SINGULAR PERTURBATIONS IN ERGODIC CONTROL OF DIFFUSIONS

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The control problem

$\epsilon > 0$. Consider

$$\begin{aligned} dz^\epsilon(t) &= h(z^\epsilon(t), x^\epsilon(t), u(t))dt + \gamma(z^\epsilon(t))dB(t), \\ dx^\epsilon(t) &= \frac{1}{\epsilon} m(z^\epsilon(t), x^\epsilon(t), u(t))dt \\ &\quad + \frac{1}{\sqrt{\epsilon}} \sigma(z^\epsilon(t), x^\epsilon(t))dW(t). \end{aligned}$$

Here:

- For a prescribed compact metric action space A,
  $h : \mathcal{R}^d \times \mathcal{R}^s \times A \to \mathcal{R}^d$, $\gamma : \mathcal{R}^d \to \mathcal{R}^{d\times d}$,
  $m : \mathcal{R}^d \times \mathcal{R}^s \times A \to \mathcal{R}^s$, $\sigma : \mathcal{R}^d \times \mathcal{R}^s \to \mathcal{R}^{s\times s}$,
  are Lipschitz in the first and second (if any) arguments uniformly w.r.t. the third (if any),

- The least eigenvalues of $\gamma(z)\gamma(z)^T, \sigma(z,x)\sigma(z,x)^T$ are uniformly bounded away from zero
(non-degeneracy assumption).
- The initial values are fixed: $(z^\epsilon(0), x^\epsilon(0)) = (z_0, x_0)$,
- $B(\cdot), W(\cdot)$ are resp. $d-$ and $s-$dimensional independent standard Brownian motions,

- $u(\cdot)$ is an $A-$valued control process with measurable paths satisfying the *non-anticipativity* condition:

  for $t \geq s$, $(B(t) - B(s), W(t) - W(s))$ is independent of $\mathcal{F}_s \stackrel{def}{=}$ the completion of

$$\cap_{s'>s} \sigma(z^\epsilon(y), x^\epsilon(y), u(y), y \leq s').$$

  We call such $u(\cdot)$ an *admissible control.*

Ergodic control problem:

Minimize over all admissible $u(\cdot)$ the 'ergodic cost'

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t E[k(z^\epsilon(s), x^\epsilon(s), u(s))]ds. \tag{1}$$

Here $k : \mathcal{R}^d \times \mathcal{R}^s \times A \to \mathcal{R}^+$ is a continuous map satisfying

$$\lim_{||(z,x)||\to\infty} \inf_u k(z, x, u) = \infty.$$

Assume:

- $\forall\ \epsilon \in (0,1)$, the cost for at least one admissible $u(\cdot)$ is $\leq M < \infty$.
- *Weak formulation*
- *Relaxed control:* $A = \mathcal{P}(A')$ and all functions $f(\cdots, u(t))$ are of the form $\int f'(\cdots, y)u(t, dy)$
- *Stochastic Liapunov condition:* Define

$$\mathcal{L} : C^2(\mathcal{R}^s) \to C_b(\mathcal{R}^d \times \mathcal{R}^s \times A')$$

by

$$\begin{aligned} \mathcal{L}f(z,x,u) &\stackrel{def}{=} \frac{1}{2} tr\left(\sigma(z,x)\sigma^T(z,x)\nabla^2 f(x)\right) \\ &\quad + \langle \nabla f(x), m'(z,x,u)\rangle \end{aligned}$$

$\forall\ f \in C^2(\mathcal{R}^s)$.

Then there exists a $V \in C^2(\mathcal{R}^s)$, $g \in C(\mathcal{R}^d \times \mathcal{R}^s)$, such that

$$\lim_{||x|| \to \infty} V(x) = \infty, \quad \lim_{||x|| \to \infty} g(z, x) = \infty,$$

uniformly in z belonging to any compact subset of $\mathcal{R}^d$, and

$$\mathcal{L}V(z, x, u) \leq -g(z, x). \tag{2}$$

Ergodic control: review

- Markov control: a $u(\cdot)$ of the form

$$u(t) = v(z^\epsilon(t), x^\epsilon(t))\ \forall t$$

  for a measurable $v : \mathcal{R}^d \times \mathcal{R}^s \rightarrow A$.

  'Stable' Markov control if the resulting Markov diffusion is positive recurrent.

- $\zeta_v(dzdx) \stackrel{def}{=}$ the unique invariant probability measure. Then 'cost' $= \int \int k'(z, x, u)v(du|z, x)\zeta_v(dzdx)$.

- *ergodic occupation measure:*

$$\Phi_v(dzdxdu) \stackrel{def}{=} \zeta_v(dzdx)v(du|z, x).$$

- $\mathcal{G} \stackrel{def}{=}$ the set of all ergodic occupation measures Φ_v as v varies over all stable Markov controls.

- Define the empirical measures $\nu_t, t > 0$, and the average empirical measures $\bar{\nu}_t, t > 0$, by

$$\begin{aligned}\int f d\nu_t &\overset{def}{=} \frac{1}{t}\int_0^t f(z^\epsilon(s), x^\epsilon(s), u(s))ds,\\ \int f d\bar{\nu}_t &\overset{def}{=} \frac{1}{t}\int_0^t E[f(z^\epsilon(s), x^\epsilon(s), u(s))]ds,\end{aligned}$$

  for $f \in C_b(\mathcal{R}^{d+s} \times A')$.

- $\mathcal{R}^* \overset{def}{=}$ denote the one point compactification of $\mathcal{R}^d \times \mathcal{R}^s$ with '∞' the point at infinity.

$$
\begin{aligned}
\hat{\mathcal{L}} f(z,x,u) &\stackrel{def}{=} \\
& \frac{1}{2} tr\left(\gamma(z)\gamma^T(z)\nabla_z^2 f(z,x)\right) + \langle \nabla_z f(z,x), h'(z,x,u)\rangle \\
+ \quad & \frac{1}{2\epsilon} tr\left(\sigma(z,x)\sigma^T(z,x)\nabla_x^2 f(z,x)\right) \\
& + \frac{1}{\epsilon}\langle \nabla_x f(z,x), m'(z,x,u)\rangle.
\end{aligned}
$$

Lemma

$$
\mathcal{G} = \{\Phi \in \mathcal{P}(\mathcal{R}^{d+s} \times A') : \int \hat{\mathcal{L}} f d\Phi = 0 \ \forall f \in C_0^2(\mathcal{R}^{d+s})\}.
$$

Define

$$\mathcal{G}^* \stackrel{def}{=} \{\Phi \in \mathcal{P}(R^* \times A') : \text{there exist some } 0 \leq a \leq 1,$$

$\phi \in \mathcal{G}$ and $\phi' \in \mathcal{P}(\{\infty\} \times A')$ such that:

$$\begin{aligned} \Phi(B \times B') &= a\phi((B \times B') \cap (\mathcal{R}^{d+s} \times A')) \\ &\quad +(1-a)\phi'((B \times B') \cap (\{\infty\} \times A')) \end{aligned}$$

$\forall\ B$ Borel in $\mathcal{R}^*, B'$ Borel in$A'\}$.

Lemma As $t \uparrow \infty$, $\bar{\nu}_t \to \mathcal{G}^*$ and $\nu_t \to \mathcal{G}^*$ a.s. in $\mathcal{P}(R^* \times A')$.

Theorem There exists a stable optimal Markov control v_ϵ^* such that if Φ_ϵ^* is the corresponding ergodic occupation measure, then under any admissible $u(\cdot)$,

$$\liminf_{t \uparrow \infty} \frac{1}{t} \int_0^t k(z^\epsilon(s), x^\epsilon(s), u(s))ds \quad \geq \quad \int k' d\Phi_\epsilon^* \quad \text{a.s.},$$

$$\liminf_{t \uparrow \infty} \frac{1}{t} \int_0^t E[k(z^\epsilon(s), x^\epsilon(s), u(s))]ds \quad \geq \quad \int k' d\Phi_\epsilon^*.$$

Remark One can in fact show that the v_ϵ^* can be taken to be *precise*, i.e., $v_\epsilon^*(z, x)$ is a Dirac measure for all z, x. This is because the extreme points of $\mathcal{G}$ correspond to precise controls.

The associated system

Set $\tau = \frac{t}{\epsilon}, x'(\tau) = x^{\epsilon}(\epsilon\tau), z'(\tau) = z^{\epsilon}(\epsilon\tau)$,
$u'(\tau) = u(\epsilon\tau), W'(\tau) = \frac{1}{\sqrt{\epsilon}}W(\epsilon\tau)$.

Then the fast dynamics on the new timescale is:

$$dx'(\tau) = m(z'(\tau), x'(\tau), u'(\tau))d\tau + \sigma(z'(\tau), x'(\tau))dW'(\tau).$$

To this we associate the '*associated system*'

$$dx'(\tau) = m(z', x'(\tau), u'(\tau))d\tau + \sigma(z', x'(\tau))dW'(\tau).$$

Let $D_z \stackrel{def}{=}$

$$\{\mu \in \mathcal{P}(\mathcal{R}^s \times A') : \int \mathcal{L}f(z,x,u)\mu(dxdu) = 0 \ \forall \ f \in C_0^2(\mathcal{R}^s)\}.$$

Lemma $D_z =$ the set of $\mu(dxdu)$ of the form

$$\mu(dxdu) = \eta(dx)v(du|x),$$

where η is the unique stationary distribution for $X(\cdot)$ when $u(\cdot) = v(X(\cdot)) \stackrel{def}{=} v(du|X(\cdot))$. The set valued map $z \rightarrow D_z$ is convex compact valued and continuous. Furthermore, for compact $B \subset \mathcal{R}^d$, $\cup_{z\in B} D_z$ is compact.

The averaged system

Define

$$\bar{h}(z,\mu) \stackrel{def}{=} \int h'(z,x,u)\mu(dxdu),$$
$$\bar{k}(z,\mu) \stackrel{def}{=} \int k'(z,x,u)\mu(dxdu).$$

The *averaged system* is defined by

$$dz(t) = \bar{h}(z(t),\mu(t))dt + \gamma(z(t))dB'(t),$$
$$\mu(t) \in D_{z(t)} \ \forall\ t.$$

The control problem is to minimize over admissible $\mu(\cdot)$,

$$\limsup_{t\uparrow\infty} \frac{1}{t} E[\int_0^t \bar{k}(z(s),\mu(s))]ds.$$

- $\mu(\cdot)$ a Markov control if

$$\mu(t) = q(z(t)) \stackrel{def}{=} q(dxdu|z(t)) \ \forall t.$$

- 'stable Markov control' if the resulting $z(\cdot)$ is positive recurrent $\Rightarrow$ a unique invariant probability distribution $\varphi_q(dz)$, ergodic occupation measure

$$\Gamma(dzdxdu) \stackrel{def}{=} \varphi_q(dz)q(dxdu|z).$$

- $\mathcal{Q} \stackrel{def}{=}$ the set of such Γ.

Define

$$\tilde{\mathcal{L}}f(z,\mu) = \frac{1}{2}tr\left(\gamma(z)\nabla^2 f(z)\right) + \langle \nabla f(z), \bar{h}(z,\mu)\rangle.$$

Then:

Lemma

$$\mathcal{Q} \quad = \quad \{\xi = q(dxdu|z)\phi(dz) \in \mathcal{P}(\mathcal{R}^d \times \mathcal{R}^s \times A') :$$

$$q(\cdot|z) \in D_z \text{ and } \forall z, \forall\ f \in C_0^2(\mathcal{R}^d),$$

$$\int \tilde{\mathcal{L}}f(z, q(dxdu|z))\phi(dz) = 0\ \}.$$

Theorem There exists a stable optimal Markov control q^* for the averaged system such that if $\Gamma^*(dzdxdu) = q^*(dxdu|z)\varphi^*(dz)$ is the corresponding ergodic occupation measure, then for any admissible $\mu(\cdot)$ as above,

$$\begin{aligned}
\liminf_{t\uparrow\infty} \frac{1}{t}\int_0^t \bar{k}(z(s),\mu(s))ds &\geq \int k' d\Gamma^* \text{ a.s.}, \\
\liminf_{t\uparrow\infty} \frac{1}{t}\int_0^t E[\bar{k}(z(s),\mu(s))]ds &\geq \int k' d\Gamma^*.
\end{aligned}$$

$\{\Phi^*_\epsilon, \epsilon \in (0,1)\}$ tight

$\Rightarrow$ Let Φ^*_0 be a limit point thereof in $\mathcal{P}(\mathcal{R}^{d+s} \times A')$.

Theorem $\Phi^*_0 \in \mathcal{Q}$.

Proof Disintegrate Φ^*_0 as

$$\begin{aligned} \Phi^*_0(dzdxdu) &= \varphi(dz)\mu(dxdu|z) \\ &= \varphi(dz)\eta(dx|z)v(du|z,x). \end{aligned}$$

(In particular, $\mu(dxdu|z) = \eta(dx|z)v(du|z,x)$.)

Let $f \in C_0^2(\mathcal{R}^d), g \in C_0^2(\mathcal{R}^s)$. Let $\epsilon \downarrow 0$ in the equation

$$\epsilon \int \hat{\mathcal{L}}(fg) d\Phi_\epsilon^* = 0$$

to obtain

$$\int f(z) \int \mathcal{L}g(z,x,u)\mu(dxdu|z)\varphi(dz) = 0. \tag{3}$$

Then as (3) holds for all $f \in C_0^2(\mathcal{R}^d)$, we conclude that for $\varphi-$a.s. z,

$$\int \mathcal{L}g(z,x,u) d\mu(dxdu|z) = 0,$$

implying that $\mu(dxdu|z) \in D_z$.

Now for $h \in C_0^2(\mathcal{R}^d)$ (i.e., h is a function of $z \in \mathcal{R}^d$ alone), let $\epsilon \downarrow 0$ in

$$\int \hat{\mathcal{L}} h d\Phi_\epsilon^* = 0$$

to obtain

$$\int \hat{\mathcal{L}} h d\Phi_0^* = \int \tilde{\mathcal{L}} h(z, \mu(\cdot|z))\varphi(dz) = 0. \tag{4}$$

(4) implies that φ is the unique stationary distribution under μ for the averaged system. It follows that $\Phi_0^* \in \mathcal{Q}$. □

Corollary $\liminf_{\epsilon\downarrow 0} \int k' d\Phi_\epsilon^* \geq \int \bar{k} d\Gamma^*$.

(This provides a **lower bound**.)

Notation:

- $\mathcal{Q}_{opt} \stackrel{def}{=} \text{Argmin}\{\int k' d\xi : \xi \in \mathcal{Q}\}$.
- Write $q^*(dxdu|z) = v^*(du|z,x)\eta^*(dx|z)$, where v^* is the optimal Markov control.

The affine case

Assume:

- **(*)** A' is a compact subset of $\mathcal{R}^m$ for some $m \geq 1$ and for each z, x, $h'(z, x, \cdot)$, $m'(z, x, \cdot)$ are componentwise affine and $k'(z, x, \cdot)$ is strictly convex.
- **(**)** $||h'(z, x, u)|| = o(k'(z, x, u))$ as $||(z, x)|| \uparrow \infty$ and

$$\sup_u |k'(z, x, u)|^{1+a} \leq Kg(z, x)$$

  for some $K, a > 0$ and g as before.

- **(***)** $v^*(z, x) \stackrel{def}{=} v^*(du|z, x)$ (unique by Lemma below) is a stable Markov control for sufficiently small $\epsilon > 0$ (say, $\epsilon < \epsilon_0$) and the corresponding stationary distributions, denoted $\zeta^\epsilon(dzdx), 0 < \epsilon < \epsilon_0$, are tight.

A stochastic Liapunov condition can be given to ensure this.

Lemma $v^*(du|z, x)$ above is unique and continuous in z, x.

Remark: Note that $v^*(z, x)$ will in fact be Dirac for all z, x.

Recall the measures q^*, Γ^*.

Corollary q^*, Γ^* are unique.

Let

$$\tilde{\Phi}_\epsilon(dzdxdu) \stackrel{def}{=} \zeta^\epsilon(dzdx)v^*(du|z,x), \epsilon \in (0, \epsilon_0),$$

and v^* as above.

Theorem 6.1 $\tilde{\Phi}_\epsilon \to \mathcal{Q}_{opt}$ in $\mathcal{P}(\mathcal{R}^d \times \mathcal{R}^s \times A')$.

Proof It suffices to prove that

$$\lim_{\epsilon \downarrow 0} \int k' d\tilde{\Phi}_\epsilon = \int k' d\Gamma^*. \tag{5}$$

Let $\zeta^\epsilon(dzdx) \to \hat{\zeta}(dzdx) = \hat{\varphi}(dz)\hat{\eta}(dx|z)$ along a subsequence as $\epsilon \downarrow 0$. In view of the continuity of $v^*(du|\cdot,\cdot)$, we may pass to the limit along this subsequence in

$$\epsilon \int \hat{\mathcal{L}} f(z,x,u) v^*(du|z,x) \zeta^\epsilon(dzdx) = 0, \; f \in C_0^2(\mathcal{R}^{d+s}),$$

to obtain

$$\int \mathcal{L} f(z,x,u) v^*(du|z,x) \hat{\zeta}(dzdx) = 0, \; f \in C_0^2(\mathcal{R}^{d+s}).$$

Conclude that $\hat{\eta}(dx|z)$ is in fact the unique stationary distribution for the associated system controlled by $v^*(du|z,x)$ (i.e., $\hat{\eta}(dx|z) = \eta^*(dx|z)$) for $\hat{\varphi}-$a.s. z. The latter qualification may be dropped by choosing an appropriate version.

Recall that $q^*(dxdu|z) = \eta^*(dx|z)v^*(du|z,x)$ for all z. Let $\epsilon \downarrow 0$ in

$$\int \hat{\mathcal{L}} f(z,x,u)v^*(du|z,x)\zeta^\epsilon(dzdx) = 0,$$

for $f \in C_0^2(\mathcal{R}^d)$ (i.e., f is a C^2 function of the z variable alone). An argument similar to the above then yields

$$\int \tilde{\mathcal{L}} f(z, q^*(\cdot|z))\hat{\varphi}(dz) = 0,\ f \in C_0^2(\mathcal{R}^d).$$

Thus $\hat{\varphi}(dz)$ is the unique stationary distribution for the averaged system controlled by the stable Markov control q^*, i.e., $\hat{\varphi} = \varphi^*$. Then

$$v^*(du|z,x)\hat{\zeta}(dzdx) = \Gamma^*(dzdxdu).$$

That is, $\tilde{\Phi}_\epsilon \to \Gamma^*$. It can be shown that $\int g d\Phi$ is uniformly bounded as Φ varies over $\mathcal{Q}$. By the second half of **(**)**, it then follows that k' is uniformly integrable over $\mathcal{Q}$. Hence (5) holds. $\square$

The general case

Define $v_\delta^*(du|z,x), \delta > 0$ small (say, $\delta \in (0, \delta_0]$), by

$$\int f v_\delta^*(du|z,x) \stackrel{def}{=}$$

$$\int\int f v^*(du|z',x')\pi_\delta(z-z',x-x')dz'dx',\ \ f \in C(A'),$$

where $\{\pi_\delta : \mathcal{R}^{d+s} \to \mathcal{R}, \delta \in (0, \delta_0]\}$ are smooth approximations to the Dirac measure, i.e., compactly supported C^∞ probability density functions such that

$$\pi_\delta(z,x)dzdx \to \delta_{(0,0)}$$

in $\mathcal{P}(\mathcal{R}^{d+s})$ as $\delta \downarrow 0$.

In the following,

$$v_0^*(du|z,x) \stackrel{def}{=} v^*(du|z,x)$$

and all quantities with subscript $\delta = 0$ correspond to it. Replace **(***)** by **(A1)**, **(A2)** below:

(A1) $v_\delta^*(z,x) \stackrel{def}{=} v_\delta^*(du|z,x)$ is a stable Markov control for $\delta \in [0,\delta_0], \epsilon \in (0,\epsilon_0)$. Furthermore, there exists a $\hat{g} \in C(\mathcal{R}^{d+s})$ satisfying:

$$\sup_u |k'(z,x,u)|^{1+a} \leq K\hat{g}(z,x),$$

such that the stationary distributions corresponding to $\{v_\delta^*\}$, denoted by $\zeta_\delta^\epsilon(dzdx), 0 < \epsilon < \epsilon_0$, satisfy: $\forall\ \delta \in [0,\delta_0]$,

$$\sup_{0<\epsilon<\epsilon_0} \int \hat{g}(z,x)\zeta_\delta^\epsilon(dzdx) < \infty.$$

By our non-degeneracy assumption, the transition probabilities for $t > 0$ of the time-homogeneous Markov process described under Markov control $v^*_\delta, \delta \in [0, \delta_0]$, have densities w.r.t. the Lebesgue measure. Therefore so do the corresponding invariant probability measures $\hat{\eta}_\delta(dx|z)$. Let $\chi_\delta(x|z)$ denote this density.

Let $\bar{\mu}_\delta(dxdu|z) \stackrel{def}{=} \hat{\eta}_\delta(dx|z)v^*_\delta(du|z,x)$ and $\bar{\varphi}_\delta$ the unique stationary distribution under the Markov control $\bar{\mu}_\delta$.

Let

$$\begin{aligned} \zeta^0_\delta(dzdx) &\stackrel{def}{=} \hat{\eta}_\delta(dx|z)\bar{\varphi}_\delta(dz), \\ \Phi^0_\delta(dzdxdu) &\stackrel{def}{=} \zeta^0_\delta(dzdx)v^*_\delta(du|z,x), \end{aligned}$$

for δ as above. Note that $\Phi^0_0 \in \mathcal{Q}_{opt}$.

We also assume:

(A2) $\bar{\mu}_\delta(dxdu|z)$ is a stable Markov control for $\delta \in [0, \delta_0]$, and for $\hat{g}$ as above,

$$\sup_{\delta \in [0,\delta_0]} \int \hat{g}(z,x)\zeta_\delta^0(dzdx) < \infty.$$

This implies in particular that $\zeta_\delta^0, \delta \in [0, \delta_0]$, and therefore $\bar{\varphi}_\delta, \delta \in [0, \delta_0]$, form tight sets.

Lemma As $(\delta_n, z_n) \to (\delta, z)$ in $[0, \delta^*] \times \mathcal{R}^d$,

$$\hat{\eta}_{\delta_n}(dx|z_n) \to \hat{\eta}_\delta(dx|z)$$

in total variation.

Proof This follows by an argument based on Harnack inequality, using the fact that $\chi_\delta(\cdot|z)$ will be equicontinuous pointwise bounded. $\square$

Lemma $\int k' d\Phi^0_\delta \to \int k' d\Phi^0_0$ as $\delta \downarrow 0$.

Proof Can show that $\bar{\varphi}_\delta, \delta \in [0, \delta_0]$, are tight. Let $\bar{\varphi}$ be any limit point of $\bar{\varphi}_\delta$ as $\delta \downarrow 0$. Since $\bar{\varphi}_\delta$ is characterized by

$$\int \tilde{\mathcal{L}} f(z, \bar{\mu}_\delta(\cdot|z))\bar{\varphi}_\delta(dz) = 0,\ \ f \in C^2(\mathcal{R}^s), \tag{6}$$

an argument based on the Harnack inequality analogous implies that this convergence is in fact in total variation. Now for

$f \in C_b(\mathcal{R}^d \times \mathcal{R}^s \times A')$,

$$\int f(z,x,u)v_\delta^*(du|z,x) \to \int f(z,x,u)v_0^*(du|z,x) \quad \text{a.e.}$$

Hence,

$$\int\int f(z,x,u)v_\delta^*(du|z,x)\hat{\eta}_\delta(dx|z) \to$$

$$\int f(z,x,u)v_0^*(du|z,x)\hat{\eta}_0(dx|z)$$

a.e., which in turn leads to:

$$\int\int\int f(z,x,u)v_\delta^*(du|z,x)\hat{\eta}_\delta(dx|z)\bar{\varphi}_\delta(dz)$$

$$\to \quad \int\int\int f(z,x,u)v_0^*(du|z,x)\hat{\eta}_0(dx|z)\bar{\varphi}_0(dz).$$

In particular, letting $\delta \downarrow 0$ along an appropriate subsequence in (6), we have

$$\int \tilde{\mathcal{L}} f(z, \bar{\mu}_0(\cdot|z))\bar{\varphi}(dz) = 0, \;\; f \in C^2(\mathcal{R}^s), \tag{7}$$

i.e., $\bar{\varphi} = \bar{\varphi}_0$. Thus $\Phi^0_\delta = \mu_\delta(dxdu|z)\bar{\varphi}_\delta(dz) \to \Phi^0_0 = \mu_0(dxdu|z)\bar{\varphi}_0(dz)$ as $\delta \downarrow 0$. Uniform integrability of k' under these implies the claim. $\Box$

Let $u(\cdot) = v_\delta^*(z^\epsilon(\cdot), x^\epsilon(\cdot))$ and $\Phi_\delta^\epsilon \in \mathcal{P}(\mathcal{R}^d \times \mathcal{R}^s \times A')$ the corresponding ergodic occupation measure for $\delta > 0$.

Lemma $\int k' d\Phi_\delta^\epsilon \to \int k' d\Phi_\delta^0$ as $\epsilon \downarrow 0$.

Theorem $\lim_{\epsilon \downarrow 0} \int k' d\Phi_\epsilon^* = \int k' d\Phi_0^*$.

Proof Fix $\alpha > 0$ and take $\delta > 0$ small enough such that

$$|\int k' d\Phi_\delta^0 - \int k' d\Phi_0^0| < \frac{\alpha}{2}.$$

Then pick $\epsilon > 0$ small enough so that

$$|\int k' d\Phi_\delta^\epsilon - \int k' d\Phi_\delta^0| < \frac{\alpha}{2}.$$

Thus

$$\begin{aligned}
\limsup_{\epsilon \downarrow 0} \int k' d\Phi_\epsilon^* &\leq \limsup_{\epsilon \downarrow 0} \int k' d\Phi_\delta^\epsilon \\
&\leq \int k' d\Phi_0^0 + \alpha.
\end{aligned}$$

Since $\alpha > 0$ is arbitrary, the claim follows. $\square$

(Some relaxation of hypotheses possible.)

- ‘Stable case’ can be handled similarly.
- need to allow dependence of σ on ‘u’, of γ on ‘x, u’.
- degenerate case?