

# **GEOMETRY AND DISTRIBUTIONS OF SHAPES**

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- 1. PROBAB. MEASURES ON MANIFOLDS
  - (a) FRÉCHET MEAN
  - (b) EXTRINSIC MEAN
  - (c) INTRINSIC MEAN
  - (d) ASYMP. DISTR. OF SAMPLE MEANS
- 2. SHAPES OF  $k$ -ADS
  - (a) KENDALL'S SPACES:  $\Sigma_m^k$
  - (b) REFLECTION SHAPE SPACES:  $R\Sigma_m^k$
  - (c) AFFINE SHAPE SPACES:  $A\Sigma(m, k)$
  - (d) PROJECTIVE SHAPE SPACES:  $P\Sigma(m, k)$
- 3. NONPARAMETRIC TESTS

- ASSUMPTION: CLOSED BOUNDED SUBSETS OF  $(M, \rho)$  ARE COMPACT.
- FRÉCHET FUNCTION OF A PROBAB.  $Q$  IS

$$F(p) = \int \rho^2(p, x)Q(dx) \quad p \in M.$$

- FRÉCHET MEAN SET IS THE SET OF MINIMIZERS OF  $F$ . A UNIQUE MINIMIZER IS CALLED THE FRÉCHET MEAN OF  $Q$ , SAY  $\mu_F$ . SAMPLE FRÉCHET MEAN  $\mu_n, F$  IS A MEASURABLE SELECTION FROM THE MEAN SET OF THE EMPIRICAL  $Q_n$  BASED ON I.I.D.  $X_1, \dots, X_n \sim Q$ .
- **PROPOSITION 1.** LET  $F$  BE FINITE. (i) THEN THE FRÉCHET MEAN SET IS NONEMPTY COMPACT. (ii) IN CASE OF A UNIQUE MINIM.  $\mu_F, \mu_{n,F} \rightarrow \mu_F$  (WITH PROBAB. ONE). (ZIEZOLD,1977; BP,2003).

# 1(b). EXTRINSIC MEAN $\mu_E$ OF $Q$

- $M$ : COMPLETE ( $d$ -DIM.) DIFF. MANIFOLD. THE METRIC  $\rho = \rho_E$  IS INDUCED BY AN EMBEDDING  $J : M \rightarrow \mathbb{R}^N$ .
- $\mu^J =$  MEAN OF IMAGE  $Q^J$  OF  $Q$  IN  $\mathbb{R}^N$ .
- $\mu_E =$  PROJECTION OF  $\mu^J$  ON  $J(M)$  (IF UNIQUE) IS EXTRINSIC MEAN OF  $Q$ .
- EXAMPLE.

$$M = S^d = \{p \in \mathbb{R}^{d+1} : \|p\| = 1\}.$$

$J =$  INCLUSION MAP,  $\rho_E =$  CHORD DISTANCE,  $\mu_E =$  UNIQUE IFF  $\mu^J \neq 0$  ( $\in \mathbb{R}^{d+1}$ ).

- **PROPOSITION 2.** SUPPOSE  $Q^J$  HAS FINITE SECOND MOMENTS, AND  $\mu_E$  IS UNIQUE. THEN THE PROJECTION OF  $\mu_{n,E}$  ON THE TANGENT SPACE  $T_{\mu_E} M$  AT  $\mu_E$  IS ASYMP. GAUSSIAN  $N(0, \Gamma/n)$ .  $\square$

# 1(c). INTRINSIC MEAN $\mu_I$ OF $Q$

- $M$ : COMPLETE RIEMANNIAN MANIFOLD, METRIC TENSOR  $g$ ,  $\rho = \rho_g$  GEODESIC DIST.
- INTRINSIC MEAN OF  $Q$  IS  $\mu_I =$  MINIMIZER OF  $F$ , IF UNIQUE.
- TERMS: (i)  $\gamma(t)$  IS A GEODESIC IF  $(D/dt)\gamma(t) = 0$  (ZERO ACCELERATION).  
(ii) CUT LOCUS OF  $p$  ( $CUT(p)$ ).  
(iii) INJECTIVITY RADIUS ( $INJ(M)$ ).  
(iv)  $\text{Exp}_p (T_p M \rightarrow M)$ :  $\text{Exp}_p(v) = \gamma(1)$ ,  $\gamma$  GEODESIC,  $\gamma(0) = p$ ,  $\dot{\gamma}(0) = v$ .  
(v)

$$\text{Log}_p = \text{Exp}_p^{-1} : M \setminus \text{CUT}(p) \rightarrow T_p M.$$

- (vi) SECTIONAL CURVATURE AT  $p$ .
- EXAMPLE.  $M = S^d$ . GEODESICS ARE GREAT CIRCLES,  $\rho_g$  IS ARC DISTANCE.  $CUT(p) = \{-p\}$ .  $INJ(M) = \pi$ .

- $$\text{Exp}_p(v) = \cos(|v|)p + \sin(|v|)\frac{v}{\|v\|}, \quad (v \neq 0).$$

- $$\text{Log}_p(x) = (1 - (p \cdot x)^2)^{-1/2} \arccos(p \cdot x)(x - (p \cdot x)p), \quad x \neq p.$$

- SECTIONAL CURVATURE = 1 (CONSTANT)
- LET  $C$  DENOTE THE L.U.B. OF SECTIONAL CURVATURES ON  $M$  IF L.U.B  $> 0$ , AND  $C = 0$  IF  $L.U.B. \leq 0$ .

$$r^* := \min\{INJ(M), \pi/\sqrt{C}\}.$$

- PROPOSITION 3.** IF  $\text{SUPP}(Q) \subset B(p, r^*/2)$ , THERE EXISTS A UNIQUE INTRINSIC MEAN  $\mu_I$  OF  $Q$  ON THE METRIC SPACE  $\overline{B(p, r^*/2)}$ . IF  $\text{SUPP}(Q) \subset B(p, r^*/4)$ , THEN  $\mu_I$  IS THE INTRINSIC MEAN OF  $Q$  ON  $M$ .  $\square$   
(KARCHER, 1977; KENDALL, 1990)

- PROPOSITION 4.** (a) ASSUME  $\text{SUPP}(Q) \subset B(p, r^*/2)$ ,  $Q(\text{CUT}(p)) = 0$ . THEN  $\text{Log}_{\mu_I}(\mu_{n,I})$  IS ASYMP. GAUSSIAN  $N(0, \Gamma/n)$ .

(b) SUPPOSE (i)  $Q$  IS ABS. CONT., (ii) THE INTRINSIC MEAN  $\mu_I$  EXISTS, (iii)  $F$  IS TWICE CONT. DIFF. IN A NBD. OF  $\mu_I$ . THEN  $\text{Log}_{\mu_I}(\mu_{n,I})$  IS ASYMP. GAUSSIAN  $N(0, \Gamma/n)$ .  $\square$
- IN NORMAL COORDINATES (UNDER  $\text{Log}_{\mu_I}$ ), WITH THE IMAGE  $Q_L$  OF  $Q$ , ONE HAS

$$\int_{T_{\mu_I}M} v Q_L(dv) = 0, \quad \Gamma = \Lambda^{-1} \Sigma \Lambda^{-1},$$

$$[\Sigma = \text{COV}(Q_L), \quad \Lambda = (\text{HESSIAN OF } F \text{ AT } \mu_I) \geq K] \quad (*)$$

$$K_{ij} = \int [2 \left( \frac{1 - f|v|}{|v|^2} \right) v^i v^j + f(|v|) \delta_{ij}] Q_L(dv);$$



$$f(r) = \begin{cases} 1 & \text{if } C = 0 \\ \sqrt{C}r \frac{\cos(\sqrt{C}r)}{\sin(\sqrt{C}r)} & \text{if } C > 0 \\ \sqrt{-C}r \frac{\cosh(\sqrt{-C}r)}{\sinh(\sqrt{-C}r)} & \text{if } C < 0 \end{cases}$$

- THERE IS EQUALITY IN (\*) IF  $M$  HAS CONSTANT SECTIONAL CURVATURE.

## 2. SHAPE SPACES M OF $k$ -ADS

- EACH OBSERVATION  $\mathbf{x} = (x_1, \dots, x_k)$  OF  $k > m$  POINTS IN  $m$ -DIMENSION (NOT ALL THE SAME) - $k$  LOCATIONS ON AN  $m$ -DIM. OBJECT.  $k$ -ADS ARE EQUIVALENT MOD  $G$ : A GROUP  $G$  OF TRANSFORMATIONS.
- (a).  $\Sigma(m, k)$ . [KENDALL, 1984, KENT, LE]
- $G$  IS GENERATED BY TRANSLATIONS, SCALING (TO UNIT SIZE), ROTATIONS.
- PRESHAPE

$$\mathbf{u} = (x_1 - \langle x \rangle, \dots, x_k - \langle x \rangle) / \|\mathbf{x} - \langle \mathbf{x} \rangle\|$$

- SHAPE OF  $k$ -AD  $\sigma(x) \in S^{m(k-1)-1} / SO(m) = \Sigma(m, k)$ .

- CASE  $m = 2$ . PLANAR SHAPES.  $M = \Sigma(2, k)$ .
- $M \simeq S^{2k-3}/S^1 \simeq \mathbb{C}P^{k-2}$  (COMPLEX PROJ. SPACE)
- **EXTRINSIC MEAN**  $\mu_E$ : EMBEDDING:

$$J : \sigma(x) \mapsto \mathbf{u}\mathbf{u}^* \in S_0(k, \mathbb{C})$$

[EQUIVARIANT]

- **PROPOSITION 5.**  $\mu_E$  EXISTS IFF THE LARGEST EIGENVALUE OF  $E(UU^*)$  IS SIMPLE. [ $J(\mu_E) = mm^*$ ,  $m$  UNIT EIGENVEC.]
- **INTRINSIC MEAN**  $\mu_I$ .
- CASE  $m > 2$ .  $\Sigma(m, k)$  HAS SINGULARITIES. ACTION OF  $SO(m)$  IS NOT FREE ON  $M$ .

## (b) REFLECTION-SHAPE SPACE $R\Sigma(m, k)$ .

- ASSUME AFFINE SPAN OF EACH  $k$ -AD  $\mathbf{x}$  IS  $\mathbb{R}^m$ , WITH PRESHAPE  $\mathbf{u} = (u_1, \dots, u_k) \in \mathcal{S}^{m(k-1)-1}$ .
- SHAPE  $\sigma(x) \in \mathcal{S}^{m(k-1)-1} / O(m) = M$ .  
EMBEDDING

$$J : \sigma(x) \mapsto ((u_i \cdot u_j)) (M \rightarrow S_{0+}(k, R))$$

- **PROPOSITION 6.** LET  $\lambda_1 \geq \dots \geq \lambda_k$  BE EIGENVALUES OF  $E((U_i \cdot U_j))$ , WITH EIGEN- VECTORS  $v_1, \dots, v_k$ , WHERE  $|v_j|^2 = \lambda_j / (\lambda_1 + \dots + \lambda_m)$  ( $j = 1, \dots, m$ ). (i)  $\mu_E$  EXISTS IFF  $\lambda_m > \lambda_{m+1}$ , AND THEN (ii)  
 $J(\mu_E) = (v_1, \dots, v_m)(v_1, \dots, v_m)^t$ .

## (c) AFFINE SHAPE SPACE $A\Sigma(m, k)$ , $k > m + 1$ .

- $C(k, m) = k$ -ADS  $u = x - \langle x \rangle$  WITH SPAN  $\mathbb{R}^m$
- SHAPE

$$\sigma(x) = \{Au = (Au_1, \dots, Au_k) : A \in GL(m, \mathbb{R})\}.$$

- $A\Sigma(m, k) = C(k, m)/GL(m, \mathbb{R})$ .
- THE  $m$  ROWS OF  $u$  LIE IN A HYPERPLANE  $H$  OF  $\mathbb{R}^k$ ,  $H = \mathbf{1}^\perp \simeq \mathbb{R}^{k-1}$ , AND SPAN A SUBSPACE  $L$  OF DIM.  $m$  OF  $H$ . NOW  $Au = v$  FOR SOME  $A$  IFF THE ROWS OF  $v$  SPAN  $L$ . HENCE  $\sigma(x) \simeq L$ .
- THUS  $A\Sigma(m, k) \simeq$  GRASSMANNIAN  $G_m(k-1)$ .  
EMBEDDING  $J : A\Sigma(m, k) \rightarrow S_+(k-1, \mathbb{R})$ .

- $\sigma(x) \rightarrow$  MATRIX OF PROJ.  $P_L : H \mapsto L = B^t B$ .
- HERE  $L$  IS SPANNED BY ORTHONORMAL  $w_1, \dots, w_m$ ;  
 $w_i = \sum b_{ij} f_j$  ( $\{f_1, \dots, f_{k-1}\}$  ORTH. BASIS OF  $H$ );  
 $B = ((b_{ij}))_{1 \leq i \leq m, 1 \leq j \leq k-1}$ .
- **PROPOSITION 7.** (i)  $\mu_E$  EXISTS IFF AMONG EIGENVALUES  $\lambda_1 \geq \dots \geq \lambda_{k-1}$  OF THE MEAN OF  $B^t B$ ,  $\lambda_m > \lambda_{m+1}$ , AND (ii) THEN  $\mu_E$  IS THE SUBSPACE OF  $H$  SPANNED BY THE FIRST  $m$  EIGENVECTORS.  
 [CHIKUSE, BP]

# (d) PROJECTIVE SHAPE SPACES $P\Sigma(m, k)$ , $k > m + 1$ .

- $k$ -AD  $\mathbf{x} = (x_1, \dots, x_k) \in (R^{m+1})^k$ ; FOR  $x \in R^{m+1}$ ,  $x \neq 0$ ,  
DEFINE

$$[x] = \{\lambda x : \lambda \neq 0\} \text{ (THE LINE THROUGH } 0 \text{ \& } y \text{).}$$



$$RP^m = \{[x] : x \in R^{m+1} \setminus \{0\}\}.$$

- FOR  $A \in GL(m+1, R)$ , A PROJ. LINEAR  
TRANSFORMATION  $\alpha$  ON  $RP^m$  IS

$$\alpha[x] = [Ax]; \alpha \in PGL(m) \text{ [GROUP].}$$

- PROJ. SHAPE OF A  $k$ -AD  $x$  IS

$$\sigma(x) = \{(\alpha[x_1], \dots, \alpha[x_k]) : \alpha \in PGL(m)\}.$$

- A  $k$ -AD  $\{y_1, \dots, y_k\}$  OF POINTS IN  $RP^m$  IS IN *GENERAL POSITION* IF THE LINEAR SPAN OF  $\{y_1, \dots, y_k\}$  IS  $RP^m$ .
- THE SPACE OF PROJ. SHAPES OF  $k$ -ADS IN GENERAL POSITION IS  $P\Sigma(m, k)$ .
- A PROJ. FRAME IN  $RP^m$  IS AN ORDERED SYSTEM OF  $m + 2$  POINTS IN GENERAL POSITION.
- LET  $I$  BE AN ORDERED SET OF INDICES  $i_1 < i_2 < \dots < i_{m+2} \leq k$ . LET  $P_I\Sigma(m, k)$  BE THE SET OF PROJ. SHAPES OF  $k$ -ADS  $x$  FOR WHICH  $\{[x_{i_1}], [x_{i_2}], \dots, [x_{i_{m+2}}]\}$  IS A PROJ. FRAME.
- GIVEN TWO PROJ. FRAMES  $(p_1, \dots, p_{m+2})$ ,  $(q_1, \dots, q_{m+2})$ , THERE EXISTS A UNIQUE  $\alpha \in PGL(m)$  SUCH THAT  $\alpha(p_j) = q_j$  FOR ALL  $j$ .

- BY ORDERING THE POINTS IN A  $k$ -AD SUCH THAT THE FIRST  $m + 2$  POINTS ARE IN GENERAL POSITION, ONE MAY BRING THIS ORDERED SET, SAY,  $(p_1, \dots, p_{m+2})$  TO THE STANDARD FORM  $\{[e_1], \dots, [e_{m+1}], [e_1 + \dots + e_{m+1}]\}$  BY A UNIQUE  $\alpha$  IN  $PGL(m)$ .
- THUS MODULO  $PGL(m)$ , PROJECTIVE SHAPES IN  $P_I\Sigma(m, k)$  ARE DISTINGUISHED ONLY BY THE REMAINING  $k - m - 2$   $RP^m$ -VALUED COORDINATES.
- THUS  $P_I\Sigma(m, k) \simeq (RP^m)^{k-m-2}$ .
- ONE MAY NOW USE THE V-W EMBEDDING FOR EXTRINSIC ANALYSIS.

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