Spectra of Random Linear Combination of Representations

at Positive Root System of Classical Coxeter Groups

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Representation of a Finite Group

- Let G be a finite group. A finite dimensional representation of G is a homomorphism ρ from G to $\mathrm{GL}_n(\mathbb{C})$ for some $0 < n < \infty$. n is the dimension of the representation.
- ρ is called unitary if $\rho(g)$ is unitary for all $g \in G$.
- ρ is called irreducible if $\mathbb{C}^n$ and $\{0\}$ are the only two invariant subspace under all the operators $\rho(g), g \in G$.
- Two representations ρ and ρ' are equivalent if there exists an invertible matrix A such that $\rho(g) = A\rho'(g)A^{-1}$ for all $g \in G$.

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- Fact: Any unitary representation of G can be decomposed into a direct sum of irreducible representations.
- Theorem (Peter-Weyl): Any continuous function on G is a uniform limit of finite linear combinations of coefficients of irreducible representations.
- Theorem (Schur): Coefficients of finite dimensional unitary irreducible representations form an orthogonal basis of the Hilbert space $\mathbb{L}^2(G)$.
- Unitary irreducible representations of the Permutation group $\mathcal{S}_n$ are indexed by partitions of n.

What are we trying to prove?

- Let $S = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$ be a set of involutions. *i.e.* $\sigma_i^2 = e$ for all i.
- Let ρ be a finite dimensional unitary representation of G with dimension d and character $\chi(\cdot) = \text{Trace}(\rho(\cdot))$.
- The matrices $\rho(\sigma_i)$ are self-adjoint and have real eigenvalues.
- Suppose $a_1, a_2, \dots, a_n$ are real numbers with $\sum a_i^2 = 1$ and $Z_1, Z_2, \dots, Z_n$ are i.i.d. N$(0, 1)$.
- Let Ξ be the random probability distribution where we put mass d^{-1} at each of the d eigenvalues of $A = \sum_{i=1}^{n} a_i Z_i \rho(\sigma_i)$.
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Known Result

Theorem (Evans'07)

For $n \geq 1$, let λ_n be a partition of n. Let ρ_n be an irreducible unitary representation of $\mathcal{S}_n$ corresponding to λ_n. Let Ξ_n be the spectral distribution of the random matrix

$$\frac{1}{\sqrt{n-1}} \sum_{k=1}^{n-1} Z_k \rho_n((k, k+1))$$

where Z_i's are i.i.d. $N(0,1)$. Suppose $n \to \infty$ and

$$\lim \frac{\chi_n(1,2)}{\dim(\rho_n)} = \theta$$

exists. Then Ξ_n converges to a random probability measure Ξ_∞ that is Gaussian with mean θZ and variance $1 - \theta^2$, where Z is a standard Gaussian random variable.

Why is this result expected?

- Suppose all $\rho(\sigma_i)$ commute among themselves.
- $\{\rho(\sigma_i)\}$'s are simultaneously diagonalizable.
- So the eigenvalues of $\sum_{i=1}^{n} a_i Z_i \rho(\sigma_i)$ are $\sum_{i=1}^{n} a_i Z_i \varepsilon_{ij}$, $j = 1, 2, \ldots, d$ where $\varepsilon_{ij} \in \{+1, -1\}$.
- If all σ_i's belong to same conjugacy class, for all $i = 1, \ldots, n$ we have $d^{-1} \sum_{j=1}^{d} \varepsilon_{ij} = d^{-1} Tr \rho(\sigma_i) = c$.
- Suppose ε_{ij}'s are i.i.d. $\{\pm 1\}$ valued random variable with $\mathbb{E}\, \varepsilon_{ij} = c$.

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Why is this result expected? (contd.)

- Then the spectral distribution is like

$$d^{-1}\sum_{j=1}^{d}\delta\{c\sum a_i Z_i + \sum a_i Z_i(\varepsilon_{ij} - c)\} \approx \mathrm{N}(cZ, 1 - c^2).$$

- In our case "commutativeness" is missing.

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Sparseness Conditions

- Given a set $S = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ of involutions, consider it's "commutativity" graph $H = ([n], E)$ where

$$(i,j) \in E \Leftrightarrow \sigma_i \sigma_j \neq \sigma_j \sigma_i.$$

- Suppose $\{a_i\}$ satisfy

$$\text{Normalization constraint: } \sum_{i=1}^{n} a_i^2 = 1,$$

$$\text{Uniformity Condition: } \sum_{i=1}^{n} a_i^4 = o(1)$$

$$\text{Sparseness Condition: } \sum_{(i,j)\in E} a_i^2 a_j^2 = o(1)$$

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Conditions (contd.)

- If $I = \{i_1, i_2, \dots, i_k\}$ is an independent set of vertices in H, then asymptotically $\chi(\prod_{j \in I} \sigma_j)$ is same as $\prod_{j \in I} \chi(\sigma_j)$, which we will prove for positive root system for finite classical Coxeter group.

Little bit of Group Theory

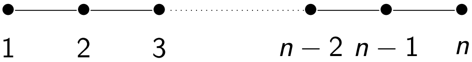
- Coxeter groups are groups of the form

$$W = \langle s \in S | s^2 = e \text{ for } s \in S \text{ and } (st)^{m_{st}} = e \text{ for } s \neq t \rangle.$$

- The Coxeter graph is the graph with vertex set S and (s, t) is an edge iff $m_{st} > 2$ and the edge is marked by m_{st} if $m_{st} > 3$.
- Finite Coxeter groups arise naturally as reflection groups of regular polytopes and Weyl group of simple Lie algebras.
- Building blocks of finite Coxeter groups are 3 classical types A_n, B_n, D_n and 7 exceptional types.

Little bit of Group Theory (contd.)

- Type A_n: (permutation group S_{n+1}) Symmetry group of n-simplex with Coxeter graph



- Type B_n: (signed permutation group W_n) Symmetry group of n-hypercube with Coxeter graph

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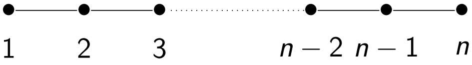

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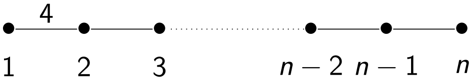

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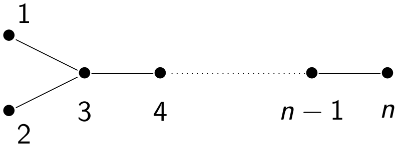
- Type D_n: Symmetry group of n-demihypercube with Coxeter graph



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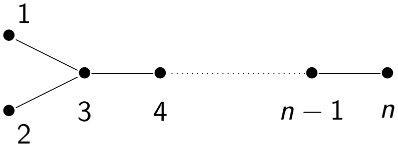

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Main Theorem

Theorem (One Conjugacy Class)

Let G_n be a sequence of Classical Coxeter groups of fixed type and S_n be a subset of the set of reflections corresponding to the root system. Let ρ_n be a irreducible unitary representation of G_n with $d_n = \dim(\rho_n) \to \infty$. Suppose $\{a_s^{(n)} : s \in S\}$ satisfies "sparseness" conditions and $\lim \chi_n(s)/d_n = \theta$ exists for $s \in S$. Then Ξ_n converges to Ξ_∞, where $\Xi_\infty = N(\theta Z, 1 - \theta^2)$ and $Z \sim N(0, 1)$.

Idea of the proof:

- We can take $Z_i = a_i^{-1}(B_{b_i} - B_{b_{i-1}})$, where B is a SBM and $b_i = \sum_{j=1}^{i} a_j^2$.
- Proof by method of moments.
- $$\begin{aligned} \Xi(x^s) = \int x^s d\Xi(x) = \frac{1}{d}\text{Trace}(A^s) \\ = \sum_{i_1,i_2,\ldots,i_s=1}^{N_n} \prod_{j=1}^{s} a_{i_j} Z_{i_j} \cdot \frac{\chi(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_s})}{\chi(e)}. \end{aligned}$$

Idea of the proof (Contd.):

- Given $\mathbf{i} = (i_1, i_2, \ldots, i_s)$ construct the edge labelled graph $G_{\mathbf{i}} = ([s], E_{\mathbf{i}})$ as follows:

  $$(p, q) \in E_{\mathbf{i}} \Leftrightarrow i_p = i_q \text{ or } (i_p, i_q) \in E(H)$$

  and the edge (p, q) is marked zero if $i_p = i_q$ and one otherwise. Let $\hat{G}_{\mathbf{i}}$ be the skeleton of $G_{\mathbf{i}}$ without the edge labels.
- Note that number of $G_{\mathbf{i}}$'s is finite independent of n and $|\chi(g)| \le \chi(e)$ for all $g \in G_n$.
- $\Xi(x^s) = \sum_{G} \sum_{\mathbf{i}:G_{\mathbf{i}}=G} \prod_{j=1}^{s} a_{i_j} Z_{i_j} \cdot \frac{\chi(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_s})}{\chi(e)}.$

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Idea of the proof (Contd.):

- The main contribution comes from graphs with all component sizes ≤ 2 and edges labeled by 0. Number of graphs with r 1-components and $\frac{s-r}{2}$ 2-components is $\binom{s}{r}\mathbb{E}[V^{s-r}]$ where $V \sim N(0,1)$.
- $\Xi(x^s) - \sum_{r=0}^{s}\binom{s}{r}\mathbb{E}[V^{s-r}]\sum_{\mathbf{i}:G_{\mathbf{i}}=K_r}\prod_{j=1}^{s} a_{i_j} Z_{i_j}\cdot\frac{\chi(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_s})}{\chi(e)} \to 0$ in L^2 where

$$K_r = ([s], \{(r+1, r+2), (r+3, r+4), \ldots, (s-1, s)\}).$$

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- Using the fact that $\sum a_i^2 Z_i^2 \to 1$ in L^2, it can be shown that for fix r,

$$\sum_{\mathbf{i}: G_{\mathbf{i}} = K_r} \prod_{j=1}^{s} a_{i_j} Z_{i_j} \cdot \frac{\chi(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_s})}{\chi(e)} - r! \sum_{i_1 < i_2 < \cdots < i_r} \prod_{j=1}^{r} a_{i_j} Z_{i_j} \cdot \frac{\chi(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r})}{\chi(e)} \to 0 \text{ in } L^2.$$

- If we can prove $\chi(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r})/\chi(e) = \kappa_{r,n}$ depends only on r, n then we need limit of $\sum_{i_1<i_2<\cdots<i_r} \prod_{j=1}^{r} a_{i_j} Z_{i_j}$.

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Idea of the proof (Contd.):

- Now $\sum_{i_1 < i_2 < \cdots < i_r} \prod_{j=1}^{r} a_{i_j} Z_{i_j}$ is nothing but the multiple-Ito integral
$$\int_0^1 \cdots \int_0^{t_1} f_n(t_1, \ldots, t_r) dB_{t_1} \ldots dB_{t_r}$$
of the function
$$f_n(t_1, \ldots, t_r) = \sum_{i_1 < i_2 < \ldots < i_r} \prod_{j=1}^{r} \mathbf{1}_{(b_{i_j - 1}, b_{i_j}]}(t_j).$$

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  $$H_r = \frac{(-1)^r}{r!} \exp(\frac{x^2}{2}) \frac{d^r}{dx^r} \exp(-\frac{x^2}{2})$$

  is the r-th Hermite polynomial.
- If we assume $\kappa_{1,n} \to \theta$ then using subsequence argument and $\limsup \kappa_{2,n} \leq \theta^2$, we have $\kappa_{r,n} \to \theta^r$.
- Combining everything we have the limiting distribution is $N(\theta B_1, 1 - \theta^2)$.

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- If we assume $\kappa_{1,n} \to \theta$ then using subsequence argument and $\limsup \kappa_{2,n} \leq \theta^2$, we have $\kappa_{r,n} \to \theta^r$.
- Combining everything we have the limiting distribution is $N(\theta B_1, 1 - \theta^2)$.

Idea of the proof (Contd.):

- Now $f_n \to \mathbf{1}\{t_1 < t_2 < \cdots < t_r\}$ in L^2. Hence the above sum converges in L^2 to $\int_0^1 \cdots \int_0^{t_1} dB_{t_1} \ldots dB_{t_r} = H_r(B_1)$ where

$$H_r = \frac{(-1)^r}{r!} \exp(\frac{x^2}{2}) \frac{d^r}{dx^r} \exp(-\frac{x^2}{2})$$

  is the r-th Hermite polynomial.
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- Combining everything we have the limiting distribution is $N(\theta B_1, 1 - \theta^2)$.

When $G = W_n$, for the set S consisting of all roots of the group we have two conjugacy classes one corresponding to the positive permutation and other corrs. to the diagonal matrices. So we have to find the limiting value of the character for all interaction terms, which in this particular case have a nice form.

Thank You