SELF-INTERACTING RANDOM WALKS ON $\mathbb{Z}$

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Notation: Discrete time setting:

$n \mapsto X(n)$ n.n. RW on $\mathbb{Z}$, with $X(0) = 0$.

Local time *on sites*:

$$\ell(n,k) := \left|\left\{0 \le m \le n : X(m) = k\right\}\right|$$

Local time *on unoriented edges*:

$$\ell_{\pm}(n,k) := \Big|\Big\{0 \le m \le n :$$
$$\{X(m), X(m+1)\} = \{k, k\pm 1\}\Big\}\Big|$$

Local time *on oriented edges*:

$$\widetilde{\ell}_{\pm}(n,k) := \Big|\Big\{0 \le m \le n :$$
$$(X(m), X(m+1)) = (k, k\pm 1)\Big\}\Big|$$

Obvious identities:

$$\begin{aligned}
\ell_{+}(n,k) &= \widetilde{\ell}_{+}(n,k) + \widetilde{\ell}_{-}(n,k+1) \\
&= \ell_{-}(n,k+1), \\
\ell(n,k) &= (\ell_{+}(n,k) + \ell_{-}(n,k))/2
\end{aligned}$$

Notation: Continuous time setting:

$t \mapsto X(t)$ continuous time RW on $\mathbb{Z}$, with n.n. jumps, with $X(0) = 0$.

Local time on sites:

$$\ell(t,k) := \Big|\Big\{0 \le s < t : X(s) = k\Big\}\Big|$$

Notation: Continuous space and continuous time setting:

$t \mapsto X(t)$ continuous time process on $\mathbb{R}$, with $X(0) = 0$.

Local time, or occupation time density:

$$\ell(t,x) := \partial_x \Big|\Big\{0 \le s < t : X(s) < x\Big\}\Big|$$

Example: Reinforced random walk, with edge reinforcement (ERRW)

[D. Coppersmith, P. Diaconis: 1987],
[B. Davis: 1990-1996],
[R. Pemantle: 1988-2007],
[F. Merkl, S. Rolles: 2004-2007],
. . . . . .

$$\mathbf{P}\Big(X(n+1) = X(n) \pm 1 \,\Big|\, \mathcal{F}_n \Big) \sim 1 + \alpha \ell_{\pm}(n, X(n))$$

Facts about:

- Almost surely

$$\lim_{n\to\infty} \frac{\ell(n,k)}{n} = \xi_k \in (0,1), \qquad \sum_{k\in\mathbb{Z}} \xi_k = 1.$$

  The joint distribution of the occupation densities ξ_k is explicitly known.

- In particular: recurrence and tightness of $X(n)$ follows.

Example: Reinforced Random Walk, with vertex reinforcement (VRRW):

[R. Pemantle, S. Volkov: 1999],

[P. Tarrés: 2004],

. . . . . .

$$\mathbf{P}\Big(X(n+1) = X(n) \pm 1 \,\Big|\, \mathcal{F}_n \Big)$$
$$\sim 1 + \alpha \ell(n, X(n) \pm 1)$$

Facts about:

- The *asymptotic range*

$$\mathcal{AR} := \{k \in \mathbb{Z} : \lim_{n\to\infty} \ell(n,k) = \infty\}$$

Almost surely $\exists K \in \mathbb{Z}$ so that

$$\mathcal{AR} = \Big\{K-2, K-1, K, K+1, K+2\Big\}$$

Example: Continuous time reinforced random walk with vertex reinforcement (CTVRRW):

[B. Davis, S. Volkov: 2002-2004],

. . . . . .

$$\mathbf{P}\Big(X(t+dt) = X(t) \pm 1 \,\Big|\, \mathcal{F}_t \Big)$$
$$= \Big(1 + \alpha \ell(t, X(t) \pm 1)\Big) dt + o(dt)$$

Facts about:

- Almost surely

  $$\lim_{t\to\infty} \frac{\ell(t,k)}{t} = \xi_k \in (0,1), \qquad \sum_{k\in\mathbb{Z}} \xi_k = 1.$$

  The joint distribution of the occupation densities ξ_k is explicitly known.

- In particular: recurrence and tightness of $X(t)$ follows.

Example: "True" self-avoiding random walk with vertex repulsion (VTSAW):
[D. Amit, G. Parisi, L. Peliti: 1983],
[L. Peliti, L. Pietronero: 1987],
. . . . . .

$$\mathbf{P}\Big(X(n+1) = X(n) \pm 1 \,\Big|\, \mathcal{F}_n \Big)$$
$$\sim \exp\{-\beta \ell(n, X(n) \pm 1)\}$$
$$\sim \exp\{\beta(\ell(n, X(n)) - \ell(n, X(n) \pm 1))\}$$

Facts about: Only guesses:

- Recurrence (?)

- Nondegenerate scaling of $n^{-2/3}X(n)$ (with no hint about the scaling limit) (?)
  RG argument

- $\dim \geq 3$: diffusive scaling (?)
  RG argument

Example: "True" self-avoiding random walk with edge repulsion (ETSAW):
[B. Tóth: 1995-97],
[B. Tóth, W. Werner: 1997]

$$\mathbf{P}\Big(X(n+1) = X(n) \pm 1 \,\Big|\, \mathcal{F}_n \Big)$$

$$\sim \exp\{-2\beta\ell_\pm(n, X(n))\}$$

$$\sim \exp\{\beta(\ell_\mp(n, X(n)) - \ell_\pm(n, X(n)))\}$$

Facts about:

- Local limit theorem for

$$n^{-2/3} X(\theta_{n/s})$$

where

$$\mathbf{P}\Big(\theta_{n/s} = m \Big) =$$

$$\Big(1 - \exp\{-s/n\}\Big) \exp\{-(sm)/n\}.$$

This means "almost"

$$\lim_{n\to\infty} \mathbf{P}\Big(n^{-2/3} X(n) < x \Big) = \int_{-\infty}^{x} \pi(y)dy,$$

with $\pi(y)$ identified.

- Scaling limit of the process:

$$n^{-2/3} X([nt]) \stackrel{?}{\Rightarrow} \mathcal{X}(t)$$

[B. Tóth, W. Werner: 1997]: Construction and analysis of the "true self-repelling process" $t \mapsto \mathcal{X}(t)$.

$$\begin{aligned} d\mathcal{X}(t) \text{ ``=`` } & -\partial_x \ell(t, \mathcal{X}(t))dt \\ \text{ ``=`` } & \Big\{ \int_{-\infty}^{\infty} \delta'(z)\, \ell(t, X(t) - z) dz \Big\} dt \end{aligned}$$

Continuous space-time models ("Brownian polymers"):

[J.R. Norris, L.C.G. Rogers, D. Williams: 1987],
[R.T. Durrett, L.C.G. Rogers: 1992],
[M. Cranston, T.S. Mountford: 1996],
[T.S. Mountford, P. Tarrés: 2007]
. . . . . .

A function $f:\mathbb{R}\to\mathbb{R}$ with suitable regularity, fixed.

$$X(t)=B(t)+\int_0^t\Big\{\int_0^s f\Big(X(s)-X(u)\Big)du\Big\}ds$$
$$=B(t)+\int_0^t\Big\{\int_{-\infty}^{\infty} f(z)\,\ell(s,X(s)-z)dz\Big\}ds$$
$$dX(t)=dB(t)+\Big\{\int_{-\infty}^{\infty} f(z)\,\ell(t,X(t)-z)dz\Big\}dt$$

Facts about:

- If f is bounded and of compact support, $f \geq 0$, $f \not\equiv 0$, then

$$\lim_{t\to\infty} X(t)/t = c \in (0,\infty) \quad \text{a.s.}$$

- If $f(x) = x/(1+|x|^{\beta+1})$, $\beta \in (0,1)$ $\alpha := 2/(1+\beta)$ then

$$\mathbf{P}\Big(\lim_{t\to\infty} X(t)/t^{\alpha} = \pm c\Big) = 1/2$$

Natural generalizations of the ERRW

[B. Davis: 1992],

[B. Tóth 1994-2000],

. . . . . .

Fixed weight function $w : \mathbb{Z}_+ \to (0, \infty)$ monotone $\uparrow$ or $\downarrow$, $w(l) \not\equiv$ const.

$$\mathbf{P}\Big(X(n+1) = X(n) \pm 1 \Big| \mathcal{F}_n \Big)$$

$$\sim w(\ell_\pm(n, X(n)))$$

$$= \frac{w(\ell_\pm(n, X(n)))}{w(\ell_+(n, X(n))) + w(\ell_-(n, X(n)))}$$

Character:

$w(l+1) \geq w(l)$: self-attractive trajectories

$w(l+1) \leq w(l)$: self-repelling trajectories

Facts about:

Recurrence [Davis: 1992]:

- If $\sum_{k=1}^{\infty} w(k)^{-1} = \infty$ then a.s

$$\forall k \in \mathbb{Z} : \qquad \lim_{n \to \infty} \ell(n,k) = \infty.$$

- If $\sum_{k=1}^{\infty} w(k)^{-1} < \infty$ then a.s $\exists K \in \mathbb{Z}$ so that

$$\mathcal{AR} = \Big\{K, K+1\Big\}.$$

Scaling limits [Tóth 1994-1999]:

- $w(l) = \exp\{-\beta l\}$: see later ...

- $w(l) = \exp\{-\beta l^{\kappa}\}$, $0 < \kappa < 1$: limit theorem for

$$n^{-(\kappa+1)/(\kappa+2)} X(\theta_{n/s})$$

- $w(l) \asymp l^{-\alpha}$, $0 < \alpha < \infty$ or $w(l) = 1 + bl^{-1} + \mathcal{O}(l^{-2})$: limit theorem for

$$n^{-1/2} X(\theta_{n/s})$$

  [Diffusive but non-Gaussian.]

- $w(l) \asymp l^{\alpha}$, $0 < \alpha < 1$: limit theorem for

$$n^{-(1-\alpha)/(2-\alpha)} X(\theta_{n/s})$$

Natural generalizations of the ETSAW

Discrete time, edge repulsion:
[Tóth: 1996]

Weight function $w : \mathbb{Z} \to (0, \infty)$
nondecreasing, $w(l) \not\equiv$ const.

$$\mathbf{P}\Big(X(n+1) = X(n) \pm 1 \,\Big|\, \mathcal{F}_n \Big)$$
$$\sim w(\ell_{\mp}(n, X(n)) - \ell_{\pm}(n, X(n)))$$

Continuous time, vertex repulsion
[Tóth, Vető: 2007]

Weight function: $w : \mathbb{R} \to (0, \infty)$,
non-decreasing, $w(l) \not\equiv$ const.

$$\mathbf{P}\Big(X(t+dt) = X(t) \pm 1 \,\Big|\, \mathcal{F}_t \Big)$$
$$= w(\ell(t, X(t) \pm 1) - \ell(t, X(t)))dt + o(dt)$$

Facts about:

- Recurrence.

- Scaling limit: 1-d marginals: limit theorem for

$$n^{-2/3} X(\theta_{n/s})$$

- Scaling limit:

$$n^{-2/3} X(nt) \stackrel{?}{\Rightarrow} \mathcal{X}(t)$$

  The limit process ("true self-repelling motion")

$$t \mapsto \mathcal{X}(t)$$

  constructed and analyzed, [Tóth, Werner: 1998]

ETSAW with self-repellence on oriented edges

[B. Tóth, B. Vető: 2007]

Weight function $w : \mathbb{Z} \to (0, \infty)$
nondecreasing, $w(l) \not\equiv \text{const.}$

$$\mathbf{P}\Big(X(n+1) = X(n) \pm 1 \,\Big|\, \mathcal{F}_n \Big)$$
$$\sim w(\widetilde{\ell}_{\mp}(n, X(n)) - \widetilde{\ell}_{\pm}(n, X(n)))$$

Facts about:

- Scaling limit, 1-d marginals: limit theorem for

$$n^{-1/2} X(\theta_{n/s})$$

$$n^{-1/2} X(nt) \Rightarrow \text{UNI}[-\sqrt{t}, \sqrt{t}]$$

Idea of proof:

The model: Continuous time, vertex repulsion:

$w : \mathbb{R} \to (0, \infty)$, non-decr., non-const.

$$\mathbf{P}\Big(X(t + dt) = X(t) \pm 1 \,\Big|\, \mathcal{F}_t \Big)$$

$$= w(\ell(t, X(t) \pm 1) - \ell(t, X(t)))dt + o(dt)$$

An auxiliary Markov process:

$$t \mapsto (\alpha(t), \xi(t)) \in \{-1, +1\} \times \mathbb{R}$$

$$\mathbf{P}\Big(\alpha(t+dt) = -\alpha(t) \,\Big|\, \mathcal{F}_t\Big) = \\ w(\alpha(t)\xi(t))dt + o(dt),$$

$$\partial_t \xi(t) = \alpha(t),$$

Stationary measure:

$$\mu(\pm 1, du) = \frac{1}{2Z} e^{-W(u)} du,$$

$$W(u) := \int_0^u \Big(w(v) - w(-v)\Big) dv,$$

$$Z := \int_{-\infty}^{\infty} e^{-W(v)} dv < \infty.$$

Some paricular stopping times:

$$\sigma^{\pm}(s) := \inf\{t : \int_0^t 1\!\!1\{\, \alpha(u) = \pm 1 \,\} du \geq s\}$$

$$\zeta^{\pm}(s) := \xi(\sigma^{\pm}(s))$$

The construction:

Local time on (the endpoints of) the edge $\langle k, k+1\rangle$, $x \in \mathbb{Z}$:

$$\tau(t,k) := \ell(t,k) + \ell(t,k+1)$$

Its inverse:

$$\theta(s,k) := \inf\{t \geq 0 : \tau(t,k) \geq s\}$$

Let

$$\alpha_k(s) :=$$
$$1\!\!1\{ X(\theta(s,k)) = k+1 \} - 1\!\!1\{ X(\theta(s,k)) = k \},$$

$$\xi_k(s) := \ell(\theta(s,k), k+1) - \ell(\theta(s,k), k)$$

Fact: The processes $s \mapsto (\alpha_k(s), \xi_k(s))$, $k \in \mathbb{Z}$, are *independent copies* of the Markov process $s \mapsto (\alpha(s), \xi(s))$ started with initial states $(\alpha_k(0), \xi_k(0)) = (-\mathrm{sgn}(k+1/2), 0)$.

Ray-Knight approach:

Inverse local times:

$$T_{k,h} := \min\{t \geq 0 : \ell(t,k) \geq h\}$$

Local time profile at $T_{k,h}$:

$$\Lambda_{k,h}(l) := \ell(T_{k,h}, l)$$

From the basic representation it follows that

$$l \mapsto \Lambda_{k,h}(l) \quad \text{is Markov}$$

Moreover:

$$l \geq k : \quad \Lambda_{k,h}(l+1) - \Lambda_{k,h}(l) = \zeta_k^-(\Lambda_{k,h}(l))$$

$$l \leq k : \quad \Lambda_{k,h}(l-1) - \Lambda_{k,h}(l) = \zeta_{k-1}^+(\Lambda_{k,h}(l)).$$