

Long Rare Segments and Ruin Probabilities, Large Deviations for Moving Average Processes

Souvik Ghosh

Joint with Gennady Samorodnitsky

Cornell University

December 13, 2007

Moving Average Process

- $\{Z_i, i \in \mathbb{Z}\}$, are \mathbb{R}^d -valued, iid, centered, var-covar matrix Σ .
- $\{Z_i\}$ are light tailed: $\Lambda(\lambda) = \log E(e^{\lambda Z_0}) < \infty$, in a nbhd of 0.
- $X_n = \sum_{i=-\infty}^{\infty} \phi_i Z_{n-i}$, where $\sum_{i=-\infty}^{\infty} \phi_i^2 < \infty$.
- $S_n = \sum_{i=1}^n X_i$.

Moving Average Process

- $\{Z_i, i \in \mathbb{Z}\}$, are \mathbb{R}^d -valued, iid, centered, var-covar matrix Σ .
- $\{Z_i\}$ are light tailed: $\Lambda(\lambda) = \log E(e^{\lambda Z_0}) < \infty$, in a nbhd of 0.
- $X_n = \sum_{i=-\infty}^{\infty} \phi_i Z_{n-i}$, where $\sum_{i=-\infty}^{\infty} \phi_i^2 < \infty$.
- $S_n = \sum_{i=1}^n X_i$.

Memory in a MA Process

- Short Memory: $\sum_{i=-\infty}^{\infty} |\phi_i| < \infty$.
- Long Memory: $\sum_{i=-\infty}^{\infty} |\phi_i| = \infty$.

Long Strange Segments

- $L_n(u) = \sup \left\{ j - i : 0 \leq i < j \leq n, \frac{S_j - S_i}{j - i} > u \right\}$
- Z_i has balanced regular varying tails with index $-\beta, \beta > 1$.

Long Strange Segments

- $L_n(u) = \sup \left\{ j - i : 0 \leq i < j \leq n, \frac{S_j - S_i}{j - i} > u \right\}$
- Z_i has balanced regular varying tails with index $-\beta, \beta > 1$.
- Mansfield et al.(2001): If $\sum_{i=-\infty}^{\infty} |\phi_i| < \infty, \exists \{b_n\} \in RV_{1/\beta}$

$$P(b_n^{-1} L_n(u) \leq x) \rightarrow \exp\{-C_s u^{-\beta} x^{-\beta}\}$$

Long Strange Segments

- $L_n(u) = \sup \left\{ j - i : 0 \leq i < j \leq n, \frac{S_j - S_i}{j - i} > u \right\}$
- Z_i has balanced regular varying tails with index $-\beta, \beta > 1$.
- Mansfield et al.(2001): If $\sum_{i=-\infty}^{\infty} |\phi_i| < \infty, \exists \{b_n\} \in RV_{1/\beta}$

$$P(b_n^{-1}L_n(u) \leq x) \rightarrow \exp\{-C_s u^{-\beta} x^{-\beta}\}$$

- Rachev et al.(2001): If $\{\phi_i\}$ is balanced regular varying with index $-\alpha, 1 \geq \alpha > \max(1/\beta, 1/2), \exists \{b_n\} \in RV_{1/\alpha\beta}$,

$$P(b_n^{-1}L_n(u) \leq x) \rightarrow \exp\{-C_l u^{-\beta} x^{-\alpha\beta}\}$$

Large Deviation Principle

Definition

$\{\mu_n\}$ satisfies LDP with speed b_n if for any set A

$$-\inf_{x \in A^\circ} I_l(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{b_n} \log \mu_n(A) \leq \limsup_{n \rightarrow \infty} \frac{1}{b_n} \log \mu_n(A) \leq -\inf_{x \in \bar{A}} I_u(x)$$

I_l and I_u (nonnegative, lower semi-continuous) are called lower and upper rate functions, respectively.

The History

- Donsker and Varadhan (1985)

If $\{X_i\}$ is a Gaussian process for which the spectral density is continuous, then $n^{-1} \sum_{i \leq n} \delta_{X_i}$ satisfy LDP with speed n .

The History

- Donsker and Varadhan (1985)

If $\{X_i\}$ is a Gaussian process for which the spectral density is continuous, then $n^{-1} \sum_{i \leq n} \delta_{X_i}$ satisfy LDP with speed n .

- Burton and Dehling (1990)

If $X_n = \sum_{i \in \mathbb{Z}} \phi_i Z_{n-i}$, with $\sum_{i \in \mathbb{Z}} |\phi_i| < \infty$, $\sum_{i \in \mathbb{Z}} \phi_i = 1$, and

$$\Lambda(\lambda) := \log E[e^{\lambda Z_0}] < \infty, \text{ for all } \lambda \in \mathbb{R}.$$

Then $n^{-1} S_n$ satisfy LDP with speed n and rate fn. $\Lambda^*(\cdot)$.

Absolute Summable Coefficients

- Jiang, Rao and Wang (1992, 1995)
 - $\Lambda(\lambda) < \infty$ for λ in a neighborhood of 0.
 - Large and moderate deviations.
- Dong, Xi-Li and Yang (2005)
 - $\{Z_i\}$ need not be iid for moderate deviation.

Continuity of Spectral Density

- Djellout and Guillin (2001)
- Wu (2004)
 - Gaussian like tails.

Short Memory

Assumption

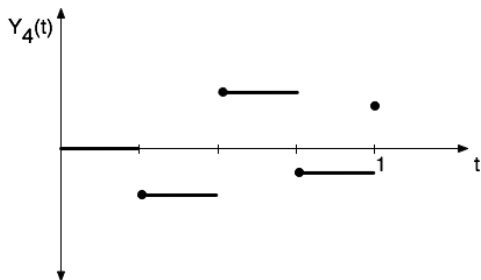
$$\sum_{i \in \mathbb{Z}} |\phi_i| < \infty \text{ and } \sum_{i \in \mathbb{Z}} \phi_i = 1.$$

The Sample Paths

$$Y_n(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} X_i, t \in [0, 1].$$

$$Y_n \in \mathcal{BV}$$

$$\mu_n = \mathcal{L}(Y_n)$$



Functional Large Deviations

Theorem

If $a_n = n$ then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with speed n , with the topology where $f_n \rightarrow f$ iff $f_n \rightarrow f$ pointwise and in $L_p, \forall p \geq 1$.

Functional Large Deviations

Theorem

If $a_n = n$ then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with speed n , with the topology where $f_n \rightarrow f$ iff $f_n \rightarrow f$ pointwise and in $L_p, \forall p \geq 1$.

If $\Lambda(\lambda) < \infty, \forall \lambda \in \mathbb{R}^d$, then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with sup-norm topology, with speed n and good rate function

$$H^{sl}(f) = \begin{cases} \int_0^1 \Lambda^*(f'(x)) dx. & f(0) = 0, f \in \mathcal{AC} \\ \infty & \text{otherwise} \end{cases}$$

Functional Large Deviations

Theorem

If $a_n = n$ then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with speed n , with the topology where $f_n \rightarrow f$ iff $f_n \rightarrow f$ pointwise and in $L_p, \forall p \geq 1$.

If $\Lambda(\lambda) < \infty, \forall \lambda \in \mathbb{R}^d$, then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with sup-norm topology, with speed n and good rate function

$$H^{sl}(f) = \begin{cases} \int_0^1 \Lambda^*(f'(x)) dx. & f(0) = 0, f \in \mathcal{AC} \\ \infty & \text{otherwise} \end{cases}$$

Mogulski(1976)

If $\{X_i\}$ are iid, $a_n = n$ and $\Lambda(\lambda) < \infty, \forall \lambda$, then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with sup-norm topology, speed n and good rate fn $H^{sl}(\cdot)$.

Theorem

If $a_n/\sqrt{n} \rightarrow \infty$ and $a_n/n \rightarrow 0$ then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with sup-norm topology, with speed a_n^2/n and rate function

$$H^{sm}(f) = \begin{cases} \int_0^1 \frac{1}{2} f'(x) \cdot \Sigma^{-1} f'(x) dx & f(0) = 0, f \in \mathcal{AC} \\ \infty & \text{otherwise} \end{cases}$$

Moderate Deviations

Theorem

If $a_n/\sqrt{n} \rightarrow \infty$ and $a_n/n \rightarrow 0$ then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with sup-norm topology, with speed a_n^2/n and rate function

$$H^{sm}(f) = \begin{cases} \int_0^1 \frac{1}{2} f'(x) \cdot \Sigma^{-1} f'(x) dx & f(0) = 0, f \in \mathcal{AC} \\ \infty & \text{otherwise} \end{cases}$$

Corollaries

- 1 $n^{-1}S_n$ satisfies LDP with speed n and rate function $\Lambda^*(\cdot)$.
- 2 If $a_n = o(n)$ and $\sqrt{n} = o(a_n)$ then $a_n^{-1}S_n$ satisfy LDP with speed a_n^2/n and rate function $\frac{1}{2}x \cdot \Sigma^{-1}x$.

Long Memory

- $\{\phi_i\}$ is not absolutely summable.

- $\{\phi_i\}$ is not absolutely summable.
- Assume that $\{\phi_i\}$ is balanced regular varying with index $-\alpha$.
 - There exists $\psi : [0, \infty) \rightarrow [0, \infty)$, regular varying with index $-\alpha$, $1/2 < \alpha \leq 1$.
 - $$\frac{\phi_n}{\psi(n)} \rightarrow p, \frac{\phi_{-n}}{\psi(n)} \rightarrow q = 1 - p, 0 \leq p \leq 1$$
 - $\Psi(n) = \sum_{i=1}^n \psi(i) \in RV_{1-\alpha}$

Functional Large Deviations

Recall

μ_n is the law of $\frac{1}{a_n} \sum_{i=1}^{\lfloor nt \rfloor} X_i$, $t \in [0, 1]$ in \mathcal{BV} .

Theorem

If $a_n = n\Psi(n)$ then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with speed n , with the topology where $f_n \rightarrow f$ iff $f_n \rightarrow f$ pointwise and in L_p , $\forall p \geq 1$.

Functional Large Deviations

Recall

μ_n is the law of $\frac{1}{a_n} \sum_{i=1}^{\lfloor nt \rfloor} X_i$, $t \in [0, 1]$ in \mathcal{BV} .

Theorem

If $a_n = n\Psi(n)$ then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with speed n , with the topology where $f_n \rightarrow f$ iff $f_n \rightarrow f$ pointwise and in L_p , $\forall p \geq 1$.

If $a_n = n\Psi(n)$ and $\Lambda(\lambda) < \infty, \forall \lambda \in \mathbb{R}^d$ then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with sup-norm topology, speed n and a good rate function.

Theorem

If $\frac{a_n}{n\Psi(n)} \rightarrow 0$ and $\frac{a_n}{\sqrt{n\Psi(n)}} \rightarrow \infty$, then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with sup-norm topology, speed $\frac{a_n^2}{n\Psi^2(n)}$ and good rate function H^{rm} .

Theorem

If $\frac{a_n}{n\Psi(n)} \rightarrow 0$ and $\frac{a_n}{\sqrt{n\Psi(n)}} \rightarrow \infty$, then $\{\mu_n\}$ satisfies LDP in \mathcal{BV} with sup-norm topology, speed $\frac{a_n^2}{n\Psi^2(n)}$ and good rate function H^{rm} .

Corollaries

- 1 $n^{-1}S_n$ satisfies LDP in \mathbb{R}^d with good rate function $C_X \cdot \Sigma^{-1}X$ and speed $n/\Psi^2(n) \in RV_{2\alpha-1}$.
- 2 $\frac{1}{n\Psi(n)}S_n$ satisfies LDP in \mathbb{R}^d with speed n .

Summary

μ_n is the law of $\frac{1}{a_n} \sum_{i=1}^{[nt]} X_i$, $t \in [0, 1]$ in \mathcal{BV} .

Short Memory

- If $a_n = n$ then $\{\mu_n\}$ satisfies LDP with speed n .
- If $a_n/n \rightarrow 0$ and $a_n/\sqrt{n} \rightarrow \infty$, then $\{\mu_n\}$ satisfies LDP with speed a_n^2/n .

Long Memory

- $\{\phi_n\} \in RV_{-\alpha}$ and $\Psi \in RV_{1-\alpha}$.
- If $a_n = n\Psi(n)$ then $\{\mu_n\}$ satisfies LDP with speed n .
- If $a_n/n\Psi(n) \rightarrow 0$ and $a_n/\sqrt{n}\Psi(n) \rightarrow \infty$ then $\{\mu_n\}$ satisfies LDP with speed $a_n^2/n\Psi^2(n)$.

Long Strange Segments

Definition

$$T_n(A) = \inf \left\{ l : \exists k, 0 \leq k \leq l - n, \frac{S_l - S_k}{a_{l-k}} \in A \right\}$$

Definition

$$T_n(A) = \inf \left\{ l : \exists k, 0 \leq k \leq l - n, \frac{S_l - S_k}{a_{l-k}} \in A \right\}$$

Theorem

For all the different situations considered in the earlier theorems

$$l_* \leq \liminf_{n \rightarrow \infty} \frac{\log T_n(A)}{b_n} \leq \limsup_{n \rightarrow \infty} \frac{\log T_n(A)}{b_n} \leq l^* - a.s.$$

Long Strange Segments

Definition

$$T_n(A) = \inf \left\{ l : \exists k, 0 \leq k \leq l - n, \frac{S_l - S_k}{a_{l-k}} \in A \right\}$$

Theorem

For all the different situations considered in the earlier theorems

$$l_* \leq \liminf_{n \rightarrow \infty} \frac{\log T_n(A)}{b_n} \leq \limsup_{n \rightarrow \infty} \frac{\log T_n(A)}{b_n} \leq l^* - a.s.$$

where

$$l_* = - \limsup_{n \rightarrow \infty} \frac{1}{b_n} \log P \left(a_n^{-1} S_n \in A \right),$$

$$l^* = - \lim_{\eta \downarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{b_n} \log P \left(a_n^{-1} S_n \in A(\eta) \right).$$

- Consider the classical case when when $a_n = n$.

$$T_n(A) = \inf \left\{ l : \exists k, 0 \leq k \leq l - n, \frac{S_l - S_k}{l - k} \in A \right\}$$

- Consider the classical case when when $a_n = n$.

$$T_n(A) = \inf \left\{ l : \exists k, 0 \leq k \leq l - n, \frac{S_l - S_k}{l - k} \in A \right\}$$

- In presence of short memory $\log T_n(A)$ grows linearly in n .

- Consider the classical case when when $a_n = n$.

$$T_n(A) = \inf \left\{ l : \exists k, 0 \leq k \leq l - n, \frac{S_l - S_k}{l - k} \in A \right\}$$

- In presence of short memory $\log T_n(A)$ grows linearly in n .
- In presence of long memory $\log T_n(A)$ grows like $n/\Psi(n)^2$, which is $RV_{2\alpha-1}$.

- Consider the classical case when when $a_n = n$.

$$T_n(A) = \inf \left\{ l : \exists k, 0 \leq k \leq l - n, \frac{S_l - S_k}{l - k} \in A \right\}$$

- In presence of short memory $\log T_n(A)$ grows linearly in n .
- In presence of long memory $\log T_n(A)$ grows like $n/\Psi(n)^2$, which is $RV_{2\alpha-1}$.
- If the process is long range dependent and $a_n = n\Psi(n)$ then $\log T_n(A)$ grows linearly in n .

Definition

$$R(u) = P\left[\sum_{i \leq n} X_i - n\mu \geq u \text{ for some } n \geq 1\right]$$

Definition

$$R(u) = P \left[\sum_{i \leq n} X_i - n\mu \geq u \text{ for some } n \geq 1 \right]$$

Theorem

- In presence of short memory, there exists $C_s^1, C_s^2 > 0$, so that

$$-C_s^1 \leq \liminf_{u \rightarrow \infty} \frac{1}{u} \log R(u) \leq \limsup_{u \rightarrow \infty} \frac{1}{u} \log R(u) \leq -C_s^2$$

Definition

$$R(u) = P \left[\sum_{i \leq n} X_i - n\mu \geq u \text{ for some } n \geq 1 \right]$$

Theorem

- In presence of short memory, there exists $C_s^1, C_s^2 > 0$, so that

$$-C_s^1 \leq \liminf_{u \rightarrow \infty} \frac{1}{u} \log R(u) \leq \limsup_{u \rightarrow \infty} \frac{1}{u} \log R(u) \leq -C_s^2$$

- In the long memory case, there exists $C_l^1, C_l^2 > 0$, such that

$$-C_l^1 \leq \liminf_{u \rightarrow \infty} \frac{\Psi^2(u)}{u} \log R(u) \leq \limsup_{u \rightarrow \infty} \frac{\Psi^2(u)}{u} \log R(u) \leq -C_l^2.$$

- $S_n = \sum_{i \in \mathbb{Z}} (\phi_{1-i} + \cdots + \phi_{n-i}) Z_i$

$$\frac{1}{b_n} \log E \left[\exp \left(\lambda \frac{b_n}{a_n} S_n \right) \right] = \frac{1}{b_n} \sum_{i \in \mathbb{Z}} \Lambda \left(\lambda \frac{b_n}{a_n} (\phi_{1-i} + \cdots + \phi_{n-i}) \right)$$

- Gartner-Ellis gives us LDP for $(Y_n(t_1), \dots, Y_n(t_k))$.

- $S_n = \sum_{i \in \mathbb{Z}} (\phi_{1-i} + \cdots + \phi_{n-i}) Z_i$

$$\frac{1}{b_n} \log E \left[\exp \left(\lambda \frac{b_n}{a_n} S_n \right) \right] = \frac{1}{b_n} \sum_{i \in \mathbb{Z}} \Lambda \left(\lambda \frac{b_n}{a_n} (\phi_{1-i} + \cdots + \phi_{n-i}) \right)$$

- Gartner-Ellis gives us LDP for $(Y_n(t_1), \dots, Y_n(t_k))$.
- Extend it to LDP for Y_n in \mathcal{BV} using tools for projective limits.
- Improve on the topology by showing exponential tightness.

Thank You!