Axiomatic Geometry: Euclid and Beyond

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A Talk for High School Students
Geometry:

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- Field of knowledge concerned with spatial relations; shapes and sizes of figures; structure of space; ...
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Geometry arose in ancient civilizations which built structures and studied astronomy.

- A collection of empirically discovered principles concerning lengths, angles, areas and volumes.
- Developed to meet practical needs in surveying, construction, astronomy, and various crafts.
Geometry: The Beginnings

- **Indus valley civilization**: had discovered obtuse angled triangles.
- **Babylon**: rules for computing areas and volumes.
- **Egypt**: approximate area of a circle; volume of frustum of a square pyramid.
- Both the Babylonians and the Egyptians knew the Pythagoras theorem about 1500 years before Pythagoras.
- **China**:
  - The oldest existent work on geometry in China is Mo Jing by Mozi (around 400 BCE).
  - There may have been earlier works which were destroyed in the infamous ‘Burning of the Books’ around 200 BCE.
Geometry in Ancient India

Shulbasutras (800 to 200 BCE):

- Part of the Shrauta Sutras which are considered to be appendices to the Vedas.
- The only sources of knowledge of Indian mathematics from the Vedic period.
- Mathematical problems arose to solve the requirement of creating fire altars.
  - Fire altars having different shapes but occupy the same area.
  - Each fire altar consisted of five layers of brick; each layer consisting of 200 bricks and no two adjacent layers have congruent arrangements of bricks.
- Pythagoras theorem and some triplets were known.
Geometry in Ancient Greece

Thales (635-543 BCE) and his disciple Pythagoras (582-496 BCE).

Plato (427-347 BCE): ‘Let none ignorant of geometry enter here.’

Euclid (c. 325-265 BCE): Wrote ‘The Elements of Geometry’ consisting of 13 books: rigorous proofs; axiomatic development of geometry; foundations of number theory. Also wrote other works on geometry. ‘There is no Royal Road to geometry.’ – Euclid to King Ptolemy
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School Geometry and Beyond
**Theorem:** The bisectors of the base angles of an isosceles triangle are equal.
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**Proof:**

The base angles of an isosceles triangle are equal (theorem)

In triangles BEC and DBC

\[ \angle ECB = \angle DBC \]
\[ \angle EBC = \angle DCB \]
side BC is common.

So triangle BEC is congruent to DBC (theorem ASA)

Therefore BE = DC.  

QED
Theorem: If the bisectors of the base angles of a triangle are equal then the triangle is isosceles.
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Proof (sketch): Assume $a > b$.

Construction: DF parallel to BE and BD parallel to EF.

In triangles DBC and BEC:
- BC is common; BE = DC; $a > b$; so, DB > EC (theorem).
- EF = DB > EC; so $d > c$ (theorem).

Hence, $a + d > b + c$ and so, $DF > DC$.

But, $DF = BE = DC < DF$.

Contradiction!
Theorem: A triangle is isosceles if and only if the bisectors of its base angles are equal.
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Exercises: Show

- A triangle is isosceles if and only if two altitudes drawn from vertices at the base to the sides are equal.
- A triangle is isosceles if and only if the two medians drawn from vertices at the base to the sides are equal.

An online reference:

http://www.algebra.com/
A theorem is a statement (about geometrical objects).
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A proof is a sequence of statements such that each statement is one of the following:
- A previously proved theorem.
- One of the axioms.
- Follows ‘logically’ from the statements appearing so far in the proof.
- The last statement in the proof is the statement of the theorem to be proved.
**Modus Ponens:**
- if $p$ then $q$;
- $p$;
- therefore $q$. 
Logical Deduction: Examples

*Modus Ponens:*
- if $p$ then $q$;
- $p$;
- therefore $q$.

*Modus Tollens:*
- if $p$ then $q$;
- not $q$;
- therefore not $p$.

These principles are independent of geometry.
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**Disjunctive Syllogism:**
- $p$ or $q$;
- not $p$;
- therefore $q$. 
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Contradiction:
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Geometry and Mathematical Rigour
Euclid’s Elements: Definitions and Common Notions

Definitions: point, line, right angle, plane, circle, quadrilateral, parallel lines, ... (all together 23).

- Parallel lines: co-planar straight lines which when extended in both directions do not meet in either.

Common notions:
- Things equal to the same thing are equal to one another.
- If equal things are added to equal things, then the wholes are equal.
- If equal things are subtracted from equal things, then the differences are equal.
- Things coinciding with one another are equal.
- The whole is greater than the part.
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Euclid’s Elements: Axioms

- To draw a straight-line from any point to any other point.
- To produce a finite straight-line continuously in a straight-line.
- To draw a circle with any center and radius.
- All right-angles are equal to one another.
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- To draw a straight-line from any point to any other point.
- To produce a finite straight-line continuously in a straight-line.
- To draw a circle with any center and radius.
- All right-angles are equal to one another.
- Parallel axiom: If a straight line $\ell$ cuts two other straight lines $m$ and $n$ such that the sum of the internal angles on one side is less than two right angles, then $m$ and $n$ produced on that side will eventually meet.

\[ \begin{array}{c}
m \\
\hline
\end{array} \]

\[ \begin{array}{c}
n \\
\hline
\end{array} \]

\[ \begin{array}{c}
m \\
\hline
\end{array} \]

\[ \begin{array}{c}
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\hline
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All theorems in Euclidean geometry can be deduced from the axioms and the common notions (well, not quite).

A figure is a visual aid to the proof; it is not essential to the correctness of the proof. (It is essential, though, for preserving the sanity of the mind.)
Theorem: An equilateral triangle can be constructed on any finite straight line.

Proof: Given line AB.
Draw circle with centre A and radius AB (ax. 3).
Draw circle with centre B and radius BA (ax. 3).
Let the circles intersect at C (requires an additional assumption).
Draw CA and CB (ax. 1).
CA=AB and CB=AB (defn of circle).
So, CA=CB (c.n. 1).
Therefore CA=AB=BC.         QED
Axioms:

- Two circles cut one another; two straight lines do not share a common segment (used in Theorem 1).
- One figure can be applied to another (used in Theorem 4).
- Point lying within the interior of an angle (used in Theorem 16).
- First three theorems of Book 11.
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Common notions:

- Halves of equal things are equal; doubles of equal things are equal.
- Squares of equal things are equal; also square-roots of equal things are equal.
Axiomatisation of Solid Geometry


- Three *primitive terms*: point, straight line; plane.
- Six *primitive relations*: betweenness (ternary relation linking points); containment (linking points and lines, linking points and planes, linking lines and planes); congruence (of lines, of angles).
- Line segments, angles and triangles can be defined in terms of points and lines using betweenness and containment.

- Three *primitive terms*: point, straight line; plane.
- Six *primitive relations*: betweenness (ternary relation linking points); *containment* (linking points and lines, linking points and planes, linking lines and planes); *congruence* (of lines, of angles).
- Line segments, angles and triangles can be defined in terms of points and lines using betweenness and containment.
- Twenty *axioms*: 7 axioms of *combination* (of points, lines and planes); 3 axioms of *order* (of points on a line); Playfair’s axiom; 6 axioms of *congruence* (of line segments and angles); 2 axioms of *continuity*.
Axiomatization of solid geometry: notions of completeness and consistency of axioms; use of models to show independence of axioms.

Development of metamathematics, i.e., mathematics about a (mathematical) theory.

Theoretically, one may consider that it is possible to mechanically determine whether a given (geometrical) statement is a theorem or not, i.e., whether there is a proof for the statement or not. (Not quite; some deep algorithmic issues are involved.)
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Other complete sets of axioms had also been proposed earlier.
A false proof (due to Lewis Caroll) that a right angle is equal to any obtuse angle.

Proof(?): Draw AB and construct angle DAB to be a right angle and angle ABC to be angle a such that AD=BC. G and E are mid-points of AB and DC respectively. Perpendicular to AB from G and perpendicular to DC from E meet at H. (This is easy to prove, since AB is not parallel to DC). Triangles HDE and HEC are congruent (SAS) and so HD=HC. Triangles HAG and HGB are congruent (SAS) and so HA=HB and \( \angle HAG = \angle HBG \).

Triangles AHD and HBC are congruent (SSS) and so \( \angle HAD = \angle HBC \).

\[
\angle HAG + \angle GAD = \angle HAD = \angle HBC = \angle HBG + \angle GBC.
\]

So, \( \angle GAD = \angle GBC \).
A false proof (due to Lewis Caroll) that a right angle is equal to any obtuse angle.

Proof(?): Draw $AB$ and construct angle $DAB$ to be a right angle and angle $ABC$ to be angle $a$ such that $AD=BC$. G and E are mid–points of $AB$ and $DC$ respectively. Perpendicular to $AB$ from $G$ and perpendicular to $DC$ from $E$ meet at $H$. (This is easy to prove, since $AB$ is not parallel to $DC$). Triangles HDE and HEC are congruent (SAS) and so $HD=HC$. Triangles HAG and HGB are congruent (SAS) and so $HA=HB$ and $\angle HAG = \angle HBG$.

Triangles AHD and HBC are congruent (SSS) and so $\angle HAD = \angle HBC$.

$\angle HAG + \angle GAD = \angle HAD = \angle HBC = \angle HBG + \angle GBC$.

So, $\angle GAD = \angle GBC$.

Find the fallacy. (Hint: take $a = 120^\circ$ and construct the resulting figure.)
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\end{align*}
\]

Find the fallacy. (Hint: take \( a = 120^\circ \) and construct the resulting figure.)

**Moral:** Be careful about intuition; but, by no means discard it.
The Parallel Axiom
Euclid’s Axioms Revisited

If axioms are taken to be ‘self-evident truths’, then of all the axioms, the fifth one is the least self-evident.
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- The first 28 theorems in the Elements were proved without using the fifth axiom.
- Thm 27 (briefly): equality of alternate angles implies parallelism.
- Thm 28 (briefly): external angle equal to interior opposite angle implies parallelism.
- Thm 29 (briefly): parallelism implies equality of alternate angles; equality of external angle with internal opposite angle; sum of internal angles on the same side to be equal to two right angles.
**Axiom statement:** If \( m \) is a line and \( P \) is a point outside \( m \), then there is exactly one line which passes through \( P \); is coplanar with \( m \) and is also parallel to \( m \).

```
\begin{center}
\includegraphics[width=0.5\textwidth]{parallel_line.png}
\end{center}
```

\( m \)
Playfair’s Axiom

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```
  ________● P __________

m
```

Equivalence:

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$\hspace{1cm} \bullet \hspace{1cm} P$

$m$

Equivalence:

- The five axioms of Euclid imply Playfair’s axiom, i.e., Playfair’s axiom is a theorem in Euclidean geometry.
- The first four axioms of Euclid and Playfair’s axiom together imply the fifth axiom of Euclid.
Playfair’s Axiom

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\[ \begin{array}{c}
\text{m} \\
\text{-----------------} \\
\text{P}
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**Equivalence:**
- The five axioms of Euclid imply Playfair’s axiom, i.e., Playfair’s axiom is a theorem in Euclidean geometry.
- The first four axioms of Euclid and Playfair’s axiom together imply the fifth axiom of Euclid.
- The fifth axiom of Euclid does not by itself imply Playfair’s axiom.
Theorem: The five axioms of Euclid together imply Playfair’s axiom.

Proof: Through P draw perp to m meeting at Q. Through P draw n perp to PQ. \( \angle RPQ = \angle BQP = \text{rt angle} \) and so \( PR \parallel QB \).

Let p be any other line through P and S be such that \( \angle SPQ < \text{rt angle} \). Therefore \( \angle SPQ + \angle BQP < 2 \text{ rt angles} \).

So, m and p intersect and n is the only line parallel to m which passes through P.
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So, \( m \) and \( p \) intersect and \( n \) is the only line parallel to \( m \) which passes through \( P \).

Converse: exercise.
'Proving’ the Fifth Axiom

‘The scandal of elementary geometry.’

– D’Alembert (1767)
‘Proving’ the Fifth Axiom

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- Greece: Ptolemy, Proclus;
- Persia: Khayyám, Nasir al-Din;
- Later Europe: Saccheri, Lambert, Legendre.

- Assumed that the fifth axiom does not hold and tried to derive a contradiction.
- Resulted in either failure to obtain a contradiction or in mistaken contradictions.
- Obtained interesting results but, did not realise that the implications of contradicting the fifth axiom gave rise to a new kind of geometry.
‘A Book of Verses underneath the Bough,
A Jug of Wine, a Loaf of Bread—and Thou
Beside me singing in the Wilderness—
Oh, Wilderness were Paradise enow!’

– from The Rubaiyat
– translated by Edward FitzGerald
Non-Euclidean Geometry: Fragments of a Story
‘For God’s sake, I beseech you, give it up. Fear it no less than sensual passions because it too may take all your time and deprive you of your health, peace of mind and happiness in life.’

– in a letter from Farkas Bolyai to János Bolyai
János Bolyai (1802-1860)

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I have discovered things so wonderful that I was astounded ... out of nothing I have created a strange new world.

– in a letter from János Bolyai to Farkas Bolyai.

- One of the founders of non-Euclidean (actually hyperbolic) geometry.
- Realised that contradicting Playfair’s axiom gave rise to another kind of geometry.
Nikolai Lobachevsky (1792–1856)

‘... the momentous gap in the theory of parallels, to fill which all efforts of mathematicians have so far been in vain.’
– Geometrical Investigations on the Theory of Parallels (1840)

- A Russian mathematician of repute.
- Co-founder of hyperbolic geometry.
Carl Friedrich Gauss (1777–1855)

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- Gauss had discovered non-Euclidean geometry earlier to János Bolyai. But, did not publish his ideas fearing controversy.
- Immanuel Kant (1724–1804) had argued in ‘The Critique of Pure Reason’ that Euclidean geometry was an example of *synthetic a priori knowledge*; not derived from the senses nor deduced through logic – our knowledge of space is a truth that we are born with.
‘If only I had the theorems! Then I should find the proofs easily enough.’
Bernhard Riemann (1826–1866)

‘If only I had the theorems! Then I should find the proofs easily enough.’

- Credited with creating a different kind of non-Euclidean geometry called elliptical geometry.
- Made contributions to analysis, number theory and differential geometry.
Non-Euclidean Geometry: Some Mathematics
Contradicting Playfair’s Axiom

Two ways of contradicting the axiom:

1. There are at least two straight lines through $P$ which are parallel to $m$.

2. Any straight line through $P$ will always meet $m$, i.e., there are no parallel lines.
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Non-Euclidean Geometries

Hyperbolic geometry:

- Replace Playfair’s axiom by the hyperbolic axiom and keep all other axioms intact.
- The other axioms of Euclid are consistent with the hyperbolic axiom.
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Elliptic geometry:

- Using the first four axioms of Euclid, it is possible to show the existence of parallel lines.
- So, non-existence of parallel lines is not consistent with the first four axioms.
- Obtaining a consistent set of axioms requires slightly changing the first four axioms.
- These along with the elliptic axiom give rise to elliptic geometry.
Using the first four axioms, it can be shown that the angle-sum of a triangle is at most equal to two right angles.

- Euclidean geometry: the angle-sum is *equal* to two right angles.
- Hyperbolic geometry: the angle-sum is *less* than two right angles.
- Elliptic geometry: the angle-sum is *more* than two right angles.
Lambert quadrilateral: has three right angles; fourth angle: acute if the geometry is hyperbolic; right angle if the geometry is Euclidean; obtuse if the geometry is elliptic.

- No rectangles in non-Euclidean geometry.
Non-Euclidean Geometry: Some Consequences

*Lambert quadrilateral*: has three right angles; fourth angle: acute if the geometry is hyperbolic; right angle if the geometry is Euclidean; obtuse if the geometry is elliptic.

- No rectangles in non-Euclidean geometry.

*Saccheri quadrilateral*: two sides of equal length, both perpendicular to a base side; the other two angles are equal and are called the summit angles; summit angles: acute if the geometry is hyperbolic; right if the geometry is Euclidean; obtuse if the geometry is elliptic.
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The notion of models can be made rigorous using mathematical/formal logic.
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The world as we see it can be considered to be a model for Euclidean geometry.
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A model for hyperbolic geometry was first proposed by Eugenio Beltrami (1868); later work by Felix Klein and Henri Poincaré.
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A model for hyperbolic geometry was first proposed by Eugenio Beltrami (1868); later work by Felix Klein and Henri Poincaré.

Bernhard Riemann and later Felix Klein provided models for elliptic geometry.
Models for Geometries

*Poincaré disc:* A model for hyperbolic geometry consists of the points inside and on the unit circle; straight lines are arcs of orthogonal circles.

![Diagram of Poincaré disc]
Models for Geometries

**Poincaré disc**: A model for hyperbolic geometry consists of the points inside and on the unit circle; straight lines are arcs of orthogonal circles.

**Modified Riemann sphere**: A model for elliptic geometry is the surface of a sphere where antipodal points (i.e., those opposite each other) are considered to be one and the same; straight lines are the great circles on the sphere.
Non-Euclidean Geometry: A Paradigm Shift

*The Structure of Scientific Revolutions* (1962) by Thomas Kuhn.

- The progress of revolutionary science occurs through *paradigm shifts*.

- The paradigm of Euclidean geometry: Euclidean geometry stood unchallenged as the mathematical model of space. The Euclidean point of view represented absolute authority buttressed by supporters such as Immanuel Kant. Provided the notion of absolute truth in theology.

Non-Euclidean geometry as a paradigm shift: The realisation that contradicting the fifth axiom gave rise to new kinds of geometries. This flew in the face of entrenched belief of 2200 years.
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Different Axiomatic Geometries
Description of Geometries

A geometry describes spatial relations, distance properties, properties of shapes, etcetera.

The language of geometry consists of points, lines, planes, etcetera.

Relations describe how points are related to lines and planes, lines to lines, etcetera.
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Relations describe how points are related to lines and planes, lines to lines, etcetera.

The purpose of axioms is to lay down certain rules that the basic objects (points, lines, relations) have to satisfy.

One then studies what can be derived under the given axioms.

The axioms are not arbitrary; they are supposed to capture intuition about ‘reality’ (whatever it may be!)
Basic fragment of geometry with no notion of distance.

Three axioms:
1. For any two distinct points, there is a unique line incident on both of them.
2. For every line, there are two distinct points incident on the line.
3. There exist three distinct non-collinear points.

It is possible to define collinearity, concurrence and parallelism; parallel lines: no common incident point.

Two Theorems:
1. For every line, there is a point which is not incident on the line.
2. There exist 3 distinct lines that are not concurrent.
Basic fragment of geometry with no notion of distance.

- Consists of a set of points, a set of lines and an incidence relation (as primitive or undefined terms).

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Incidence Geometry

Basic fragment of geometry with no notion of distance.
- Consists of a set of points, a set of lines and an incidence relation (as primitive or undefined terms).
- **Three axioms:**
  1. For any two distinct points, there is a unique line incident on both of them.
  2. For every line, there are two distinct points incident on the line.
  3. There exist three distinct non-collinear points.
- It is possible to define collinearity, concurrence and parallelism; parallel lines: no common incident point.

**Two Theorems:**
- For every line, there is a point which is not incident on the line.
- There exist 3 distinct lines that are not concurrent.
7 Points $= \{A, B, C, D, E, F, G\}$.

7 lines (including the one that *looks* like a circle) and the incidence relation described by the diagram.

It is possible to verify that the axioms of incidence geometry hold for the Fano plane.

(The Fano plane is also an example of a projective plane.)
Some Lessons to Take Home

- The difference between geometry as a practical tool and axiomatic geometry as a mathematical theory.
- The axiom, definition, theorem, proof structure of geometry including examples of proof by contradiction.
- Role of intuition and its pitfalls.
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- The difference between geometry as a practical tool and axiomatic geometry as a mathematical theory.
- The axiom, definition, theorem, proof structure of geometry including examples of proof by contradiction.
- Role of intuition and its pitfalls.
- The parallel postulate and Playfair’s axiom.
- Rise of non-Euclidean geometry and its consequences.
- Boldness of the mind in making a paradigm shift.
- Incidence geometry and Fano plane.
References and Links

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Thank you for your attention!