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Image enhancement and a quantitative index using fuzzy sets

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The membership function for implementing the enhancement algorithm as described by Pal and King (1981, 1983) is modified here. The modified version enables one to avoid using contrast intensification (INT) operators for enhancement and has been found significantly to reduce the time of computing the algorithm in addition to providing a better approximation to the standard S and Π membership functions. To give a quantitative evaluation of their relative performances, an edge ambiguity index has also been defined through fuzzy measures in a set. The value of the index is maximum for image input and decreases as the ambiguity in detecting edges diminishes. The $\delta(X)$ value, as expected, is also found to increase with increasing standard deviation of the noise injected on X.

1. Introduction

The theory of fuzzy sets provides a suitable algorithm in analysing complex systems and decision processes when pattern indeterminacy is due to vagueness (fuzziness) rather than randomness. Since a grey tone picture possesses some ambiguity within pixels due to the possible multivalued levels of brightness, it is justified to apply the concept and logic of the fuzzy set rather than ordinary set theory to an image processing problem. Keeping this in mind, an image can be considered as an array of fuzzy singletons each with a membership function denoting the degree of some property attribute, e.g. brightness, darkness, edginess, smoothness, etc.

The methods so far developed for image enhancement may be based either on the frequency domain, the spatial domain, or the fuzzy property domain. The technique in the first category depends on modifying the Fourier transform of an image, whereas in spatial domain methods the direct manipulation of the pixel is adopted (Rosenfeld and Kak 1982, Gonzalez and Wintz 1977). The concept in the third category involves modification of fuzzy properties in the property domain (Pal and King 1981 a, 1983, Pal and Majumder 1986) using the contrast intensification (INT) operator (Zadeh 1973).

It is to be mentioned here that all these techniques are problem oriented. When an image is processed for visual interpretation, it is ultimately up to the viewer to judge its quality for a specific application. The process of evaluation of image quality therefore becomes a subjective one.

The present paper consists of three parts. In the first part, the membership function for implementing the enhancement algorithm (Pal and Majumder 1986, Gupta *et al.* 1985) is modified. The modified version enables us to avoid using INT operators for contrast enhancement and has been found significantly to reduce the computation time in addition to being a better approximation to the standard S and Π functions (Zadeh 1973).

An attempt is then made to make the task of evaluating edge enhancement quality somewhat more objective by providing a quantitative measure of edge ambiguity in

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an image. The fuzzy measures, namely the index of fuzziness (Kaufmann 1975) and entropy (De Luca and Termini 1972) are used in defining an index of edge ambiguity. The index value is seen to decrease as edge ambiguity decreases.

Finally, the merit of the above mentioned modified enhancement algorithm is justified quantitatively on the basis of the index value when an X-ray image of a wrist is considered as input. The effect of noise on the index value is also studied in a part of the experiment.

2. Enhancement algorithm

In the authors' previous work on image enhancement (Pal and King 1981 a) and edge detection (Pal and King 1983) using fuzzy sets, the enhancement operations for enhancing an $M \times N$ dimensional, L-level grey tone image $X = \{x_{mn}, m = 1, 2, \ldots, M; n = 1, 2, \ldots, N\}$ consisted of three stages, namely

(i) extraction of fuzzy property plane $\{p_{mn}\}$ from the image plane $\{x_{mn}\}$ using a function $G(x_{mn})$;

(ii) modification of the p_{mn} plane by r ($r = 1, 2, \ldots$) successive applications of the 'INT' operator $T_r(p_{mn})$ to result in a contrast intensified property plane $\{p'_{mn}\}$; and finally

(iii) obtaining an enhanced version $X' = \{x'_{mn}\}$ of the image X using the inverse function $G^{-1}(p'_{mn})$.

The most salient feature considered for the algorithm is to find out a membership function $G(x_{mn})$ which

(*a*) has provision, unlike the ideal ones, for controlling the cross-over point (threshold level of enhancement operation) in the p_{mn} plane;

(*b*) approximates well both the ideal S function and the Π function (Zadeh 1973); and

(*c*) reduces the number r in $T_r(p_{mn})$, $r = 1, 2, \ldots$, operations (Zadeh 1973).

The ultimate aim is to reduce the computation time and complexity. The modified membership function as suggested in § 4 is found to be superior to that reported (Pal and King 1981 a, 1983) in these regards. This superiority is also quantitatively evaluated using a measure called the 'index of edge ambiguity'. The index value as defined in § 6 reflects an average amount of difficulty (ambiguity) that might arise while detecting the edges of different regions in an image.

3. Previous membership function

The membership function considered by Pal and King (1981 a, 1983) was of the form

$$p_{mn} = G(x_{mn}) = (1 + |\hat{x} - x_{mn}|/F_d)^{-F_e}, \quad m = 1, 2, \ldots, M, \quad n = 1, 2, \ldots, N \qquad (1)$$

This represents an S-type function $G_s(x_{mn})$ for $\hat{x} = x_{\max}$, maximum level $(= L - 1)$ in X, and a Π-type function $G_\pi(x_{mn})$ for $\hat{x}$ being some arbitrary level l_c, $0 < l_c < x_{\max}$. F_e and F_d are two positive constants whose values are determined from the cross-over points in the enhancement operation.

Suppose x_c is the cross-over point (threshold level) for an S-type function. Then we

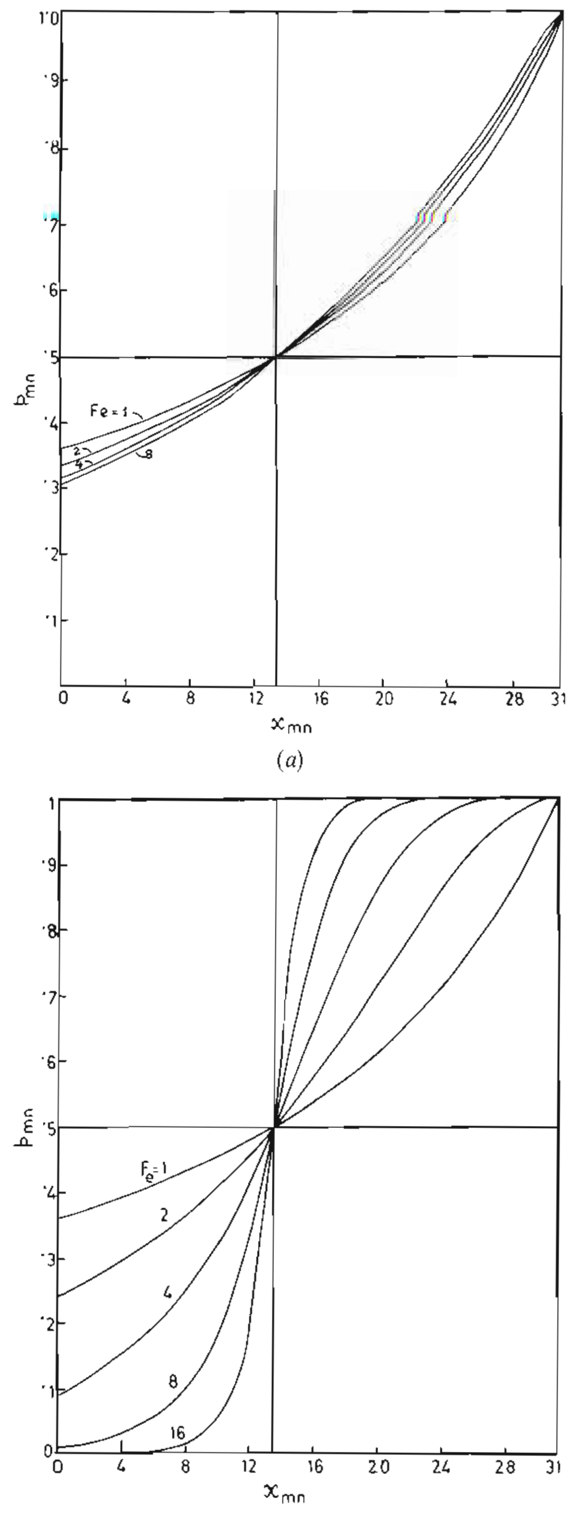


Figure 1. Plot of membership function with different values of F_e: (*a*) from (3); (*b*) from (6).

have from (1) that

$$p_{mn} = G_s(x_c) = 0{\cdot}5 = (1 + |x_{\max} - x_c|/F_d)^{-F_e}$$

or

$$F_d = (x_{\max} - x_c)/(2^{1/F_e} - 1) \tag{2}$$

In terms of the cross-over point, (1) can therefore be written as

$$p_{mn} = G_s(x_{mn}) = \frac{1}{\left[1 + \dfrac{(x_{\max} - x_{mn})(2^{1/F_e} - 1)}{x_{\max} - x_c}\right]^{F_e}} \tag{3}$$

A plot of (3) for different values of F_e is shown in Fig. 1(*a*) for a 32 ($= x_{\max} + 1$) level image when x_c is set at 13·5.

4. Modified membership function

Let us now modify (1) as

$$p_{mn} = G(x_{mn}) = (1 + (|\hat{x} - x_{mn}|/F_d)^{F_e})^{-1} \tag{4}$$

Considering an S-type function, the value of F_d for cross-over point x_c turns out here to be

$$F_d = (x_{\max} - x_c)\Big/\left(\frac{1}{0{\cdot}5} - 1\right)^{1/F_e} = x_{\max} - x_c \tag{5}$$

and (4) in terms of x_c becomes

$$p_{mn} = G_s(x_{mn}) = \frac{1}{1 + \left(\dfrac{x_{\max} - x_{mn}}{x_{\max} - x_c}\right)^{F_e}} \tag{6}$$

A plot of this equation for different values of F_e is shown in Fig. 1(*b*) for a 32 level image.

5. Relative merits

The merits of (4) over (1) with respect to the criteria (*a*)–(*c*) in § 2 are explained below.

5.1. From (2) it is seen that F_d is dependent on F_e in selecting a cross-over point. The modified version, on the other hand, makes it simpler (less constrained) where F_d is determined only by x_c (see (5)).

Comparing (3) and (6), the previous function needs some more time (say, Δt_1) for computing p_{mn} because of the additional factors $(2^{1/F_e} - 1)$ which again involves one inverse exponential operation and a difference operation. For an L-level image, the total time saved by (4) for computing $G(x_{mn})$ and $G^{-1}(p'_{mn})$ of the enhancement block of Pal and King (1981 a, 1983) is therefore

$$L\,\Delta t_1 + L\,\Delta t_1 = 2L\,\Delta t_1$$

5.2. The curves in Fig. 1(*b*) for $F_e > 1$ have better symmetry around the cross-over point and lesser α-values (the value of p_{mn} for $x_{mn} = 0$) and hence they bear a better resemblance to the ideal S-function than those of Fig. 1(*a*). As a result, the intensity levels in the enhanced image obtained with (4) would have better symmetry and contrast around the threshold level.

5.3. The contrast steepness of the curves (Fig. 1(*b*)) especially, for $F_e > 2$ closely follows that of the operator $T_r(P_{mn})$, $r = 1, 2, \ldots$ (Zadeh 1973). The property plane $p_{mn}\}$ as extracted by (4) has therefore much more contrast present within the plane as compared to that obtained with (1). The higher the contrast, the lower is the value of r in the $T_r(p_{mn})$ operator required to achieve the desired amount of enhancement.

Again, for large values of F_e one can also delete the $T_r(p_{mn})$ block from the enhancement block diagram of Pal and King (1981 a, 1983). In such a case, the block $G^{-1}(p_{mn})$ must have to use a value of F_e smaller than that used in the preceding $G(x_{mn})$ block.

The *algorithm without INT operator* can thus be written as

Step 1. Apply the G_s transformation to obtain the μ_{mn} or p_{mn} plane from the x_{mn} plane for a particular value of x_c and F_e ($= F_e''$, say).

Step 2. Reduce the value of F_e to F_e' ($F_e' \ll F_e''$) and apply the G_s^{-1} transformation on the p_{mn} or μ_{mn} plane to result in an enhanced-contrast image plane x'_{mn}.

The higher the difference between F_e'' and F_e', the greater would be the contrast around x_c.

Since this algorithm does not need the INT operator (as needed in the previously mentioned algorithms) for enhancement, the time of computation can further be reduced; the amount being saved is

$$Lr\frac{\Delta t_2}{2}, \quad r = 1, 2, \ldots$$

for an L-level image. Δt_2 is the time required for computing the INT operator.

It is to be mentioned here that the superiority of the modified membership function (see (4)) over (1) for enhancing an image has been explained above considering the S-type function $G_s(x_{mn})$. The same argument would hold good for a Π-type function $G_\Pi(x_{mn})$ as used by Pal and King (1981 a) for enhancing the edges of X.

6. Index for edge ambiguity

From the foregoing sections, it is seen analytically that the modified membership function (see (4)) leads to the saving of some computation time compared with (1) to obtain the desired amount of enhancement. It also leads to defining an algorithm without using the INT operator, as a result of which the time for computing T_r, $r = 1, 2, \ldots,$ operations can be avoided totally.

To give an objective evaluation of the relative performance of the three previously-mentioned algorithms for the edge detection problem, a quantitative measure of edge ambiguity in X may be defined as

$$\delta(X) = [1 - I(X)]^{\beta} \tag{7}$$

where $I(X)$ stands either for the linear index of fuzziness $\gamma_l(X)$ (Kaufmann 1975), the quadratic index of fuzziness $\gamma_q(X)$ (Kaufmann 1975) and the entropy $H(X)$ (De Luca and Termini 1972) of the image X. β is a positive constant.

$\gamma_l(X)$, $\gamma_q(X)$ and $H(X)$ of X are defined as:

$$\gamma_l(X) = \frac{2}{MN} \sum_m \sum_n |\mu_X(x_{mn}) - \mu_{\mathbf{X}}(x_{mn})| \tag{8 a}$$

$$= \frac{2}{MN} \sum_m \sum_n \mu_{X \cap \bar{X}}(x_{mn}) \tag{8 b}$$

$$\gamma_q(X) = \frac{2}{\sqrt{MN}} \left[\sum_m \sum_n (\mu_X(x_{mn}) - \mu_{\mathbf{X}}(x_{mn}))^2 \right]^{1/2} \tag{9}$$

and

$$H(X) = \frac{1}{MN \ln 2} \sum_m \sum_n S_n(\mu_X(x_{mn})) \tag{10 a}$$

with

$$S_n(\mu_X(x_{mn})) = -\mu_X(x_{mn}) \ln \mu_X(x_{mn}) - (1 - \mu_X(x_{mn})) \ln (1 - \mu_X(x_{mn})) \tag{10 b}$$

$\gamma(X)$ and $H(X)$ measure, in a global sense, the average amount of difficulty (fuzziness) present when one has to decide whether an element can be considered to be a member of the set or not. Both of them have the property that they increase monotonically in the interval [0, 0·5] and decrease monotonically in [0·5, 1] with a maximum of unity at $\mu = 0{\cdot}5$ in the fuzzy property plane of X. $\mathbf{X}$ denotes the two-tone version of X.

For computing $\delta(X)$, which reflects an average amount of ambiguity (difficulty) in detecting edges of various regions in X, we have considered fuzziness in the spatial domain and make the membership function $\mu_X(x_{mn})$ of the (m, n)th pixel in X (unlike (1) or (4)) dependent on its local distribution such that

$$\mu_X(x_{mn}) = \frac{0{\cdot}5}{1 + \frac{1}{N_1} \sum_Q |x_{mn} - x_{ij}|}, \quad (i, j) \in Q, \quad (i, j) \neq (m, n) \tag{11}$$

where Q is a set of N_1 neighbouring co-ordinates of the point (m, n).

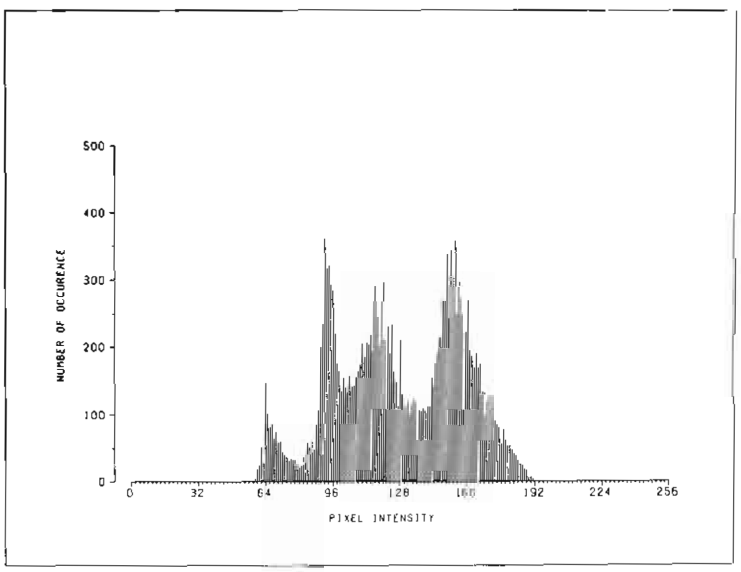
From (11), it is seen that if all the pixels have the same intensity, then $x_{mn} = x_{ij}$ for all (m, n), $\mu(x_{mn}) = 0{\cdot}5$ for all (m, n) and $I(X) = 1$. The measure of edge ambiguity would therefore be zero, as there is no edge in the image. An image with dissimilar grey levels would have a higher $\delta(X)$ value.

Since with increase in the value of r in the T_r operation, the contrast among successive regions in X increases, the dissimilarity in grey levels would decrease because the pixels in a given region would tend to possess similar intensity levels. The value of $\delta(X)$ would therefore decrease with increasing r. Or, in other words, the higher the contrast among different regions in X, the less would be the difficulty (ambiguity) in taking decisions regarding edges (contours) or in detecting edges, and hence the lower would be the value of $\delta(X)$. Therefore if one applies an edge detection operator and computes the value of $\delta(X)$ on the edge-detected output, the decrease in edge ambiguity (i.e. conversion of grey tone edges to their two-tone versions) would automatically correspond to a decrease in the value of $\delta(X)$.

7. Implementation

Figure 2 shows the radiograph along with a histogram of part of a wrist containing a radius and part of two small carpal bones. Figure 3 shows the contours of different

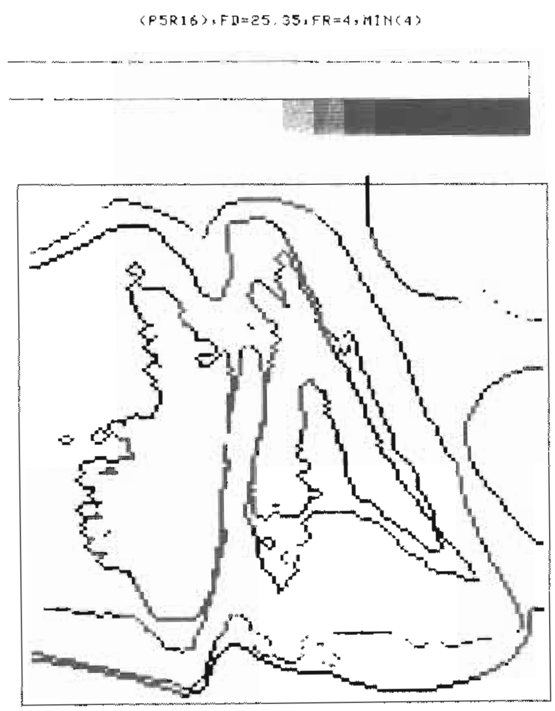
(*a*)



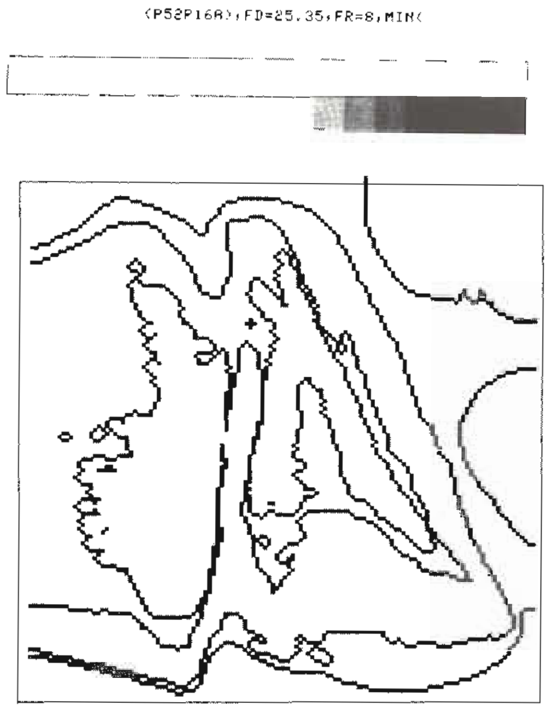
(*b*)

Figure 2. Input: (*a*) X-ray image; (*b*) histogram.

3(*a*)



3(*b*)



(*c*)

Figure 3. Edge detected output: (*a*) $r = 2$; (*b*) $r = 4$; (*c*) $r = 8$.

regions of Fig. 2. These contours were obtained by first applying a contrast enhancement technique and then a 'min' edge detection operator (Pal and King 1983). Figures 3(*a*), (*b*) and (*c*) correspond to $r = 2$, 4 and 8, respectively, of the T_r operation.

To describe the relative performance of the three previously-mentioned enhancement algorithms in extracting edges, we denote the enhancement technique of Pal and King (1981 a, 1983) using (1) by Algorithm 1, the similar technique using a modified membership function (see (4)) by Algorithm 2 and the technique without the INT operator by Algorithm 3.

Table 1 illustrates the value of $\delta(X)$ for Algorithm 1 when $I(X)$ in (7) stands for

r	$\delta(X)$
2	0·32187
4	0·17542
5	0·12828
6	0·11493
7	0·10749
8	0·10221
10	0·10025

Note. $\delta(X)$ for input image (Fig. 2) is 0·53369.

Table 1. $\delta(X)$ value for Algorithm 1.

$\gamma_I(X)$ (see (8)) and $\beta = 1$. The value of $\delta(X)$ which reflects a measure of edge ambiguity is maximum ($= 0{\cdot}53369$) for the input image (Fig. 2) and, as r increases, $\delta(X)$ decreases with a minimum of 0·10025 for two-tone edges. As a typical illustration, we have demonstrated in Fig. 3 only three edge detected outputs of Fig. 2. Algorithm 2 which uses (4) (a slight modification of the membership function used in Algorithm 1) is found to reduce the number r of INT operations to attain a desired edge detected output. This is shown in Table 2 where the value of $\delta(X)$ is seen always to be smaller than that in Table 1 for a particular value of r. As explained before, this superiority is because of the greater fuzzifying property and contrast steepness of (4) (Fig. 1(*b*)) as compared with (1) (Fig. 1(*a*)).

r	$\delta(X)$
2	0·28152
3	0·16050
4	0·10932
5	0·10045
6	0·10025

Table 2. $\delta(X)$ value for Algorithm 2.

Table 3 shows the value of $\delta(X)$ for different combinations of F''_e and F'_e when Algorithm 3 is considered as the enhancement operation. The greater the difference between F''_e and F'_e, the lower is the ambiguity in detecting edges and the smaller is the $\delta(X)$ value. Since we no longer need INT operators here, the computation time is further reduced for a desired output.

F''_e	F'_e	$\delta(X)$
40	4	0·17494
40	2	0·14251
40	1	0·11660
40	0·5	0·10850
40	0·25	0·10254
40	0·125	0·10025
60	4	0·14347
60	2	0·12184
60	1	0·11159
60	0·5	0·10361
60	0·25	0·10118
60	0·125	0·10025

Table 3. $\delta(X)$ value for Algorithm 3.

The average time of computation by an EC 1033 is found to be 2 min 38 s, 2 min 34 s and 2 min 28 s for Algorithms 1, 2 and 3, respectively.

In support of our previously mentioned claim for noisy images, and to demonstrate the effect of noise on edge-detected output, the experiment was also conducted by making Fig. 2 corrupted by random noise with zero mean and different standard deviations. Tables 4 and 5 show the $\delta(X)$ values corresponding to Algorithms 1, 2 and 3, respectively, when a standard deviation σ of noise is considered to be 1, 2, 3 and 4 with zero mean. Figure 4 shows, as an illustration, the noisy histogram corresponding to $\sigma = 4$. This is shown for comparison with the original histogram (Fig. 2(*b*)).

From Table 4 it is seen that the $\delta(X)$ value decreases as r increases and Algorithm 2 needs (as mentioned before) fewer INT operations; hence less computation time compared with Algorithm 1 to attain the same $\delta(X)$ value. The computation time is seen to be further reduced by Algorithm 3.

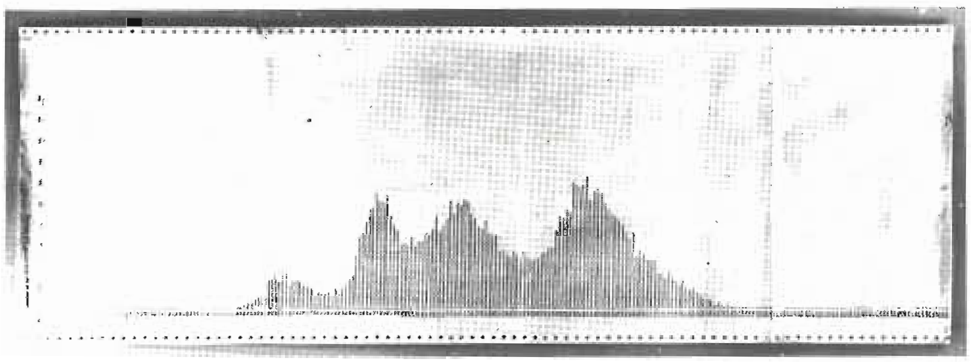


Figure 4. Noisy histogram ($\sigma = 4$ with zero mean).

Furthermore, as an effect of noise on the edge-detected output, the $\delta(X)$ value is found to increase with the standard deviation of the injected noise. For example, considering Algorithm 2 and $r = 6$, $\delta(X)$ starts increasing from a value of 0·10025 for $\sigma = 0$ to

0·10456 for $\sigma = 1$

0·11429 for $\sigma = 2$

0·12941 for $\sigma = 3$

and

0·15022 for $\sigma = 4$

This increase in $\delta(X)$ value is because of the contribution of the undesirable spurious wiggles which appeared in the edge-detected output due to injected noise at the input.

Finally, as a comparison of the effectiveness of these algorithms with an existing one, investigations have also been reported (Pal and King 1981 b) in which the pre-enhancement operation of Pal and King (1983) was replaced by the histogram equalization technique (Gonzalez and Wintz 1977) (a standard existing enhancement operation for images like X-ray pictures and landscape photographs taken under poor illumination). However, the contours of the resulting edge-detected output image (Fig. 5) (Pal and King 1981 b) show such an output for $r = 8$ for comparison with Fig. 3(*c*), compared with which the present algorithms were seen to contain more spurious wiggles which, in turn, make the task of their description and interpretation more difficult and also increase (as described before) the value of $\delta(X)$.

	Value of $\delta(X)$							
	$\sigma = 1{\cdot}0$		$\sigma = 2{\cdot}0$		$\sigma = 3{\cdot}0$		$\sigma = 4{\cdot}0$	
r	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2
2	0·35926	0·29816	0·38556	0·33932	0·44204	0·37677	0·43738	0·39521
3	0·26156	0·16785	0·28360	0·18215	0·30915	0·20438	0·34490	0·23709
4	0·18026	0·11592	0·19582	0·12473	0·21054	0·14078	0·25175	0·16380
5	0·13054	0·10482	0·14217	0·11451	0·16184	0·12947	0·18929	0·15081
6	0·11793	0·10456	0·12850	0·11429	0·14287	0·12941	0·16812	0·15022
7	0·11094	—	0·12260	—	0·13540	—	0·15887	—
8	0·10610	—	0·11831	—	0·13087	—	0·15231	—

Table 4. $\delta(X)$ value for noisy images using Algorithms 1 and 2.

F''_e	F'_e	Value of $\delta(X)$ $\sigma = 1{\cdot}0$	$\sigma = 2{\cdot}0$	$\sigma = 3{\cdot}0$	$\sigma = 4{\cdot}0$
30	4	0·21618	0·21255	0·23513	0·25178
30	2	0·15822	0·17535	0·19127	0·22731
30	1	0·12186	0·13690	0·15211	0·17713
30	0·5	0·11162	0·12624	0·13994	0·16279
30	0·25	0·10361	0·11927	0·13367	0·15523
30	0·125	0·10118	0·11524	0·13012	0·15117
60	4	0·14347	0·16075	0·17788	0·21585
60	2	0·12186	0·13690	0·15211	0·17713
60	1	0·11162	0·12624	0·13994	0·15279
60	0·5	0·10361	0·11927	0·13367	0·15523
60	0·25	0·10118	0·11524	0·13012	0·15117
60	0·125	0·10025	0·11429	0·12941	0·15022

Table 5. $\delta(X)$ value for noisy images using Algorithm 3.

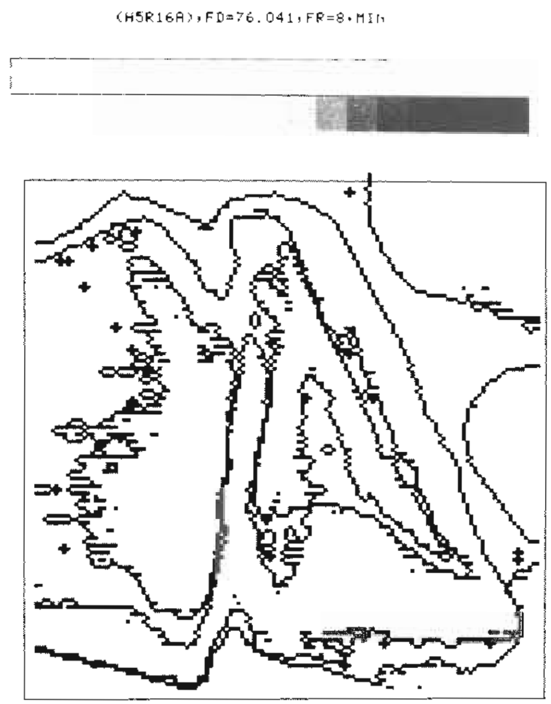


Figure 5. Edge detected output corresponding to the histogram equalization technique and $r = 8$.

8. Conclusion

A modified membership function of the form shown in (4) is suggested for implementing onr reported enhancement algorithm (Pal and King, 1981 a, 1983). The modified version has a better resemblance to the ideal membership functions and has been found to reduce the computation time significantly for a desired output. Because of the inherent fuzzifying property and contrast steepness of the function, one may avoid using the INT operation and as a result the computation time can be reduced further.

The greater fuzzifying property and contrast steepness of the modified function (see (4)) can also be shown analytically as follows.

Let

$$p_{mn}^{\text{old}} = (1 + B)^{-F_e} \tag{12}$$

and

$$p_{mn}^{\text{new}} = (1 + B^{F_e})^{-1} \tag{13}$$

with

$$B = \frac{|\hat{x} - x_{mn}|}{F_d} \tag{14}$$

corresponding to (1) and (4), respectively.

Expanding (12) and (13) by the binomial series we have

$$p_{mn}^{\text{old}} = 1 - F_e B + \frac{1}{2!} F_e(F_e + 1)(B)^2 - \frac{1}{3!} F_e(F_e + 1)(F_e + 2)(B)^3 + \dots \tag{15}$$

and

$$p_{mn}^{\text{new}} = 1 - B^{F_e} + (B^{F_e})^2 - (B^{F_e})^3 + \dots \tag{16}$$

It is therefore clear from (15) and (16) that p_{mn}^{new} contains terms involving B^{F_e} (resulting in a greater fuzzifying property) while p_{mn}^{old} contains terms involving B only.

An index of edge ambiguity $\delta(X)$ is defined through the fuzzy measures in a set. The index value decreases as the fuzziness (difficulty) in detecting edges diminishes. The modified membership function is found to reduce the number of INT operations for a specific $\delta(X)$ value. Again, the index value, as expected, is found to increase with the standard deviation of the noise randomly added on the image.

Although the effectiveness of the algorithms has only been demonstrated on X-ray images including noisy versions, it can also be suitably used for any image data where one needs to increase the contrast among different regions and to provide a quantitative measure of ambiguity in detecting edges. The relative merits of the modified membership function and the Algorithms 2 and 3 would also hold good, accordingly.

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