Satisfiability threshold for random 2-SAT model

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What is k-SAT?

- $x_1, x_2, \ldots, x_n$ are n boolean variables.
- Denote the negation of x_i by $\bar{x}_i$. Thus n boolean variables gives $2n$ *literals* $\{x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_n, \bar{x}_n\}$.
- For each i, x_i and $\bar{x}_i$ are called complimentary literals. Call x_i to be positive and $\bar{x}_i$ to be negative. Convention : $\bar{\bar{x}}_i = x_i$
- <u>k-clauses</u> : k literals joined by 'OR'
  e.g. $k = 2$, $(x_1 \vee x_3)$, $(\bar{x}_5 \vee x_8)$ are 2-clauses.
  Will not allow the type : $(x_9 \vee x_9), (x_6 \vee \bar{x}_6), (\bar{x}_7 \vee \bar{x}_7)$.
- <u>k-SAT formula</u> : Conjunction ('AND') of several k-clauses.
  e.g. $F = (x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_3) \wedge (\bar{x}_2 \vee \bar{x}_3)$
  Two parameters : m = number of clauses, n = number of variables.

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SAT assignment

- For a k-SAT-formula F on n variables , $F = 1$ leads to a simultaneous system of equations on $\mathbb{Z}_2^n$.
- SAT assignments for F = the solution space of the system of equations $\subseteq \{0,1\}^n$.
- E.g. Let $n = 3$ and $F = (x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_3) \wedge (\bar{x}_2 \vee \bar{x}_3)$. The SAT assignments are $\{(0,1,0),(1,0,0),(1,1,0),(1,0,1)\}$.

- F satisfiable : if at least one SAT assignment
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k-satisfiability problem

- Case I (Cook '71, APT '79) $k = 2$: the problem is in P - linear time algorithm exists.
- Case II (Cook' 71) $k \geq 3$: the problem is NP-complete.
- What about random instances?
- Random k-SAT : Pick m k-clauses uniformly from the set of all $2^k \binom{n}{k}$ many clauses.
- Want to study the probability that such a random instance of k-SAT formula is satisfiable as $m, n \to \infty$ with the clause density $m/n \to \alpha$.

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Satisfiability Threshold: Phase Transition

- Empirical evidence suggests random k-SAT undergoes a phase transition at some value of parameter $\alpha = \alpha_c(k)$:

  If $\alpha < \alpha_c$, then $\mathrm{P}($ a random formula is satisfiable $) \rightarrow 1$ and

  If $\alpha > \alpha_c$, then $\mathrm{P}($ a random formula is satisfiable $) \rightarrow 0$.
- The above observation has been verified for $k = 2$. In this case, $\alpha_c(2) = 1$.
  - Goerdt '92, '96, Chvátal and Reed '92, and Fernandez de la Vega '92
- For $k \geq 3$, existence of limiting threshold is still open. Friedgut '99: $\exists\ \alpha_c(k, n)$ such that phase transition occurs at $\alpha_c(k, n)$ ... but threshold may depends on n.

Scaling window for random 2-SAT phase transition

- Bollobás, Borgs, Chayes, Kim and Wilson (2001)
  - Definition (Scaling window): $W(n,\delta) = (\alpha_-(n,\delta), \alpha^+(n,\delta))$ such that probability of satisfiability $> 1-\delta$ for $m/n < \alpha_-$ and $< \delta$ for $m/n > \alpha^+$.
- $W(n,\delta) = (1 - \Theta(n^{-1/3}), 1 + \Theta(n^{1/3}))$.
- Let $\alpha = (1+\epsilon_n)$. Then there exist constants $\lambda_0, \epsilon_0 > 0$ such that
  - if $\lambda_0 n^{-1/3} \leq \epsilon_n \leq \epsilon_0$,
  then probability of satisfiability $= \exp(-\Theta(n\epsilon_n^3))$.
  - if $-\epsilon_0 \leq \epsilon_n \leq -\lambda_0 n^{-1/3}$,
  then probability of satisfiability $= 1 - \Theta(\frac{1}{n\epsilon_n^3})$.

Generalized random 2-SAT model

- Equivalent model for standard random 2-SAT: Each clause is chosen independently with probability $\alpha/2n$.
  - Expected number of clauses under this model $= \binom{n}{2} 2^2 . \alpha/2n \approx \alpha n$.
- Generalized model : Pick each clause independently with probability $\alpha_i/2n$ if it has i positive literals, $0 \leq i \leq 2$.
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Main Result

- **Theorem**(Mossel and S 2007) If $F \sim$ generalized random 2-SAT$_n(\alpha_0, \alpha_1, \alpha_2)$, then
  - if $\rho > 1$, $\mathrm{P}(F$ is satisfiable$) \to 0$
  - if $\rho < 1$, $\mathrm{P}(F$ is satisfiable$) \to 1$,

  where
  $$\rho = \text{max eigenvalue of } \frac{1}{2}\left[\begin{array}{cc} \alpha_1 & \alpha_0 \\ \alpha_2 & \alpha_1 \end{array}\right].$$
- If $\alpha_0 = \alpha_1 = \alpha_2 = \alpha$, then $\rho = \alpha$ and we get back the sharp threshold for standard random 2-SAT with clause density α.

Implication Digraph: From 2-SAT to Graph

- Given a 2-SAT formula F on n variables $x_1, x_2, \ldots, x_n$, we construct the following graph.
  - $D = (V, E)$ where $V = \{x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_n, \bar{x}_n\}$ and $\bar{u} \rightarrow v, \bar{v} \rightarrow u \in E$ iff $(u \vee v) \in F$.
- Intuition : If F contains a clause $C = (u \vee v)$, then for any SAT assignment the following implications always hold : (i) $\bar{u} = 1 \Rightarrow v = 1$ and (ii) $\bar{v} = 1 \Rightarrow u = 1$.
  - In implication digraph, the directed edges between two literals correspond to these logical implications.

Implication Digraph and satisfiability

- Contradictory cycle: If there exists a literal x such that $x \rightsquigarrow \bar{x}$ and $\bar{x} \rightsquigarrow x$ in implication digraph.
- Fact: 2-SAT formula is unsatisfiable iff its implication digraph contains a contradictory cycle.
- Random 2-SAT $\Longrightarrow$ Random graph *and we are happy !!*

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The Sketch of the proof: If $\rho < 1$

- Strategy (First moment method)
  Suppose $T : \{2\text{-SAT } F\} \to \{0, 1, 2, \dots\}$ such that
  - $T(F) > 0$ if F is unsatisfiable.
  - $\mathbb{E}(T) \to 0$ as $n \to \infty$.
  Then by Markov, $\mathrm{P}(F$ is unsatisfiable $) \leq \mathbb{E}(T) \to 0$.
- What will be suitable T?
- Answer: Hooked Chain.
  Definition : $u \to y_1 \to y_2 \to \cdots \to y_s \to v$ in the implication digraph such that $var(y_1), var(y_2), \dots, var(y_s)$ are all distinct and $u, v \in \{y_1, y_2, \dots, y_s, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_s\}$.

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- Lemma: Contradictory cycle $\Rightarrow$ hooked chain.
- Define $T = \#$ of hooked chains in the digraph.
- Step 1:
$$\mathbb{E}(T) \leq \sum_{s=2}^{n} \frac{C(2s)^2}{n}[T_{s-1}^{+} + T_{s-1}^{-}]$$
where T_{s-1}^{+}(or T_{s-1}^{-}) is the expected number of directed paths of length $(s-1)$ started from x_1 (or $\bar{x}_1$) consisting of strongly distinct literals with $T_0^{+} = T_0^{-} = 1$.

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- Step 2 (Recursion): Write $\mathbf{T}_k = (T_k^+, T_k^-)^T$. Then

$$\mathbf{T}_k \leq \mathbf{M}\mathbf{T}_{k-1}$$

where

$$\mathbf{M} = \frac{1}{2}\begin{bmatrix} \alpha_1 & \alpha_0 \\ \alpha_2 & \alpha_1 \end{bmatrix}.$$

- Combining all these, we get

$$\mathbb{E}(T) = O(1/n).$$

The proof of the theorem : $\rho > 1$

- Prove that for a fixed literal x, there exists a suitable neighborhood $\mathcal{N}_x$ of size $O(\sqrt{n})$ and a variable $x_i \in \mathcal{N}_x$ such that $P(x \rightsquigarrow x_i, \bar{x}_i \text{ in } \mathcal{N}_x) \geq \eta$.
  Note that
  $$x \rightsquigarrow x_i \ \& \ x \rightsquigarrow \bar{x}_i \Rightarrow x \rightsquigarrow x_i \ \& \ x_i \rightsquigarrow \bar{x} \Rightarrow x \rightsquigarrow \bar{x}.$$
- Find a suitable neighborhood $\mathcal{N}_{\bar{x}}$ of $\bar{x}$ of size $O(\sqrt{n})$ and a variable $x_j \in \mathcal{N}_{\bar{x}}$ such that $P(x \rightsquigarrow x_j, \bar{x}_j \text{ in } \mathcal{N}_{\bar{x}}) \geq \eta$ and edges corresponding to $\mathcal{N}_x$ and $\mathcal{N}_{\bar{x}}$ are disjoint.
- Repeat the process $O(\sqrt{n})$ times in the *deleted* graph.

Exploration Process

- $(E_t, A_t, U_t); t \geq 0$
- Let $E_t =$ the set of exposed of literals and and $A_t =$ the set of active literals at time t.
  $U_t := \{var(u) : var(u) \notin var(E_t) \cup var(A_t)\}$ be the set of alive variables at time t.
- Set $E_0 = \emptyset, A_0 = \{x\}$.
- (Transition step) If $A_t \neq \emptyset$ and $\ell \in A_t$ is *exposed* from the stack, then (i) $E_{t+1} = E_t \cup \{\ell\}$
  (ii) $A_{t+1} = (A_t \setminus \{\ell\}) \cup \{u : u \text{ literal s.t. } var(u) \in U_t \text{ and } (\bar{l} \vee u) \in F\}$.
- If $A_t = \emptyset$ is empty, then $A_{t+1} = \emptyset$ and $E_{t+1} = E_t$.
- Property: For every $y \in E_t \cup A_t$, $x \rightsquigarrow y$.

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Stochastic description of Exploration Process

- Let $u_t = |U_t|$, $a_t^+ =$ the number of positive literals in A_t and $a_t^- =$ the number of negative literals in A_t.
- Let $A_t \neq \emptyset$. Given $H(t)$, the history up to t and that the current literal at time t is positive, we have

$$u_t - u_{t+1} \stackrel{d}{=} \mathrm{Bin}(u_t, \alpha_0/2n + \alpha_1/2n - \alpha_0\alpha_1/4n^2)$$

$$a_{t+1}^+ - a_t^+ \stackrel{d}{=} -1 + \mathrm{Bin}(u_t, \alpha_1/2n), \quad a_{t+1}^- - a_t^- \stackrel{d}{=} -1 + \mathrm{Bin}(u_t, \alpha_0/2n).$$

Given $H(t)$ and the current literal at time t is negative,

$$u_t - u_{t+1} \stackrel{d}{=} \mathrm{Bin}(u_t, \alpha_1/2n + \alpha_2/2n - \alpha_1\alpha_2/4n^2)$$

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Proof details

- If '$u_t = n$ for all $t \geq 0$', then (a_t^+, a_t^-) is an exploration of the Galton-Watson tree of a 2-type branching process with branching matrix $\mathbf{M}$.
- Show that $P(|u_0 - u_{\sqrt{n}}| \geq 2\alpha\sqrt{n}) \leq 2\exp(-\frac{\alpha}{2}\sqrt{n})$ where $\alpha = \max(\alpha_0, \alpha_1, \alpha_2)$. Hence, $u_t \approx n$ for all $t \leq \sqrt{n}$.
- We can find a 2-type branching process such that (a_t^+, a_t^-) stochastically dominates of the exploration process on the GW tree of that branching process up to time $\sqrt{n}$ with exponentially high probability.

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A lemma on 2-type supercritical branching process

Lemma. Suppose $X(t) = (X_1(t), X_2(t))$ is the exploration process at time t of a supercritical 2-type branching process with $X(0) = (1, 0)$ or $(0, 1)$. Then $\exists C > 0, \delta > 0$ such that

$$\mathrm{P}(X_1(\sqrt{n}) + X_2(\sqrt{n}) \geq C\sqrt{n}) \geq \delta.$$

Proof. Couple it with a random walk with positive drift.

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Hence ....

- $$P(a^+_{\sqrt{n}} + a^-_{\sqrt{n}} \geq C\sqrt{n}) \geq \delta$$
- If $\exists u$ such that both u and $\bar{u} \in A_{\sqrt{n}}$ we are done.
- Otherwise for each $u, v \in A_{\sqrt{n}}$, check whether $(\bar{u} \vee \bar{v}) \in F$. If so, then $u \rightarrow \bar{v}$ and hence $x \rightsquigarrow v, \bar{v}$.
- Probability $\geq 1 - (1 - \min(\alpha_0, \alpha_1, \alpha_2)/2n)^{\binom{C\sqrt{n}}{2}} \geq p > 0$.

Finishing the proof

- Run the exploration process from $\bar{x}$ on the variables $\{1, 2, \ldots, n\} \setminus var(E_{\sqrt{n}} \cup A_{\sqrt{n}})$.
- Get a contradictory cycle from x with probability $\geq p^2$.
- If not, again delete the variables in $var(E_{\sqrt{n}} \cup A_{\sqrt{n}})$. New graph has still $\approx n$ many variables to start with. Again run a pair of exploration processes to get a contradictory cycle.
- Repeat the process $\Theta(\sqrt{n})$ times and use independence.