{m}_k}) - 3\left((\mathfrak{m}_1, \omega_{\mathfrak{m}_1}) + \cdots (\mathfrak{m}_r, \omega_{\mathfrak{m}_r})\right),$$

which equals $(J, \omega_J) - 4\left((\mathfrak{m}_1, \omega_{\mathfrak{m}_1}) + \cdots (\mathfrak{m}_r, \omega_{\mathfrak{m}_r})\right)$, and we are done by the relation $(*)$ above. □

We are now ready to prove:

Theorem 5.5. *$MS_{d+1}(\mathbb{R}(X))$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space of dimension $|\mathcal{C}|$.*

Proof. Recall from Section 3 that $\phi_{\mathbb{R}(X)} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \longrightarrow E^d(\mathbb{R}(X))$ is injective, and in fact, it is an isomorphism onto $2E^d(\mathbb{R}(X))$.

Consider the kernel of the map $f_{\mathbb{R}(X)} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \twoheadrightarrow MS_{d+1}(\mathbb{R}(X))$. We know that $\ker(f_{\mathbb{R}(X)})$ is generated by elements of the form

$$[w] = [a_1, \cdots, a_d, \lambda][a_1, \cdots, a_d, \mu][a_1, \cdots, a_d, \lambda\mu]^{-1}.$$

Adding suitable multiples of $\lambda\mu$ to $a_1, \cdots, a_d$, we may assume that $J = (a_1, \cdots, a_d)$ is a reduced zero-dimensional ideal. Then we have $\phi_{\mathbb{R}(X)}([w]) = (J, \lambda\omega_J) + (J, \mu\omega_J) - (J, \lambda\mu\omega_J) \in 4E^d(\mathbb{R}(X))$, by the corollary proved above. As a consequence, we have an induced surjective morphism $\overline{\phi}_{\mathbb{R}(X)} : MS_{d+1}(\mathbb{R}(X)) \twoheadrightarrow 2E^d(\mathbb{R}(X))/4E^d(\mathbb{R}(X))$. As the target object is a $\mathbb{Z}/2\mathbb{Z}$-vector space of dimension $|\mathcal{C}|$, combining with Theorem 5.2 we are done. □

From the above theorem and Theorem 3.9, we have the following easy corollary.

Corollary 5.6. *Any element $[v]$ in the kernel of $f_{\mathbb{R}(X)} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \to MS_{d+1}(\mathbb{R}(X))$ is a square.*

Theorem 5.7. *The map $\overline{f} : K \longrightarrow L$ is surjective and therefore L is divisible. In fact, L is the unique maximal divisible subgroup of $MS_{d+1}(R)$.*

Proof. We take any element from L. As f_R is surjective, we will have a preimage of the form $[v]\,\theta_2([w])$ in $Um_{d+1}(R)/E_{d+1}(R)$, where $[w] \in Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ and $[v] \in K$. But $[w] \in \ker(f_{\mathbb{R}(X)})$ and by Corollary 5.6, it is a square, say, $[w] = [w_1]^2$ for some $[w_1]$. Let $[w_1] = \delta_{\mathbb{R}(X)}((J, \omega_J))$ for some $(J, \omega_J) \in E^d(\mathbb{R}(X))$. As Mennicke relations hold in $MS_{d+1}(R)$, exactly the same argument as in the proof of Theorem 4.15 will show that $f_R\theta_2([w])$ is trivial. This shows that $\overline{f} : K \longrightarrow L$ is surjective. Thus L is divisible and therefore, $L \oplus (\oplus_{C\in\mathcal{C}}(\mathbb{Z}/2\mathbb{Z})) \xrightarrow{\sim} MS_{d+1}(R)$. We can argue as in Corollary 4.5 to prove that L is the maximal divisible subgroup. □

Theorem 5.8. *The map $\overline{f} : K \longrightarrow L$ is an isomorphism.*

Proof. Let $[v] \in K$ be such that $\overline{f}([v])$ is trivial. So, $[v] \in \ker(f_R)$ and therefore $[v]$ is a finite product of elements of the form $[w] = [x_1, \cdots, x_d, zu][x_1, \cdots, x_d, z]^{-1}[x_1, \cdots, x_d, u]^{-1}$.

We apply the method of Lemma 3.1 once again. We choose appropriate $t, \lambda \in R$ such that $(t^2z)^2 \equiv (\lambda^2u)^2 \equiv 1$ modulo $(x_1, \cdots, x_d)$. We have:

(1) $[x_1, \cdots, x_d, t^2z] = [x_1, \cdots, x_d, z][x_1, \cdots, x_d, t^2]$;
(2) $[x_1, \cdots, x_d, \lambda^2u] = [x_1, \cdots, x_d, u][x_1, \cdots, x_d, \lambda^2]$;
(3) $[x_1, \cdots, x_d, (t\lambda)^2zu] = [x_1, \cdots, x_d, zu][x_1, \cdots, x_d, t^2\lambda^2]$.

Then, writing **x** for $x_1, \cdots, x_d$ and regrouping, we have

$$[w] = \left([\mathbf{x}, t^2\lambda^2]^{-1}[\mathbf{x}, t^2][\mathbf{x}, \lambda^2]\right)\left([\mathbf{x}, (t\lambda)^2zu][\mathbf{x}, t^2z]^{-1}[\mathbf{x}, \lambda^2u]^{-1}\right)$$

Note that the first bunch is trivial by [vdK 2, Lemma 3.5 (v)], and each term in the second bunch is in the image of δ_R. Then, it follows that $[v]$ is in fact a finite product of elements, each of which is in the image of δ_R. Therefore, $[v]$ is in K as well as in the image of δ_R, implying that $[v]$ is trivial. □

Corollary 5.9. *The kernel of $f_R : Um_{d+1}(R)/E_{d+1}(R) \twoheadrightarrow MS_{d+1}(R)$ is a free abelian group of rank $|\mathcal{C}|$.*

Proof. Easy to see, as we now have $K \xrightarrow{\sim} L$. □

5.1. **Cohomological methods.** We now prepare ourselves to prove that L is torsion-free if $d \geq 3$. For this purpose, we shall require some cohomological interpretation of $MS_{d+1}(R)$ from [F 2]. We shall freely use various terms and notations from [F 1, F 2], without explicitly recalling their definition. In the result below K_{d+1} is the sheafification of the pre-sheaf arising out of Milnor K-theory groups and $H^d(A, K_{d+1})$ is the K-cohomology group.

Theorem 5.10. [F 2, Theorem 1.4] *Let A be a smooth affine algebra of dimension $d \geq 3$ over a perfect field of characteristic unequal to 2. Then $MS_{d+1}(A)$ is isomorphic to $H^d(A, K_{d+1})$.*

Let $X = \text{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Assume that $X(\mathbb{R})$ is orientable. Consider $R_{\mathbb{C}} := R \otimes_{\mathbb{R}} \mathbb{C}$. Let $Y := \text{Spec}(R_{\mathbb{C}})$ and $\pi : Y \to X$ be the canonical finite morphism. We then have induced maps

$$\pi_* : H^d(R_{\mathbb{C}}, K_{d+1}) \to H^d(R, K_{d+1}) \text{ and } \pi^* : H^d(R, K_{d+1}) \to H^d(R_{\mathbb{C}}, K_{d+1})$$

such that $\pi_*\pi^*$ is multiplication by 2. This result follows from the *Projection Formula* as available in [Ro] (or see [EKM, Proposition 56.9]). By a slight abuse of notation, we record the following reformulation to suit our needs. Note that in this section we are treating the groups as additive groups.

Proposition 5.11. *Let $d \geq 3$. The finite morphism $\pi : Y \to X$ induces morphisms*

$$\pi_* : MS_{d+1}(R_{\mathbb{C}}) \to MS_{d+1}(R) \text{ and } \pi^* : MS_{d+1}(R) \to MS_{d+1}(R_{\mathbb{C}})$$

such that $\pi_*\pi^*$ is multiplication by 2.

We are now ready to prove the following theorem.

Theorem 5.12. *Let $X = Spec(R)$ be a smooth affine variety of dimension $d \geq 3$ over $\mathbb{R}$. Assume that $X(\mathbb{R})$ is orientable. Then, the divisible group $L = ker(\beta_0)$ is torsion-free.*

Proof. We have already proved that L is divisible. Let $\alpha \in MS_{d+1}(R)$ be a torsion element, say, $n\alpha = 0$. Then $0 = \pi^*(n\alpha) = n\pi^*(\alpha)$ in $MS_{d+1}(R_{\mathbb{C}})$. By [F 2, Theorem 2.2], $MS_{d+1}(R_{\mathbb{C}})$ is a torsion-free divisible group. Therefore $\pi^*(\alpha) = 0$, implying that $2\alpha = \pi_*\pi^*(\alpha) = 0$. This shows that any torsion element of $MS_{d+1}(R)$ is 2-torsion. The same is true for the subgroup L. As L is divisible, it is now easy to deduce that L is torsion-free. □

Corollary 5.13. *Let $d \geq 3$. The kernel K of the canonical surjection $\beta : Um_{d+1}(R)/E_{d+1}(R) \twoheadrightarrow Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is torsion-free.*

Proof. K and L are isomorphic by Theorem 5.8. □

Theorems 5.5, 5.7, and 5.12, yield the following structure theorem for $MS_{d+1}(R)$.

Theorem 5.14. *Let $X = Spec(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Assume that $X(\mathbb{R})$ is orientable. Then $MS_{d+1}(R) \xrightarrow{\sim} L \oplus (\oplus_{C \in \mathcal{C}} \mathbb{Z}/2\mathbb{Z})$, where L is the unique maximal divisible subgroup of $MS_{d+1}(R)$. Further, if $d \geq 3$, then L is torsion-free.*

Remark 5.15. To prove that L is torsion-free, we rely on Fasel's cohomological interpretation of $MS_{d+1}(R)$, which in turn depends on the work of Fabien Morel (see [F 1, 4.5, 4.6, 4.7] and [F 2, 1.4]). The restriction $d \geq 3$ stems from there. At the moment we do not know how to extend the final statement of the above theorem to $d = 2$.

6. An auxiliary result

In this article, the map $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ served us well when the base field is $\mathbb{R}$. However, it is completely useless if the base field is algebraically closed, as we show now. But so is its counter-part ϕ_R.

Theorem 6.1. *Let R be a smooth affine domain of dimension $d \geq 2$ over an algebraically closed field k of characteristic $\neq 2$. Then, the map $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ is the trivial morphism.*

Proof. Under the assumptions, the Euler class group is isomorphic to the Chow group $CH^d(R)$ of 0-cycles. Let $(I, \omega_I) \in E^d(R)$. As $CH^d(R)$ is uniquely divisible, it follows that there exists $(J, \omega_J) \in E^d(R)$ such that $(I, \omega_I) = 2(J, \omega_J)$. As k is algebraically closed, -1 is a square and therefore, applying [BRS 2, Lemma 3.4] we have $(J, \omega_J) = (J, -\omega_J)$ in $E^d(R)$. Therefore, $(I, \omega_I) = (J, \omega_J) + (J, -\omega_J)$. The proof is now complete by Proposition 2.12. □

After reading an earlier version of this paper, Jean Fasel suggested us this improvement, also indicating a proof. We sincerely thank him for allowing us to include this result here.

Theorem 6.2. (Fasel) *Let k be an infinite perfect field of cohomological dimension ≤ 1 and of characteristic unequal to 2. Let R be a smooth affine domain of dimension $d \geq 3$ over k. Then the map $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ is the trivial morphism.*

Proof. Under the assumptions of the theorem, by [GRa, F 2], the group $Um_{d+1}(R)/E_{d+1}(R)$ is isomorphic to $MS_{d+1}(R)$. Therefore, by [F 2, Theorem 2.2], it is uniquely 2-divisible. Consequently, the map $\kappa : [v] \mapsto [v]^2$ is an isomorphism of $Um_{d+1}(R)/E_{d+1}(R)$. As the group structure is Mennicke-like, κ is actually Vaserstein's square operation, taking an orbit $[x_1, \cdots, x_d, z]$ to $[x_1, \cdots, x_d, z^2]$.

Now let $(J, \omega_J) \in E^d(R)$ and let ω_J be induced by $(a_1, \cdots, a_d, s)$ with $s(1-s) \in (a_1, \cdots, a_d)$. Then $\kappa\delta_R((J, \omega_J)) = [2a_1, \cdots, 2a_d, (1-2s)^2]$ and the image is clearly the trivial orbit, as $(1-2s)^2 \equiv 1$ modulo $(a_1, \cdots, a_d)$. As κ is an isomorphism, the result follows. $\square$

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