Optimal buffer size and service rate for a queueing network in heavy traffic with customer abandonment.

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Outline

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   - Main result and history
2. **Approximating diffusion model**
   - Brownian control problem (BCP)
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   - Regulator maps
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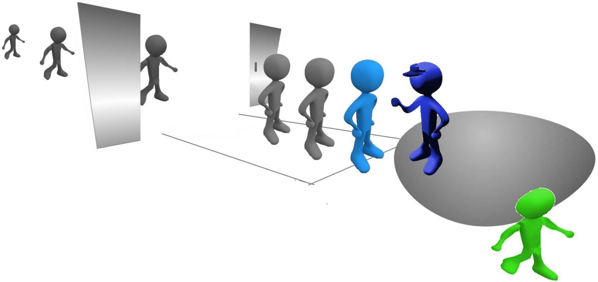
... in short



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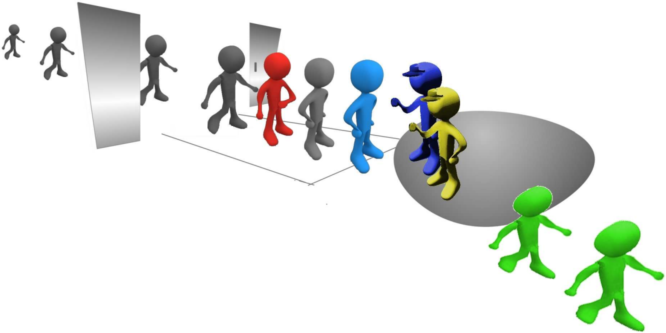

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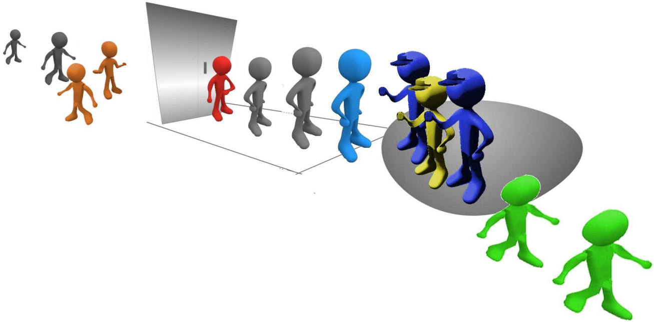

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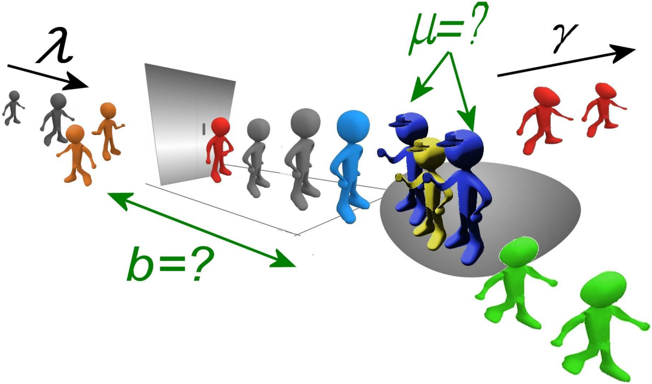

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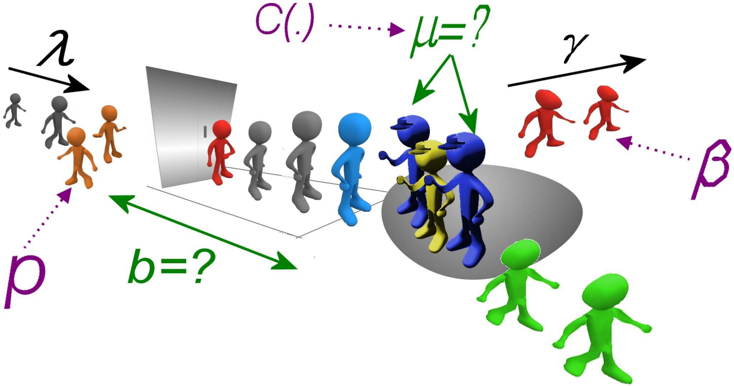

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$$q^n_{k,k+1} = \lambda_n(k) I_{\{k<\sqrt{n}b\}}, \ q^n_{k,k-1} = \mu_n(k) I_{\{k>0\}} + k\gamma_n, \ k \in \mathbb{Z}^+.$$

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If Y_n^A, Y_n^S, Y_n^R - indep. Poisson(1) processes (Ch. 6 [Kurtz-81]),

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Idle-time and Rejection processes are given by

$$L_n(t) \doteq \mu_n(0) \int_0^t I_{\{Q_n(s)=0\}} ds, \; U_n(t) \doteq \lambda \int_0^t I_{\{Q_n(s)=\sqrt{n}b\}} ds.$$

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As $n \to \infty$, $n\,\gamma_n \to \gamma > 0$, $\sup_{x \geq 0} |\mu_n(x) - \mu| \to 0$ and $\lambda = \mu$, $u_n(x) \doteq \sqrt{n}(\mu_n(\sqrt{n}x) - \lambda) \geq 0$ & $\sup_{x \geq 0} |u_n(x) - u(x)| \to 0$ and $\{u_n(\cdot)\}$ uniformly Lipschitz continuous.

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Problem: **asymptotically** minimize (among adm. policies)

$$J_p(\{\mu_n\}, b) \doteq \liminf_{n \to \infty} E \int_0^\infty e^{-\delta t} \left\{ \left[\beta(n\gamma_n)\hat{Q}_n(t) + C(u_n(\hat{Q}_n(t))) \right] dt + p\, d\hat{U}_n(t) \right\}.$$

where C is a $\mathcal{C}^2$-function, $\beta, p, \delta > 0$, and $\hat{\xi}_n(\cdot) = n^{-2}\xi_n(n\,\cdot)$.

Main Result

Let $p > 0$. There a $\mathcal{C}^2$-fn $\mathcal{V}_p$ and $b_p^* \in (0, \infty]$ s.t. if

$$\mu_n^*(x) = \lambda + \frac{1}{\sqrt{n}}(C')^{-1}\left(\mathcal{V}_p\left(\frac{1}{\sqrt{n}}x\right)\right),$$

then, $(\{\mu_n^*\}, b_p^*)$ is an optimal policy.

Also, for $p_0 = \frac{\beta\gamma}{(\delta+\gamma)}$,

$$\mathbf{p} \geq \mathbf{p}_0 \quad \Leftrightarrow \quad \mathbf{b}^* = \infty$$

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- **[Ward-Kumar-07]:** Solves **queueing control problem** explicitly, for an **adjustable** *b* but fixed μ.
- **[G-Weerasinghe]** (This work): Solves **queueing control problem** explicitly, for an **adjustable** *b* and **adjustable** μ (Also, nec. & suff.conditions for optimality of infinite buffer).

The scaled process has the following representation:

$$\hat{Q}_n(t) = 0 - \int_0^t [u_n(\hat{Q}_n(s)) + n\gamma_n \hat{Q}_n(s)]ds + \hat{W}_n(t) + \hat{L}_n(t) - \hat{U}_n(t),$$

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By f-CLT for unit Poisson processes, $(\hat{Y}_n^A, \hat{Y}_n^S, \hat{Y}_n^R) \Rightarrow (W^A, W^S, W^R)$.
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- $L(\cdot)$ is the local-time process of $X_x(\cdot)$ at 0.
- Any such (u, U) is an adm. control for BCP. **Find** (u^*, U^*) that minimizes

$$\tilde{J}_p(x, u, U) \doteq E\int_0^\infty e^{-\delta t}[(\beta\gamma X_x(t) + C(u(t)))dt + p\, dU(t)].$$

- Value function: $\mathrm{V}_p(x) = \inf \tilde{J}_p(x, u, U)$ over all adm. controls

HJB equation etc. and BCP

Let $\Phi(y) \doteq \sup_{u \in \mathcal{A}} [uy - C(u)]$. Hamilton-Jacobi-Bellman(HJB) eqn is

$$\min\left\{\frac{1}{2}\mathcal{V}''(x) - \Phi(\mathcal{V}'(x)) - \gamma x \mathcal{V}'(x) - \delta\mathcal{V}(x) + \beta\gamma x,\ \mathcal{V}'(x),\ p - \mathcal{V}'(x)\right\} = 0, \text{a.e.} x$$

Lemma 1:

If a $\mathcal{C}^2$-fn. $\mathcal{V}$ satisfies HJB with $\mathcal{V}'(0) = 0$, then $\mathrm{V}_p(x) \geq \mathcal{V}(x)$.

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Theorem 2:

There exists a unique $b_p^* \in (0, \infty]$ and a $\mathcal{C}^1$-fn $\mathcal{V}_p$ satisfying

$$\frac{1}{2}\mathcal{V}_p''(x) - \Phi(\mathcal{V}_p'(x)) - \gamma x\mathcal{V}_p'(x) - \delta\mathcal{V}_p(x) + \beta\gamma x = 0 \text{ for } 0 \leq x \leq b_p^*,$$
$$\mathcal{V}_p'(0) = 0, \mathcal{V}_p'(x) < p,\ \text{for } 0 < x < b_p^*., \quad \text{and } \mathcal{V}_p'(x) = p,\ \text{for } x \geq b_p^*.$$

And $\mathrm{V}_p(x) = \mathcal{V}_p(x)$. The solution of the BCP is given by: $u^*(x) = (C')^{-1}(\mathcal{V}_p'(x))$. If $p < p_0$, then $b^* < \infty$ and $U^* = U_b^*$ (loc. time at b^*). If $p \geq p_0$, $b^* = \infty$ and $U^* = 0$.

Solution to BCP

$\mathcal{Y} = \mathcal{V}'$. Solve **parametric family:**

$$\mathcal{Y}_r''(x) - 2\Phi'(\mathcal{Y}_r(x))\mathcal{Y}_r'(x) - 2\gamma x \mathcal{Y}_r'(x) - 2(\gamma + \delta)\mathcal{Y}_r(x) + 2\beta\gamma = 0,$$
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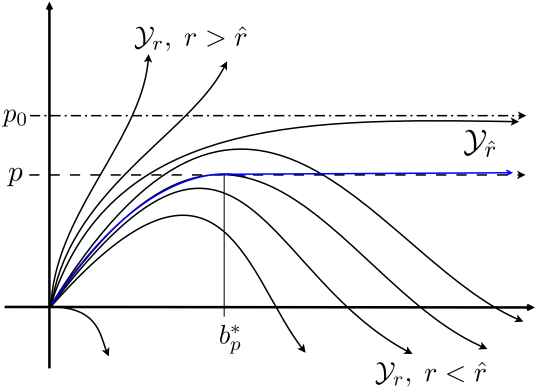

$$\mathcal{V}_p(x) = \frac{1}{2\delta}\mathcal{Y}_p'(0) + \int_0^x \mathcal{Y}_p(u)du, 0 \le x \le b_p^*; \mathcal{V}_p(x) = \mathcal{V}_p(b_p^*) + p(x - b_p^*), x > b_p^*$$

1- & 2-sided gen. regulator map $(\phi^u_\gamma, \psi^u_\gamma, \eta^u_\gamma)$: *Fix* b$\in (0, \infty]$. If $w \in \mathcal{D}$, $(\tilde{q}, \tilde{\ell}, \tilde{u}) \equiv (\phi^u_\gamma, \psi^u_\gamma, \eta^u_\gamma)(w)$ satisfies

(i) $\tilde{q}(t) = w(t) - \int_0^t [u(\tilde{q}(s)) + \gamma\tilde{q}(s)]ds + \tilde{\ell}(t) - \tilde{u}(t) \in [0, b], \ \forall t \geq 0,$
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(for $b = \infty$, $\eta \equiv 0$: 1-sided gen. reg. map)

Lemma 3

If $\gamma_n \to \gamma$, $u, u_n \geq 0$ unif. Lipschitz & $||u_n - u||_\infty \to 0$, then for $(\phi^n, \psi^n, \eta^n) = (\phi^{u_n}_{\gamma_n}, \psi^{u_n}_{\gamma_n}, \eta^{u_n}_{\gamma_n})$, $(\phi, \psi, \eta) = (\phi^u_\gamma, \psi^u_\gamma, \eta^u_\gamma)$:

(a) $||\phi^n(w)||_T \vee ||\psi^n(w)||_T \quad \leq \quad C||w||_T, \ \forall \ n \geq 1.$

(b) $\lim_n ||\phi^n(w_n) - \phi(w)||_T \vee ||\psi^n(w_n) - \psi(w)||_T = 0|| \vee ||\eta^n(w_n) - \eta(w)||_T = 0$,

if $\lim_n ||w_n - w||_T = 0$. Here, C is absolute const.

Representations

Recall the proposed policy (candidate for optimality): $(\{\mu_n^*\}, b_p^*)$ where, $\mathcal{V}_p$ & b_p^* as above ($p < p_0 \Leftrightarrow b^* < \infty$).

$$\mu_n^*(x) = \lambda + n^{-\frac{1}{2}}(C')^{-1}(\mathcal{V}_p(n^{-\frac{1}{2}}x))$$

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Note $u_n^*(x) \doteq \sqrt{n}(\mu_n^*(\sqrt{n}x) - \lambda) = u^*(x)$. In general,

$$\hat{Q}_n(t) \quad = \quad -\int_0^t [u_n(\hat{Q}_n(s)) + n\gamma_n \hat{Q}_n(s)]ds + \hat{W}_n(t) + \hat{L}_n(t) - \hat{U}_n(t)$$

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$$\begin{aligned} X_x(t) &= x - \int_0^t u(s)ds - \gamma\int_0^t X_x(s)ds + W(t) + L(t) - U(t) \\ &= (\phi, \psi, \eta)(W) \end{aligned}$$ (... also true for weak limits of proc.s from q-net.)

... if we show $W_n \Rightarrow W$.

Weak convergence

Proposition 4

For any $(\{\mu_n\}, b)$ adm. policy for the queueing network,

(i) $\hat{W}_n \Rightarrow W_0$, where W_0 is a BM, starting from 0, 0 drift, var. 2λ.

(ii) if $(X_0, L, U) \doteq (\phi, \psi, \eta)(W_0)$, $(\hat{Q}_n, \hat{L}_n, \hat{U}_n) \Rightarrow (X_0, L, U)$ and (X_0, u, U) is adm. for BCP ($x = 0$). (If $b = \infty$, $U_n = U = 0$)

(iii) $(\hat{Q}^*_n, \hat{L}^*_n, \hat{U}^*_n) \Rightarrow (X^*_0, L^*, U^*)$ ($b^* = \infty$, $U^*_n = U^* = 0$)

(iv) $E\left[\sup_{0\leq t\leq T} |\hat{W}_n(t)|^2\right] \leq \bar{C}(T^2 + T)$, $\bar{C}$ abs. constant.

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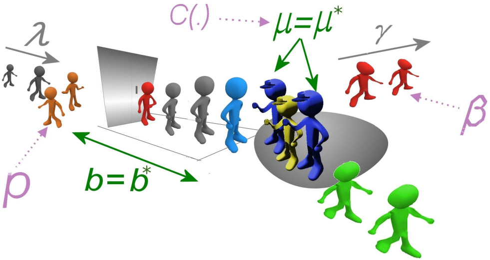
Theorem 5

$$J_p(\{\mu^*_n\}, b^*) = \mathrm{V}_p(0) \equiv \widetilde{J}_p(0, u^*, U^*) \le J_p(\{\mu_n\}, b).$$

Summary



- Computing b^* numerically - see [G-Weerasinghe-07].
- alternatively, λ control in heavy traffic.
- When is $b* = \infty$ optimal ? If and only if $p \geq \frac{\beta\gamma}{(\delta+\gamma)}$

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... **Thank you**