

# Singular Control with State Constraints

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based on joint works with R. Atar, K. Ross and R.J. Williams

## Outline.

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1. **Problem Description.** State space, Controlled dynamics, Admissible controls, Cost criterion.
2. **Nonnegative Cost, Unbounded Domain.**
  - Setup.
  - HJB Equation: Variational inequality with gradient constraints. Constrained viscosity solutions.
  - Main Result: Unique solvability by the value function.
3. **General Cost, Bounded Domain.**
  - Setup.
  - HJB equation and the main uniqueness result.
  - Remarks on assumptions and the “no arbitrage” condition.
4. **An application:** Characterization in terms of the “free boundary”.

## Problem Description.

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- **State Space:**  $\mathbb{X}$  Closed, convex subset of  $\mathbb{R}^d$ .
- **Uncontrolled Process:**  $x + B_t$ ,

$x \in \mathbb{X}$ ,  $B$  a  $(b, \Sigma)$  Brownian motion.  $b \in \mathbb{R}^d$ ,  $\Sigma$  a  $d \times d$  n.n.d. matrix.

- **Controlled Dynamics:**  $X_t = x + B_t + GY_t$ ,

$G$  is a  $d \times p$  matrix,  $d \leq p$ .

- **Control:**  $Y$  - a  $p$  dimensional RCLL process, loc. b.v. paths, adapted (to some underlying filtration  $\{\mathcal{F}_t\}$ ).

## Problem Description Contd.

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### Admissible Control:

1.  $Y$  has increments in the closed convex cone  $\mathbb{Y}$ :

$$Y(0), Y(t) - Y(s) \in \mathbb{Y}, \quad 0 \leq s \leq t < \infty.$$

(E.g.  $\mathbb{Y} = \mathbb{R}_+^p$ .)

2. **State Constraints:**

$$X_t = x + B_t + GY_t \in \mathbb{X}, \quad \text{for all } t \geq 0, \text{ a.s.}$$

Let  $\mathcal{A}(x)$  denote the family of all admissible controls.

## General Description Contd.

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**Cost Criterion:**

$$J(x, Y) = \mathbb{E} \int_{[0, \infty)} e^{-\beta s} (\ell(X_s) ds + h \cdot dY_s),$$

$$\ell : \mathbb{X} \rightarrow \mathbb{R}, h \in \mathbb{R}^p.$$

**Goal:** Characterize the value function

$$V(x) = \inf_{Y \in \mathcal{A}(x)} J(x, Y)$$

as the (unique) solution of a suitable PDE.

**Two Settings:**

- Case I:  $\ell(x) \geq 0$ ,  $h \cdot \mathbb{Y} \subset \mathbb{R}_+$ ,  $\mathbb{X}$  is a cone.
- Case II: General cost, bounded  $\mathbb{X}$ .

## Case I.

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$\ell(x) \geq 0$ ,  $h \cdot \mathbb{Y} \subset \mathbb{R}_+$ ,  $\mathbb{X}$  is a cone.

**Dynamic Programming Heuristic.** Recall:  $X_t = x + B_t + GY_t$ .

**One Candidate control:** Apply no control for  $0 \leq s \leq \delta$  and then control optimally.

$$\begin{aligned} V(x) &\leq \mathbb{E}_x \left( \int_0^\delta e^{-\beta s} \ell(X_s) ds + e^{-\beta \delta} V(X_\delta) \right) \\ &= \mathbb{E}_x \left( \int_0^\delta e^{-\beta s} \ell(X_s) ds + e^{-\beta \delta} V(x) - e^{-\beta \delta} \int_0^\delta \mathcal{L}V(X_s) ds \right) \end{aligned}$$

where  $\mathcal{L} \doteq -(1/2)\text{Tr}(\Sigma D^2) - b \cdot D$ . Dividing by  $\delta \rightarrow 0$ :

$$(\mathcal{L} + \beta)V(x) - \ell(x) \leq 0.$$

- Equality holds if it is "optimal" to apply no control around  $x$ .

## Dynamic Programming Heuristic (Contd.)

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**Another control:** Jump by  $\delta y \in \mathbb{Y}$  at time 0 & then go optimally.

$$V(x) \leq \delta h \cdot y + V(x + \delta Gy), \quad y \in \mathbb{Y}.$$

$$-DV(x) \cdot \frac{Gy}{|Gy|} - h \cdot \frac{y}{|Gy|} \leq 0.$$

Thus defining  $\mathcal{H}(p) \doteq \sup_{y \in \mathbb{Y}: |Gy|=1} \{-(p \cdot Gy + h \cdot y)\}$ :

$$\mathcal{H}(DV(x)) \leq 0.$$

- Equality holds if it is optimal to apply control at  $x$ .

## HJB equation.

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Both  $\mathcal{H}(DV(x))$  and  $(\mathcal{L} + \beta)V(x) - \ell(x)$  are at most 0 and one of them is exactly 0.

So  $V$  must satisfy:

$$\underbrace{((\mathcal{L} + \beta)\psi(x) - \ell(x)) \vee \mathcal{H}(D\psi(x))}_{F(x, \psi(x), D\psi(x), D^2\psi(x))} = 0$$

**Boundary Condition:** Fix  $x \in \partial\mathbb{X}$ .

Suppose First:  $\mathcal{H}(DV(x)) = 0$ .

Recall:  $\mathcal{H}(DV(x)) \doteq \sup_{y \in \mathbb{Y}: |Gy|=1} \{-(Gy \cdot DV(x) + y \cdot h)\}$ .

Suppose that the supremum is attained at  $y^*$ :

## HJB (contd.)

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$$0 = \mathcal{H}(DV(x)) = -(Gy^* \cdot DV(x) + y^* \cdot h).$$

Must have:  $Gy^* \cdot n(x) \leq 0$  where  $n(x)$  is the outward normal at  $x$ .

Let  $\phi \in C^2(\mathbb{X})$  be such that  $V - \phi$  is minimized at  $x$ . Then:

$$D\phi(x) = DV(x) + \theta n(x),$$

for some  $\theta \geq 0$ . Thus:

$$\mathcal{H}(D\phi(x)) = \mathcal{H}(DV(x) + \theta n(x)) \geq -\theta Gy^* \cdot n(x) \geq 0.$$

So  $F(x, V(x), D\phi(x), D^2\phi(x)) \geq 0$

– Can argue similarly if  $\mathcal{H}(DV(x)) < 0$ .

**BC:**  $V$  is a supersolution on  $\mathbb{X}$ . (Soner)

## Viscosity Solutions (Lions, Crandall, Ishii, ...)

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– In general no  $C^2$  solutions- a weaker notion of a solution is needed.

**Basic idea:** Suppose  $u$  is a classical subsolution, i.e.,

$$u \text{ is } C^2 \text{ and } F(x, u(x), Du(x), D^2u(x)) \leq 0.$$

Then if  $\phi$  is  $C^2$  and  $u - \phi$  has a maximum at  $x$  then

$$Du(x) = D\phi(x), \quad D^2u(x) \leq D^2\phi(x).$$

$$\begin{aligned} F(x, u(x), D\phi(x), D^2\phi(x)) &= F(x, u(x), Du(x), D^2\phi(x)) \\ &\leq F(x, u(x), Du(x), D^2u(x)) \leq 0. \end{aligned}$$

Thus:  $F(x, u(x), D\phi(x), D^2\phi(x)) \leq 0$ .

## Definition (Constrained Viscosity Solution).

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$$F(x, \psi(x), D\psi(x), D^2\psi(x)) = 0. \quad (1)$$

(1) For  $S \subset \mathbb{X}$ , a function  $\psi : \bar{S} \rightarrow [0, \infty)$  is a **supersolution of (1) on  $S$**  if  $\forall x \in S, \varphi \in C^2(S)$  s.t.  $\psi - \varphi$  has a **global minimum** at  $x$ :

$$F(x, \psi(x), (D\varphi)(x), (D^2\varphi)(x)) \geq 0.$$

(2) For  $S \subset \mathbb{X}$ , a function  $\psi : \bar{S} \rightarrow [0, \infty)$  is a **subsolution of (1) on  $S$**  if  $\forall x \in S, \varphi \in C^2(S)$  s.t.  $\psi - \varphi$  has a **global maximum** at  $x$ :

$$F(x, \psi(x), (D\varphi)(x), (D^2\varphi)(x)) \leq 0.$$

(3)  $\psi$  is said to be a constrained viscosity solution of (1) on  $\mathbb{X}$  if it is a **subsolution of (1) on  $\mathbb{X}^o$**  and a **supersolution of (1) on  $\mathbb{X}$** .

## Some Amusing Examples.

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$$\mathbb{X} = \mathbb{Y} = \mathbb{R}_+, G = 1, \beta = 1, h = 0$$

(a) Suppose  $\ell(x) \equiv 1, b = 0, \Sigma = 1$

$$V(x) = \inf_Y J(x, Y) = \inf_Y \mathbb{E} \int_0^\infty e^{-t} dt = 1.$$

$$0 = F(\dots) = \left(-\frac{1}{2}\psi''(x) + \psi(x) - 1\right) \vee (-D\psi(x)).$$

Then  $\psi_c(x) \equiv c$  is a solution for every  $c \in [0, 1]$ .

## Examples (Contd.).

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$$\mathbb{X} = \mathbb{Y} = \mathbb{R}_+, G = 1, \beta = 1, h = 0$$

(b) Suppose  $\ell(z) = z + 1, b = -1, \Sigma = 0: X_t = x - t + Y_t$ .

$$\begin{aligned} V(x) &= \inf_Y \mathbb{E} \int_0^\infty e^{-t} (X_t + 1) dt \\ &= \int_0^x e^{-t} (x - t + 1) dt + \int_x^\infty e^{-t} 1 dt = x + e^{-x}. \end{aligned}$$

$$0 = F(\dots) = (\psi'(x) + \psi(x) - x - 1) \vee (-D\psi(x)).$$

$V$  is a solution,

$\psi(x) = \max(x + ce^{-x}, C), c \in [0, 1], C \in (c, 1]$  is a solution as well.

- $\psi(x) = x$  solves  $F(\dots) = 0$  on  $[0, \infty)$  but doesn't satisfy the boundary condition: Subdifferential at 0 is the set  $(-\infty, 1]$ .

# Standing Assumptions

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- **Controllability:**  $\mathbb{X} \cap G\mathbb{Y} \neq \emptyset$ .
- **On the cones  $\mathbb{X}$  and  $\mathbb{Y}$ :**
  - Union of  $\mathbb{X}$  and  $G\mathbb{Y}$  is (strictly) contained in a half space:  
 $\exists$  unit vector  $e_1 \in \mathbb{R}^d$ ,  $a_1 > 0$  s.t.  $\forall x, y$  in  $\mathbb{X}, \mathbb{Y}$   
 $x \cdot e_1 \geq a_1|x|$ ,  $Gy \cdot e_1 \geq a_1|Gy|$ .
  - $\mathbb{Y}$  is (strictly) contained in a half space:  $\exists$  unit vector  $e_2 \in \mathbb{R}^p$  and  $a_2 > 0$  s.t.  $\forall y \in \mathbb{Y}$ ,  $y \cdot e_2 \geq a_2|y|$ .
- **On the running cost  $\ell$ :**
  - $\exists m_\ell \in \mathbb{N}_0$  such that  $\forall x \in \mathbb{X}$ ,

$$c_1|x|^{m_\ell} - c_2 \leq \ell(x) \leq c_3(|x|^{m_\ell} + 1).$$

- $\sup_{|x-y|<\delta} \frac{|\ell(x)-\ell(y)|}{1+|x|^{m_\ell}+|y|^{m_\ell}} \rightarrow 0$  as  $\delta \rightarrow 0$ .

## Main Result (Atar, B. *Ann. Prob.* (2006)).

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Let  $C_{\text{pol},+}(\mathbb{X})$  be the class of continuous nonnegative functions with (at most) polynomial growth.

$$C_{\text{pol},+}^c(\mathbb{X}) \doteq \{f \in C_{\text{pol},+}(\mathbb{X}) \mid f \text{ has compact sub-level sets}\}.$$

$$C_{b,+}(\mathbb{X}) \doteq \{f \in C_{\text{pol},+}(\mathbb{X}) \mid f \text{ is bounded}\}.$$

### Theorem.

(1) Recall:  $c_{\ell,1}|x|^{m_\ell} - c_{\ell,2} \leq \ell(x) \leq c_{\ell,3}(|x|^{m_\ell} + 1)$ .

- If  $m_\ell > 0$  then  $V$  is in  $C_{\text{pol},+}^c(\mathbb{X})$ .
- If  $m_\ell = 0$  then  $V$  is in  $C_{b,+}(\mathbb{X})$ .

(2) **Solvability.**  $V$  is a constrained viscosity solution of (1) on  $\mathbb{X}$ .

## Main Result (Contd.).

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(3) **Uniqueness.** Let Assumption 1 hold.

**Assumption 1.** Either  $\Sigma$  is non-degenerate or  $e_1 \cdot b > 0$ .

- If  $m_\ell > 0$ ,  $V$  is the only such solution in the class  $C_{\text{pol},+}^c(\mathbb{X})$ .
- If  $m_\ell = 0$ ,  $V$  is the maximal solution in the class  $C_{\text{pol},+}(\mathbb{X})$ .

Suppose, in addition, Assumption 2 holds.

**Assumption 2.** For some  $c_h > 0$ ,  $h \cdot y \geq c_h |Gy|$ ,  $y \in \mathbb{Y}$ .

Then uniqueness holds in  $C_{\text{pol},+}(\mathbb{X})$ ,  $m_\ell \geq 0$ .

## Remark.

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One can construct (one dimensional) examples to show:

- If Assumption 1 is dropped, in the case  $m_\ell > 0$ : Multiple solutions may exist (even) in  $C_{\text{pol},+}^c(\mathbb{X})$ .
- If Assumption 2 is dropped:
  - In the case  $m_\ell > 0$ : Uniqueness in  $C_{\text{pol},+}^c(\mathbb{X})$  but multiple solutions may exist in  $C_{\text{pol},+}(\mathbb{X})$ .
  - In the case  $m_\ell = 0$ : multiple solutions may exist in  $C_{b,+}(\mathbb{X})$  (value function is the maximal one).

## Uniqueness: Comments

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– A typical argument: If  $F(x, r, p, A)$  is  $r$ -coercive then maximum principle holds.

( $r$ -coercive:  $F(x, r_1, p, A) \leq F(x, r_2, p, A)$  implies  $r_1 \leq r_2$ .)

Eg. if  $u$  and  $v$  are  $C^2$ ,  $u$  subsoln.,  $v$  supersoln. and  $x \in \mathbb{X}^o$  is such that  $u - v$  is maximized at  $x$ , then:

$$Du(x) = Dv(x) \text{ and } D^2u(x) \leq D^2v(x).$$

$$\begin{aligned} F(x, u(x), Du(x), D^2u(x)) &\leq 0 \leq F(x, v(x), Dv(x), D^2v(x)) \\ &\leq F(x, v(x), Du(x), D^2u(x)). \end{aligned}$$

Coercivity gives  $u(x) \leq v(x)$ .

– Here coercivity of  $F$  fails... A careful construction of a suitable strict subsolution is needed.

## General cost, $\mathbb{X}$ Bounded (Atar, B., Williams, *Ann. App. Prob.* (2007))

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### Controlled Dynamics:

$$X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s + GY_t$$

–  $b, \sigma$  Lip. Control cone  $\mathbb{Y}$  and admissible controls as before.

### Cost Criterion:

$$J(x, Y) = \mathbb{E} \int_{[0, \infty)} e^{-\beta s} (\ell(X_s) ds + h \cdot dY_s),$$

$\ell$  continuous,  $\ell, h \cdot y$  may take both positive and negative values.

**Goal:** Characterize the value function  $V(x) = \inf_{Y \in \mathcal{A}(x)} J(x, Y)$  in terms of the associated HJB equation.

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## Standing Assumptions.

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- Controllability:  $GY = \mathbb{R}^d$ .
- For some  $q \in \mathbb{R}^d$ ,

$$\mathcal{H}(q) \doteq \sup_{y \in \mathbb{Y}: |Gy|=1} \{-(Gy \cdot q + y \cdot h)\} < \infty.$$

These conditions imply:

1. **Cost is bounded from below:**

$$\sup_{x \in \mathbb{X}} \sup_{Y \in \mathcal{A}(x)} \mathbb{E} \int_0^\infty e^{-\beta s} (h \cdot Y_s)^- ds < \infty. \quad (P1)$$

2. **Integration by parts holds:**  $\forall Y \in \mathcal{A}(x), x \in \mathbb{X},$

$$\beta \int_0^\infty e^{-\beta s} h \cdot Y_s ds = \int_{[0, \infty)} e^{-\beta s} h \cdot dY_s. \quad (P2)$$

## Value function Properties.

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**Proposition.**  $V(x) \in (-\infty, \infty)$  for each  $x \in \mathbb{X}$ .  $V$  is Lipschitz on  $\mathbb{X}$ .

### Sketch.

- $V(x) > -\infty$  follows from (P1).
- **Upper bound**,  $x \in \mathbb{X}^o$ : Let  $B$  be a small enough ball such that  $x \in B \subset \mathbb{X}^o$ . Let Control be the normal reflection on the boundary. Easy to check associated cost is finite.
- **Lipschitz property**: State can be moved from any  $x_1 \in \mathbb{X}$  to any other  $x_2 \in \mathbb{X}$ , instantaneously, by exercising a control and paying a cost proportional to  $|x_1 - x_2|$ .

## HJB equation.

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Let  $\mathcal{L}f = -\frac{1}{2}\text{Tr}(\sigma D^2 f \sigma') - b \cdot Df$

Recall  $\mathcal{H}(q) = \sup_{y \in \mathbb{Y}: |Gy|=1} \{-Gy \cdot q - h \cdot y\}$ .

**HJB equation:**

$$\underbrace{((\mathcal{L} + \beta)\psi(x) - \ell(x)) \vee \mathcal{H}(D\psi(x))}_{F(x, \psi(x), D\psi(x), D^2\psi(x))} = 0 \quad (2)$$

**Constrained Viscosity solution:** Subsolution on  $\mathbb{X}^o$  and supersolution on  $\mathbb{X}$ .

## Characterization.

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**Theorem.** Under the standing assumptions:

- **Solvability:**  $V$  is a constrained viscosity solution of (2).
- **Uniqueness:** Assumption (A) is necessary and sufficient for  $V$  to be only constrained viscosity solution in  $C_b(\mathbb{X})$ .

**Assumption (A):**  $\inf_{q \in \mathbb{R}^p} \mathcal{H}(q) < 0$

**Remark:** If (A) fails then every continuous function is a supersolution and if  $\psi$  is a subsolution then so is  $\psi - c$  for  $c > 0$ .

## Sufficient Condition for (A).

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1. **No Arbitrage Condition (Harrison and Williams):**

$$\{y \in \mathbb{Y} : Gy = 0 \text{ and } h \cdot y \leq 0\} = \{0\}. \quad (A1)$$

2. **Control cone Geometry:**  $\exists e \in \mathbb{R}^p$  such that

$$\inf_{y \in \mathbb{Y} : |Gy|=1} e \cdot y > 0. \quad (A2)$$

**Theorem.** (A1) and (A2) imply (A).

**Sketch.**

Recall  $V$  solves  $((\mathcal{L} + \beta)\psi(x) - \ell(x)) \vee \mathcal{H}(D\psi(x)) = 0$ .

- Set of all  $h$  for which (A1) holds is open.
- Let  $h^\varepsilon \doteq h - \varepsilon e_1$  be such that (A1) holds for  $h^\varepsilon$ .

## Sketch of Proof (Contd.)

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- $V^\varepsilon$  be the value function with  $h \mapsto h^\varepsilon$ .
- $V^\varepsilon(x) \in (-\infty, \infty)$  and  $V^\varepsilon$  solves the HJB equation (for  $h^\varepsilon$ ).
- Construct a  $C^2$  function  $\phi$  such that for some  $x_0 \in \mathbb{X}^o$ ,  $V^\varepsilon(x_0) - \phi(x_0)$  is maximized at  $x_0$ .
- Using the subsolution property of  $V^\varepsilon$  one can show that with  $q_0 = D\phi(x_0)$ ,  $\mathcal{H}^\varepsilon(q_0) \leq 0$  which implies  $\mathcal{H}(q_0) < 0$ .

## An Application.

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- **State space.**  $\mathbb{X} = \mathbb{R}_+^2$
- **State Dynamics.**  $X = x + B + Y$
- $B$  is a 2D  $(b, \Sigma)$ -Brownian motion,  $\Sigma$  positive definite.
- **Admissible Controls.**  $Y$  is an admissible control for the initial condition  $x$  if
  - It is nonnegative, nondecreasing (i.e.  $\mathbb{Y} = \mathbb{R}_+^2$ ).
  - $X(t) \in \mathbb{X}$  for all  $t \geq 0$ , a.s.
- **Cost.**  $J(x, Y) \doteq \mathbb{E} \int_0^\infty e^{-\beta t} h(X(t)) dt$ ,  
 $h : \mathbb{X} \rightarrow \mathbb{R}_+$  is a convex function of the form:

$$h(z) \doteq (\alpha \cdot z) 1_{z_2 \geq cz_1} + (\gamma \cdot z) 1_{z_2 < cz_1}, \quad z = (z_1, z_2)' \in \mathbb{X},$$

$$\alpha, \gamma \in \mathbb{R}^2, \quad c \in (0, \infty).$$

# Optimal Control.

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**Value Function.**  $V(z) = \inf_Y J(z, Y), z \in \mathbb{X}.$

## Optimal Control.

- If  $\alpha \geq 0, \gamma \geq 0$ , an optimally controlled process is a RBM in  $\mathbb{R}_+^2$ , with normal reflection.
- If  $\alpha \not\geq 0$  or  $\gamma \not\geq 0$ ???

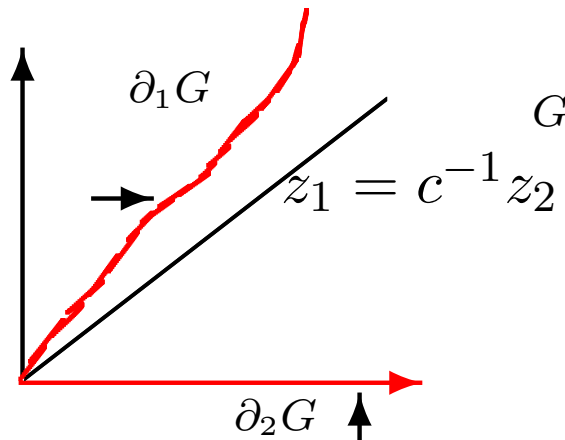
# Free Boundary Characterization.

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Martins, Shreve and Soner(1996) conjectured:

- If  $\alpha \not\geq 0$  and  $\gamma \geq 0$ , there is a continuous, increasing function  $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\Psi(z_2) \leq c^{-1}z_2$ , for all  $z_2 \in \mathbb{R}_+$ ,  $\Psi(z_2) \rightarrow \infty$  as  $z_2 \rightarrow \infty$ .
- **Optimal process is a RBM** on  $G = \{z \in \mathbb{X} : z_1 \geq \Psi(z_2)\}$  reflected along  $e_i$  on  $\partial_i G$ ,  $i = 1, 2$ , where

$$\partial_1 G = \{z \in \mathbb{X} : z_1 = \Psi(z_2)\}, \quad \partial_2 G = \{z \in \mathbb{X} : z_2 = 0\}.$$



## Optimal Control(B., Ross, *Ann. App. Prob.* (to appear))

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- $V$  is  $C^1$ .
  - Proof is probabilistic.
  - Relates the singular control problem with certain optimal stopping problems.
  - Also requires showing the existence of an optimal control (established in B. and Ross, *Ann. App. Prob.* (2006)).
- MSS Conjecture is correct and  $\Psi$  is a Lipschitz function given explicitly in terms of  $DV$ .
- **Proof** of optimality is based on a verification argument which **critically relies on the well-posedness of the HJB equation.**

## Other Issues.

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- Free boundary problems in higher dimensions: Optimal process a reflected diffusion.
- Smoothness of the free boundary.
- Interior regularity of Value function. For Singular Control (**without state constraints**): Evans, Ishii and Koike, Shreve and Soner, Menaldi and Robin.