Marginal quantiles:
Asymptotics for functions of order statistics.

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Outline

Data depth

- Data depth provides an ordering of all points from the center outward. Contours of depth are often used to reveal the shape and structure of a multivariate data set. The depth of a point x in a one-dimensional data set $\{x_1, x_2, \cdots, x_n\}$ can be defined as the minimum of the number of data points on one side of x. [J. W. Tukey, in Proceedings of the International Congress of Mathematicians (1974), Vol. 2].

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- Several multidimensional depth measures $D_n(x; x_1, \cdots, x_n)$ for $x \in R^k$ were considered by many that satisfy certain mathematical conditions. If the data is from a spherical or elliptic distribution, the depth contours are generally required to converge to spherical or elliptic shapes.

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F a k-dimensional distribution function

F_j denotes the j-th marginal distribution function.

$F_j^{-1}(u) = \inf\{x : F_j(x) \geq u\}$, $0 < u < 1$
u-th quantile of the jth marginal.

Theorem

$\mathbf{X}_1, \ldots, \mathbf{X}_n \;\; i.i.d. \;\sim$ **F**, where $\mathbf{X}_i = (X_{i1}, \ldots, X_{ik})$.
$X_{ij} \sim F_j$
$0 < q_1, \ldots, q_k < 1$. $\qquad \delta_j$ density of F_j at $F_j^{-1}(q_j)$.

$\hat{\theta}_j$ denotes the q_j-th sample quantile based on $X_{1j}, \ldots, X_{nj}$

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Theorem

Let F_j be twice continuously differentiable in a neighborhood of $F_j^{-1}(q_j)$ and $\delta_j > 0$. Then

$$\sqrt{n}(\hat{\theta}_1 - F_1^{-1}(q_1), \ldots, \hat{\theta}_k - F_k^{-1}(q_k)) \sim N(\mathbf{0}, \Sigma)$$

Covariance Matrix

The covariance matrix Σ is given by

$$\begin{pmatrix} \frac{q_1(1-q_1)}{\delta_1^2} & \sigma_{12} & \cdots & \sigma_{1k} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & \frac{q_k(1-q_k)}{\delta_k^2} \end{pmatrix},$$

where for $i \neq j$, $\sigma_{ij} = (F_{ij}(F_i^{-1}(q_i), F_j^{-1}(q_j)) - q_i q_j)/(\delta_i \delta_j)$

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In practice σ_{ij} can be estimated by bootstrap $E^*(n(\theta_i^* - \hat{\theta}_i)(\theta_j^* - \hat{\theta}_j))$, where θ_j^* denotes the bootstrapped marginal sample quantile.

Quantile process

For $(q_1, \ldots, q_k) \in (0,1)^k$, define the sample quantile process,
$Z_n(q_1, \ldots, q_k) = \sqrt{n}[\delta_1(\hat{\theta}_1 - F_1^{-1}(q_1)), \ldots, \delta_k(\hat{\theta}_k - F_k^{-1}(q_k))]$

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Theorem

Suppose for $j = 1, \ldots, k$, the marginal d.f. F_j is twice differentiable on (a_j, b_j), where

$$-\infty \leq a_j = \sup\{x : F_j(x) = 0\}$$

$$\infty \geq b_j = \inf\{x : F_j(x) = 1\}$$

If the first two derivatives of F_j satisfy some regularity conditions, then $Z_n(q_1, \ldots, q_k)$ converges weekly to a Gaussian random element $(W_1, \ldots, W_k)$ on $C[0,1]^k$

$f_j \neq 0$ on (a_j, b_j),

$$\max_{1 \leq j \leq k} \sup_{a_j < x < b_j} F_j(x)(1 - F_j(x)) \frac{|f_j'(x)|}{f_j^2(x)} < \infty$$

Each marginal of Z_n converges weakly to a Brownian bridge. The covariance is given by

$$E(W_i(t)W_j(s)) = P(F_i(X_{i1}) \leq t, F_j(X_{j1}) \leq s) - ts$$

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– *Collaboration with C. R. Rao*

Regression

How to estimate the regression coefficients when the association among the paired data is partially or completely lost?

$$Y_i = \alpha + \beta X_i + \epsilon_i$$

X_i i.i.d with mean μ and standard deviation σ_X, ϵ_i i.i.d with mean zero and standard deviation σ_ϵ and $\{X_i\}$ and $\{\epsilon_i\}$ are independent sequences. P_n the set of all permutations of $\{1, \ldots, n\}$.

$$\Phi(a, b) = \min_{\pi \in P_n} \sum_{i=1}^{n} (Y_{\pi(i)} - a - bX_i)^2$$

$$\sum (Y_i - a - bX_i)^2 \geq S_{YY}(1 - r_{XY}^2).$$

$$\Phi(\hat{\alpha}, \hat{\beta}) = \min_{a,b} \Phi(a, b)$$

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How to identify the permutation that maximizes the value?

$R_X(i)$ is the rank of X_i in $\{X_1, \ldots, X_n\}$

$\bar{R}_X(i) = n - R_X(i) + 1$. Define R_Y and $\bar{R}_Y$ similarly.

$\pi_1 = R_Y^{-1} R_X$ and $\pi_2 = \bar{R}_Y^{-1} R_X$

Lemma

$$\Phi(\hat{\alpha}, \hat{\beta}) = S_{YY}(1 - \max\{r^2_{X\pi_1(Y)}, r^2_{X\pi_2(Y)}\}).$$

$$\hat{\beta}^2 = \max\{r^2_{X\pi_1(Y)}, r^2_{X\pi_2(Y)}\}S_{YY}/S_{XX}$$

$$\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}.$$

$\frac{1}{n}\sum_{i=1}^n X_{(i)}Y_{(i)}$ and $\frac{1}{n}\sum_{i=1}^n X_{(i)}Y_{(n-i+1)}$ are used in computing $\hat{\beta}$.

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General results for functions of marginal order statistics will be discussed that would aid in establishing

$$\hat{\beta} \xrightarrow{a.e} \beta_1 = sign(\beta)\sqrt{\beta^2 + \sigma_\epsilon^2 \sigma_X^{-1}}$$

Strong Law

$\{(X_i^{(1)}, \ldots, X_i^{(k)}), i = 1, 2, \ldots\}$ random vectors
j-th marginals $\{X_1^{(j)}, X_2^{(j)}, \ldots,\}$ are i.i.d. $\sim F_j$
$X_{n:i}^{(j)}$ is the i-th order statistic of $\{X_1^{(j)}, \ldots, X_n^{(j)}\}$.

ϕ measurable function on $\mathbb{R}^k$

$$\gamma(u) = \phi(F_1^{-1}(u), \ldots, F_k^{-1}(u)),\ 0 < u < 1.$$

Theorem

If F_j are continuous, and ϕ satisfies some smoothness conditions and well behaved near the tails of F_j, then

$$\frac{1}{n} \sum_{i=1}^{n} \phi(X_{n:i}^{(1)}, \ldots, X_{n:i}^{(k)}) \xrightarrow{a.e.} \int_0^1 \gamma(y) dy$$

Assumptions for Strong Law

1. The function $\phi(F_1^{-1}(u_1), \ldots, F_k^{-1}(u_k))$ is continuous at where $u_1 = u, \ldots, u_k = u, 0 < u < 1$.
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   This implies γ is continuous.
2. For some A and $c_0 > 0$,

$$|\phi(F_1^{-1}(u_1), \ldots, F_k^{-1}(u_k))| \leq A \left(1 + \sum_{j=1}^{k} |\gamma(u_j)|\right),$$

whenever $(u_1, \ldots, u_k) \in (0, c_0)^k \cup (1 - c_0, 1)^k$

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3. γ is integrable on $(0, 1)$.

Additional assumptions for CLT

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5. $|\gamma|$ and ψ_j^2 are integrable, where for $0 < u < 1$

$$\psi_j(u) = \frac{\partial \phi(F_1^{-1}(u_1), \ldots, F_k^{-1}(u_k))}{\partial u_j}\Big|_{(u,\ldots,u)}$$

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6. ψ_j and $\phi(F_1^{-1}(u_1), \ldots, F_k^{-1}(u_k))$ are smooth functions.

Example:

$$\phi(F_1^{-1}(u), F_2^{-1}(v)) = \min(u,v)^{-\alpha}(1 - \max(u,v))^{-\alpha}$$

$$\gamma(u) = u^{-\alpha}(1-u)^{-\alpha}, \qquad 0 < \alpha < \frac{1}{2}$$

CLT

Theorem

Let $\mathbf{X}_i = (X_i^{(1)}, \ldots, X_i^{(k)})$ *be i.i.d. Assume for any pair* $(1 \leq j \neq r \leq k)$, *the joint distribution* $F_{j,r}$ *of* $(X_i^{(j)}, X_i^{(r)})$ *is continuous. Under regularity assumptions,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\phi(X_{n:i}^{(1)}, \ldots, X_{n:i}^{(k)}) - \int_0^1 \gamma(y)dy) \xrightarrow{dist} N(o, \sigma^2)$$

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$$\begin{aligned}\sigma^2 =& 2 \sum_{1 \le j \ne r \le k} \int_0^1 \int_0^1 [F_{j,r}(F_j^{-1}(x), F_r^{-1}(y)) - xy] \psi_j(x)\psi_j(y)dxdy \\ &+ 2 \sum_{j=1}^{k} \int_0^1 \int_0^y x(1-y)\psi_j(x)\psi_j(y)dxdy\end{aligned}$$

Example 1

X_i, i.i.d. $\sim F$ $\quad Y_i$, i.i.d. $\sim G$.
F, G continuous

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (X_{n:i} - Y_{n:i})^2$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (X_i^2 + Y_i^2) - \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} X_{n:i} Y_{n:i}$$

$$= E(X_1^2) + E(Y_1^2) - 2 \int_0^1 F^{-1}(u) G^{-1}(u) du$$

$$= E(X_1^2) + E(Y_1^2) - 2E(X_1 G^{-1}(F(X_1)))$$

Example 2

$(U_1, \ldots, U_k), (X_1^{(1)}, \ldots, X_1^{(k)}), (X_1^{(1)}, \ldots, X_1^{(k)}), \ldots$ i.i.d.
The marginals U_j are uniformly distributed
$F_{j,r}$ is the joint distribution of (U_j, U_r).
$\phi(u_1, \ldots, u_k) = u_1^{a_1} \cdots u_k^{a_k},\ a_j \geq 1$.

$$\frac{1}{n} \sum_{i=1}^{n} (X_{n:i}^{(1)})^{a_1} \cdots (X_{n:i}^{(k)})^{a_k} \xrightarrow{a.e.} \frac{1}{a_1 + \cdots + a_k + 1}$$

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$$\sigma^2 = 2 \sum_{1 \leq j < r \leq k} a_j a_r E(U_j U_r)^M + \frac{(2M-3)(M^2-2)}{2M+1} \sum_{j=1}^{k} a_j^2$$

$M = a_1 + \cdots + a_k$

Back to regression

$\{(X_i, Y_i), 1 \le i \le n\}$ i.i.d. bivariate normal with correlation ρ, means μ_1, μ_2, and standard deviations σ_1, σ_2.

$$X_1 \sim F, \quad Y_1 \sim G, \quad G^{-1}(F(x)) = \mu_2 + \frac{\sigma_2}{\sigma_1}(x - \mu)$$

Strong Law

$$\frac{1}{n}\sum_{i=1}^{n} X_{n:i} Y_{n:i} \xrightarrow{a.e.} E(X_1 G^{-1}(F(X_1))) = \mu_1\mu_2 + \sigma_1\sigma_2$$

Weak convergence

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_{n:i}Y_{n:i} - \mu_1\mu_2 - \sigma_1\sigma_2) \xrightarrow{dist.} N(0, \sigma^2)$$

$$\sigma^2 = \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + (1+\rho^2)\sigma_1^2\sigma_2^2 + 2\rho\,\mu_1\mu_2\,\sigma_1\sigma_2$$

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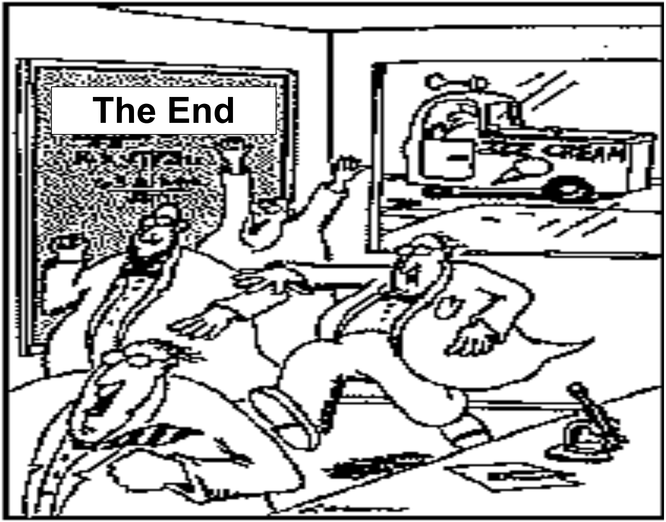
– *Collaborators*

Zhidong Bai, Kwok-Pui Choi, and Vasudevan Mangalam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