Optimal stopping times with different information levels and with time uncertainty

Arijit Chakrabarty

Joint work with Xin Guo

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The Problem

$$\sup_{\tau \leq T} E\left[e^{-r\tau} g(X_\tau)\right]$$

- X_t: the price process of a stock
- r: the interest rate
- τ: stopping time
- T: Time horizon

Existing literature

- Salminen(1985)
- Dayanik & Karatzas (2003)
- Johnson & Zervos (2005)
- Lamberton & Zervos (2006)

Our Work

$$\sup_{\tau \in \mathcal{A}} E\left[e^{-r\tau} g(X_\tau) 1_{\{\tau < T\}}\right]$$

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- T random time
- "information" about T
- possibility of recovery: $<$ replaced by $\leq$
- When is the solution of the form

$$\tau^* = \inf\{t \geq 0 : X_t \leq x_l^* \text{ or } \geq x_u^*\}?$$

Mathematical setup

- X satisfies

$$
\begin{aligned}
dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t \\
X_0 &= x
\end{aligned}
$$

- $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$=canonical filtration
- T: first arrival of Cox process.

$$P(T > t|\mathcal{F}_t) = \exp\left(-\int_0^t \lambda(s)ds\right)$$

λ progressively measurable

Different information levels

$$
\begin{aligned}
\mathbb{F} &= \text{canonical filtration} \\
\mathbb{G} &= \mathbb{F}^T \\
&\quad \text{(smallest enlargement to make } T \text{ stopping time)} \\
\mathcal{A}_0 &= \{\mathbb{F} - \text{stopping times}\} \\
&\quad \text{no information about } T \\
\mathcal{A}_1 &= \{\mathbb{G} - \text{stopping times}\} \\
&\quad \text{positive information about } T
\end{aligned}
$$

Equivalence under no recovery

Theorem

$$\begin{aligned} & \sup_{\tau \in \mathcal{A}_0} E\left[e^{-r\tau} g(X_\tau) 1_{\{\tau < T\}}\right] \\ = & \sup_{\tau \in \mathcal{A}_1} E\left[e^{-r\tau} g(X_\tau) 1_{\{\tau < T\}}\right] \\ = & \sup_{\tau \in \mathcal{A}_0} E\left[e^{-\int_0^\tau (r+\lambda(s))ds} g(X_\tau)\right] \end{aligned}$$

When there is no recovery, the extra information about T has no impact.

Distinction under positive recovery

Theorem

$$\sup_{\tau \in \mathcal{A}_1} E\left[e^{-r\tau} g(X_\tau) 1_{\{\tau \le T\}}\right]$$
$$= \sup_{\tau \in \mathcal{A}_0} E\left[e^{-r\tau} g(X_T) 1_{\tau > T} + e^{-\int_0^\tau (r+\lambda(s))ds} g(X_\tau)\right]$$

In particular, when $T \sim \exp(\lambda)$, *and is independent of* X,

$$\sup_{\tau \in \mathcal{A}_1} E\left[e^{-r\tau} g(X_\tau) 1_{\{\tau \le T\}}\right]$$
$$= \sup_{\tau \in \mathcal{A}_0} E\left[\lambda \int_0^\tau e^{-(r+\lambda)t} g(X_t) dt + e^{-(r+\lambda)\tau} g(X_\tau)\right]$$

In short

Suppose $T \sim \exp(\lambda)$, and is independent of X

	$E\left[e^{-r\tau}g(X_\tau)1_{\{\tau\le T\}}\right]$	$E\left[e^{-r\tau}g(X_\tau)1_{\{\tau< T\}}\right]$
$\sup_{\tau\in\mathcal{A}_1}$	$V_{r+\lambda}(\lambda g, g)$	$V_{r+\lambda}(0, g)$
$\sup_{\tau\in\mathcal{A}_0}$	$V_{r+\lambda}(0, g)$	$V_{r+\lambda}(0, g)$

where

$$V_\beta(f,h) = \sup_{\tau\in\mathcal{A}_0} E\left[\int_0^\tau e^{-\beta s}f(X_s)ds + e^{-\beta\tau}h(X_\tau)\right]$$

for $\beta > 0$ and functions f, h

The setup

$$V(x) = \sup_{\tau \in \mathcal{A}_0} E_x \left[\int_0^\tau e^{-\beta s} f(X_s) ds + e^{-\beta \tau} g(X_\tau) \right]$$

where $\beta > 0$, f, g "nice" functions and X recurrent diffusion process with state space (a, b) satisfying

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t \\ X_0 &= x \end{aligned}$$

Objectives:

- 'Evaluate' V
- Find the optimal stopping time τ^*

Notations

Let ψ and ϕ be the strictly increasing and decreasing solutions respectively of

$$\frac{\sigma^2(x)}{2}u''(x) + \mu(x)u'(x) - ru(x) = 0, \; x \in (a, b)$$

Define

$$\begin{aligned} l_a &= \limsup_{x \downarrow a} \frac{g^+(x)}{\phi(x)} \\ l_b &= \limsup_{x \uparrow b} \frac{g^+(x)}{\psi(x)} \\ \text{(the continuation region) } \mathcal{C} &= \{x \in (a, b) : V(x) > g(x)\} \\ \text{(the stopping region) } \mathcal{S} &= \{x \in (a, b) : V(x) = g(x)\} \end{aligned}$$

Results

Theorem

$V(x) < \infty$ for all x iff $I_a + I_b < \infty$ and in that case V is continuous on (a, b).

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$V(x) < \infty$ for all x iff $l_a + l_b < \infty$ and in that case V is continuous on (a, b).

Theorem

If g and V are continuous on (a, b), then

$$\tau^* := \inf\{t \geq 0 : X_t \in \mathcal{S}\}$$

is an optimal stopping time if one of the following conditions hold:

- $l_a = l_b = 0$
- $0 < l_a < \infty$ *and* $l_b = 0$, *and there is no* $c \in (a, b)$ *such that* $(a, c) \subset \mathcal{C}$.
- $0 < l_b < \infty$ *and* $l_b = 0$, *and there is no* $c \in (a, b)$ *such that* $(c, b) \subset \mathcal{C}$.

A verification lemma

Theorem

Suppose that there exists a C^1 function $\tilde{V}$ on $(0, \infty)$ such that $\tilde{V}$ is the difference of two convex functions, and $\tilde{V}'$ is bounded and absolutely continuous. If

$$\min\left\{\tilde{V}(x) - g(x), \beta\tilde{V}(x) - \mu x\tilde{V}'(x) - \frac{\sigma^2}{2}x^2\tilde{V}''(x) - f(x)\right\}$$

$$= 0 \ a.e.$$

in the classical sense with $\tilde{V}''$ being the Radon-Nikodym derivative of $\tilde{V}'$,
if $\{e^{-\beta\tau}\tilde{V}(X_\tau) : \tau$ finite stopping time$\}$ is P_x-uniformly integrable
and if, $\lim_{t\to\infty} e^{-\beta t}g(X_t) = 0$ P_x-a.s. for all x;
Then $\tilde{V} = V$.

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- In general, just the fact that τ^* is the first hitting time of $\mathcal{S}$ is not immensely helpful.
- When is τ^* of the form

$$\tau^* = \inf\{t \geq 0 : X_t \leq x_l^* \text{ or } \geq x_u^*\}?$$

- Equivalently, when is $\mathcal{C}$ connected?

g is C^2

Assumptions:

- $g \in C^2(a, b)$
- V is finite and continuous on (a, b)
- $\tau^* = \inf\{t \geq 0 : X_t \in \mathcal{S}\}$ is the optimal stopping time

Define

$$\mathcal{U} := \{x \in (a, b) : \mathcal{L}g(x) < f(x)\}$$

where

$$\mathcal{L}\phi(x) := \beta\phi(x) - \mu(x)\phi'(x) - \frac{\sigma^2(x)}{2}\phi''(x)$$

g is C^2 (contd.)

For an open set O, $n(O)$ denotes the number of connected components of O

Theorem

$n(\mathcal{C}) \leq 2 + n(\mathcal{U})$

Theorem

$n(\mathcal{C}) \leq 1 + n(\mathcal{U})$ *if one of the following holds*

- $\limsup_{x \downarrow a} \mathcal{L}g(x) - f(x) < 0$, *or* g *is bounded on* (a, a') *for some* $a' \in (a, b)$
- $\limsup_{x \uparrow b} \mathcal{L}g(x) - f(x) < 0$, *or* g *is bounded on* (b', b) *for some* $b' \in (a, b)$

If both hold, then $n(\mathcal{C}) \leq 1 + n(\mathcal{U})$

Theorem

If the family $\{e^{-\beta(\tau^* \wedge t)} g(X_{\tau^* \wedge t}) : t \geq 0\}$ *is uniformly integrable, then* $n(\mathcal{C}) \leq n(\mathcal{U})$

g is the difference of two convex functions

$$
\begin{aligned}
\mathcal{L}g(dx) &:= \beta g(x)dx - \mu(x)g'_-(x)dx - \frac{\sigma^2(x)}{2}g''(dx) \\
|A|_t &:= \int_a^b \frac{L_t^z}{\sigma^2(z)}|f(z)dz - \mathcal{L}g(dz)| \\
A_t &:= \int_a^b \frac{L_t^z}{\sigma^2(z)}(f(z)dz - \mathcal{L}g(dz))
\end{aligned}
$$

where L^z is the local time process of the diffusion X at level z.

Assumptions:

- V is finite and continuous on (a, b)
- $\tau^* = \inf\{t \geq 0 : X_t \in \mathcal{S}\}$ is the optimal stopping time
- $E\left[\int_0^\infty e^{-\beta s} d|A|_s\right] < \infty$

g is the difference of two convex functions (contd.)

Theorem

If there exists $\mathcal{V}$ such that for all $U \subset \mathcal{V}^c, \int_U f(z)dz \le \mathcal{L}g(U)$, and for all $U \subset \mathcal{V}$, $\int_U f(z)dz > \mathcal{L}g(U)$; Then $n(\mathcal{C}) \le 1 + n(\mathcal{U})$ if one of the following conditions holds:

- *$(a, a') \subset \mathcal{V}$ or g is bounded on (a, a') for some $a' \in (a, b)$;*
- *$(b', b) \subset \mathcal{V}$ or g is bounded on (b', b) for some $b' \in (a, b)$.*

Furthermore, if both of the above conditions hold, then $n(\mathcal{C}) \le n(\mathcal{U})$.

g is the difference of two convex functions (contd.)

Theorem

If there exists $\mathcal{V}$ such that for all $U \subset \mathcal{V}^c, \int_U f(z)dz \leq \mathcal{L}g(U)$, and for all $U \subset \mathcal{V}$, $\int_U f(z)dz > \mathcal{L}g(U)$; and if the family $\{e^{-\beta(\tau^* \wedge t)} g(X_{\tau^* \wedge t}) : t \geq 0\}$ is uniformly integrable. Then $n(\mathcal{C}) \leq n(\mathcal{U})$.

Examples

- $g(x) = (x - K)^+$ (American Call)
- $g(x) = x - K$
- $g(x) = K - x$

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Does not work when

$$g(x) = (K - x)^+$$

summarizing

- Introduction of random time horizon and possibility of recovery
- Different levels of information: filtration expansions
- Structural results for the value function
- (strong) Sufficient conditions for threshold type solution