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INVITED PAPER

Ambiguity and Decision Making in Image Analysis and Expert System : Fuzzy Mathematical Approach

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The paper presents the different fuzzy tools those are useful for taking decision in image processing and vision problems, and in designing expert system when the patterns are ill-defined or the input data does not have complete, precise or reliable information. These are of significant use in designing intelligent interfaces which is one of the major functions of FGCS. Algorithms for fuzzy enhancement/segmentation of both gray tone image and colour image, front end compiler and representing rules and facts in knowledge base are demonstrated with various examples of real life problem.

Indexing terms : Pattern recognition, Image processing, Vision, Expert systems, Fuzzy sets

THE KEY technologies for the Fifth Generation Computer System (FGCS) consist of VLSI architecture, parallel processing, logic programming, knowledge based on relational data base, applied artificial intelligence and pattern processing/analysis. With this machine, the expected generational change is more like a 'generic change' which involves not only a change in device technology, to VLSIs, but also simultaneous changes in the design philosophy and in the fields of application [1].

In other words, an environment needs to be created in which man and computer find it easy to communicate freely (and hence fastly) using a wide range of information media, such as speech, handwritting, text and graph which represent man's natural ways of communicating information. The current computers are extremely weak in basic functions for processing and understanding speech, text, graphs and other non-numerical data, and for artificial intelligence type processing such as inference, association and learning. Therefore, among the various major functions of FGCSs (*eg*, problem-solving and inference, knowledge-base management and intelligent interface), one of the requirements is the advantageous use of research achievements of pattern recognition, image processing/vision and artificial intelligence in order to realise man/machine interfaces that are natural to man. Indian Statistical Institute is one of the five nodal centres of Indian FGCS project to work on pattern recognition, image analysis and vision problems.

The present work describes some fuzzy tools those are useful for the management of uncertainty and for taking decision in pattern analysis and vision problems, and in designing expert system when the patterns are ill-defined or the input data does not have complete, precise or reliable information. The problems discussed here are:

(*a*) selection of thresholds without committing ourselves to a specific segmentation when the regions in an image are ill-defined

(*b*) to provide fuzzy transforms for enhancement and segmentation of colour (including pseodo colour) images

(*c*) to equip an expert system with a computational capability in order to analyse the transmission of uncertainty in information from knowledge base to the uncertainty in the validity of its conclusions.

The fuzzy tools considered here are measures of fuzziness (such as, index of fuzziness [2], entropy [3], index of nonfuzziness [4], π-ness [5] and dispersion [6]), fuzzy geometry [7] (such as area, perimeter and compactness), fuzzy expected value and fuzzy expected intervals [8]. The fuzzy measures are optimised in some of the problems in order to take decision for ill-defined patterns.

Fuzzy transforms for colour image processing and graphics are defined using the operator 'bounded difference' [9,10]. Fuzzy expected value and intervals are used in designing expert system when the input data does not have complete or precise information. The use of fuzzy and fractionally fuzzy grammars [11] for developing production rules is also discussed.

DISPERSION MEASURE OF FUZZY SETS [6]

A fuzzy set A in the universe of discourse $X=\{x\}$ is defined by its membership function $\mu_A(x)$ which assigns to each element $x \in X$ a real number in the interval [0,1]. The value of $\mu_A(x)$ represents the grade of membership of x in A. In other words, a fuzzy set A on X is denoted by its membership function

$$\mu_A : X \rightarrow [0,1] \text{ or } \mu_A(X), \text{ for all } x \in X$$

An energy measure E of a fuzzy set A satisfies the following axioms:

$$E(A)=0 \text{ iff } \mu_A(x)=0 \text{ for all } x \in X \quad (1a)$$

$$\text{If } \mu_A \leqslant \mu_B \text{ then } E(A) \leqslant E(B) \quad (1b)$$

$E(A)$ reaches its maximum iff $\mu_A(x)=1$ for all $x \in X$ (1c)

The Power (or cardinality) which represents an energy measure is defined as:

$$P(A) := \sum_{x \in X} \mu_A(x) = \sum_i \mu_A(x_i), \quad i = 1,2,\ldots,n \quad (2)$$

where X is a finite set of cardinality n.

Dispersion measure of a fuzzy set gives a measure of the size (cardinality in case of finite supports) in which almost all the energy of A is concentrated. Let $\hat{A}$ denote the fuzzy set obtained from A by rearranging its membership values $\mu_A(x_i)$, $i=1,2\ldots\ldots,n$ in a nonincreasing way. In other words,

$$\mu_{\hat{A}}(x) \geqslant \mu_{\hat{A}}(x+1), \; 1 \leqslant x \leqslant n-1 \quad (3)$$

One would obviously have

$$P(\hat{A}) = P(A) = P, \text{ say} \quad (4)$$

$$\text{and } d(\hat{A}) = d(A) \quad (5)$$

where d stands for measures of fuzziness [2-5]

This dispersion of A may then be defined as

$$\delta_e(A) = \min \{k \in [n] \mid \sum_{x \leqslant k} \mu_{\hat{A}}(x) > P - e\} \quad (6)$$

where $e > 0$

$[n] = \{1,2,\ldots\ldots,n\}$

This definition implies that

$$\sum_{x > \delta_e} \mu_{\hat{A}}(x) < e \quad (7a)$$

$$\sum_{x > \delta_e} \mu_{\hat{A}}(x) \geqslant e \quad (7b)$$

$$\text{and} \quad \delta_e(A) = \delta_e(\hat{A}) \quad (7c)$$

If $e = P/n$ so that $0 < e \leqslant 1$, then $\delta(A)$ gives a measure of the minimal cardinality of a subset of the universe X in which an amount of power greater than $P - \frac{P}{n}$ is concentrated.

FUZZY GEOMETRY

Rosenfeld [7] extended the concept of digital picture geometry to fuzzy subsets and generalised some of the standard geometric properties of and relationships among regions to fuzzy subsets. Among the extensions of the various properties, we will only discuss here the area, perimeter and compactness of a fuzzy image subset which may be used for pattern recognition and image processing problems. In defining the above mentioned parameters we replace $\mu_A(x)$ by μ for simplicity.

The area of μ is defined as

$$a(\mu) \triangleq \int \mu \quad (8)$$

where the integral is taken over any region outside which $\mu = 0$.

If μ is piecewise constant (for example, in a digital image) $a(\mu)$ is the weighted sum of the areas of the regions on which μ has constant values, weighted by these values.

For the piecewise constant case, the perimeter of μ is defined as

$$p(\mu) \triangleq \sum_{i,j} \sum_k |\mu_i - \mu_j| \; |A_{ijk}| \quad (9)$$

$$i, j = 1,2,\ldots,r; \; i < j; \; k = 1,2,\ldots, r_{ij}.$$

This is just the weighted sum of the length of the arcs A_{ijk} along which the ith and jth regions having constant μ values μ_i and μ_j respectively meet, weighted by the absolute difference of these values.

The compactness of μ is defined as

$$\text{Comp}(\mu) \triangleq a(\mu)/p^2(\mu) \quad (10)$$

For the crisp sets, this is largest for a disk, where it is equal to $1/4\pi$. For a fuzzy disk where μ depends only on the distance from the origin (centre), it can be shown that

$$a(\mu)/p^2(\mu) \geqslant \frac{1}{4\pi} \quad (11)$$

In other words, of all possible fuzzy disks, the compactness is smallest for its crisp version.

FUZZY EXPECTED VALUE AND INTERVAL

Kandel and Byatt [10] defined fuzzy expected value (FEV) of a membership function μ over a fuzzy set A with respect to a fuzzy measure χ as follows.

Let μ_A be a B-measurable function such that $\mu_A \varepsilon$ [0,1]. The fuzzy expected value of μ_A over A, with respect to the fuzzy measure $\chi(.)$ is defined as

$$\text{FEV}(\mu_A) = \sup_{T \in [0,1]} \{\text{Min}\,[T, \chi(\xi_T)]\} \quad (12)$$

$$\text{where} \quad \xi_T = \{x \mid \mu_A(x) \geqslant T\}. \quad (13)$$

Now, $\chi(x|\mu_A(x) \geqslant T) = f_A(T)$ is a function of the theshold T and the function χ maps ξ into [0,1]. In other words, the method of evaluating FEV(μ_A) consists of finding the point of intersection of the curves $g(T) = T$ and $f_A(T)$. These curves will therefore intersect at $T = H$ so that $\text{FEV}(\mu_A) = H \in [0,1]$.

FEV can thus be regarded as an indicative measure of me sort of central tendency.

xample

For a given population and a given membership function for the set 'Old', let us consider the following data].

10 people are 20 years old *ie*, μ=0.20

15 people are 30 years old *ie*, μ=0.30

25 people are 45 years old *ie*, μ=0.45

30 people are 55 years old *ie*, μ=0.55

20 people are 60 years old *ie*, μ=0.60

Here we have T: 0.20, 0.30, 0.45, 0.55, 0.60. For a given threshold, we can now determine the number of people (in percentage terms) who are above the threshold. For example, the numbers are 100, 90, 75, 50 and 20 corresponding to the thresholds 0.20, 0.30, 0.45 0.55 and 0.60. Thus we have χ: 1.0, 0.90, 0.75, 0.50 and 0.20.

Now, the minimum value of each (T,χ) pair is 0.20, .30, 0.45, 0.50 and 0.20. The FEV(μ_A) which is maximum of all these minima is thus 0.50.

The fuzzy expected age of the population is 50.

Suppose we have the following data for a population:

More or less 20 people are between the ages of 20 and 30.

20 to 25 people are 15 years old.

25 people are almost 40 years old.

The FEV denoting the typical age of the group of people is not applicable here, because the data do not have complete information about the distribution of the population and their grades of membership. In order to tackle this kind of problem the concept of fuzzy expected interval (FEI) is introduced [8].

The upper and lower bounds of any χ_j are defined as

$$\text{U.B. } \chi_j \quad \frac{\sum_{i=j}^{n} \text{Max}\,(p_{i1}, p_{i2})}{\sum_{j}^{n} \text{Max}\,(p_{i1}, p_{i2}) + \sum_{i=1}^{j-1} \text{Min}\,(p_{i1}, p_{i2})} \tag{14}$$

$$\text{L.B. } \chi_j \quad \frac{\sum_{i=j}^{n} \text{Min}\,(p_{i1}, p_{i2})}{\sum_{i=j}^{n} \text{Min}\,(p_{i1}, p_{i2}) + \sum_{i=1}^{j-1} \text{Max}\,(p_{i1}, p_{i2})} \tag{15}$$

where p_{i1} and p_{i2} are the lower bound and upper bound respectively of group i.

Therefore, arranging the data in order of increasing age, we may write

20 to 25 people are 15 years old

more or less 20 people are between the ages of 20 and 30

25 people are almost 40 years old.

Let us assume that the adjectives 'almost' and 'more or less' for the variable x have the lower bound and upper bound as $x-10\%$ and $x-1$, and $x-10\%$ and $x+10\%$ respectively. Therefore, we have

20 to 25 people are of μ_1 : 0.15—0.15

more or less 20 people are of μ_2 : 0.2—0.3

25 people are of μ_3 : 0.36—0.39

The corresponding upper and lower bounds of χ_j values are (using equations (14) and (15))

$$\chi_1 : \frac{20+18+25}{20+18+25+0} - \frac{25+22+25}{25+22+25+0} = 1-1$$

$$\chi_2 : \frac{18+25}{18+25+25} - \frac{22+25}{22+25+20} = 0.63\text{—}0.70$$

$$\chi_3 : \frac{25}{25+25+22} - \frac{25}{25+20+18} = 0.34\text{—}0.397$$

Taking minimum of each (χ_j, μ_j) pair we have

Min([1—1], [0.15—0.15]) = Min(1,0.15)—

Min(1,0.5) = [0.15—0.15]

Min([0.63—0.70],[0.2—0.3]) = Min(0.63,0.2)—

Min(0.70,0.3) = [0.2—0.3] and

Min([0.36—0.39],[0.34—0.397]) = [0.34—0.39]

Finally, taking maximum over all these minima we get

[0.34—0.39]

which gives the fuzzy expected interval of the group.

IMAGE PROCESSING AND VISION

Image definition [10]

With the concept of fuzzy sets, an image X of $M \times N$ dimension and L levels can be considered as an array of fuzzy singletons, each with a value of membership function denoting the degree of having brightness relative to some brightness level l, l = 0,1,2,......L—1. In the notion of fuzzy sets we may therefore write

$$X = \bigcup_m \bigcup_n p_{mn}/x_{mn} = \bigcup_m \bigcup_n \mu_{mn}/x_{mn} \tag{16}$$

$$m = 1,2,\ldots\ldots M, \; n = 1,2\ldots\ldots N$$

where p_{mn}/x_{mn} ($0 \leqslant p_{mn} \leqslant 1$) represents the grade of possessing some property p_{mn} by the (m,n)th pixel x_{mn}. This fuzzy property may, for example, be 'bright image', 'semibright image', 'dark image', 'edgy image', 'smooth image', etc. As a result, p_{mn} (or μ_{mn}) may be defined in a number of ways as shown in the following sub-sections depending on the problems in hand.

Of the several applications of fuzzy sets in image processing and vision problems, our discussion will mainly concentrate on the fuzzy segmentation of image where it is not possible to commit ourselves to a specific segmentation, and fuzzy transforms for colour image enhancement and segmentation when it is difficult to discriminate visually the colours between regions.

Image thresholding

Algorithms based on minimisation of fuzziness and compactness are developed by Pal and Rosenfeld [12] whereby it is possible to obtain both fuzzy and nonfuzzy (thresholded) versions of ill-defined images.

Let us describe the algorithm based on minimization of fuzziness in grey level (r-value, say,) and in spatial domain (comp(μ)) in an image $X = \{\mu_{mn}/x_{mn}\}$, $m = 1,2,\ldots\ldots M$, $n = 1,2,\ldots\ldots N$ with l_{max} and l_{min} denoting the maximum and minimum grey levels.

Step 1: Constract the 'bright image' membership μ_X, using S-function [13] where

$$\mu_X(l) = S(l;\, a, l_i, c)$$

$$l_{min} \leqslant l,\quad l_i \leqslant l_{max} \qquad (17)$$

with cross-over point $b = l_i$ and a particular bandwidth $\Delta b = l_i - a$.

Step 2: Compute the amount of fuzziness in μ_X corresponding to $b = l_i$ with

$$r(X)\,|\,l_i = \frac{2}{MN} \sum_l \min\{S(l;\, a, l_i, c),\ 1 - S(l;\, a, l_i, c)\}\ h(l)$$

$$= \frac{2}{MN} \sum_l T_i(l)\, h(l) \qquad (18)$$

where $\quad T_i(l) = \min\{S(l;a, l_i, c),\ 1 - S(l;a, l_i, c)\} \qquad (19)$

and $h(l)$ denotes the number of occurrences of the level l. $r(X)$ denotes linear index of fuzziness [10] of X.

Step 3: Compute the area and perimeter of μ_X corresponding to $b = l_i$ with

$$a(\mu)\,|\,l_i = \sum_m \sum_n \mu_{mn} = \sum_l S(l;\, a, l_i, c)\ h(l) \qquad (20)$$

and

$$p(\mu)\,|\,l_i = \sum_{m=1}^{M} \sum_{n=1}^{N-1} |\mu_{mn} - \mu_{mn+1}| + \sum_{n=1}^{N} \sum_{m=1}^{M-1} |\mu_{mn} - \mu_{m+1n}| \qquad (64)$$

(excluding the frame of the image)

Step 4: Compute the compactness of μ_X corresponding to $b = l_i$ with

$$\text{Comp}(\mu)\,|\,l_i = \frac{a(\mu)\,|\,l_i}{p^2(\mu)\,|\,l_i} \qquad (21)$$

Step 5: Compute the product

$$\theta_{l_i} = r(X)\,|\,l_i \cdot \text{Comp}(\mu)\,|\,l_i \qquad (22)$$

at each value of l_i and select $l_i = l_c$, say, as threshold for which θ_{l_i} is a minimum. The corresponding μ_{mn} represents the fuzzy segmented version of the image as far as minimization of its fuzziness in grey level and in the spatial domain is concerned.

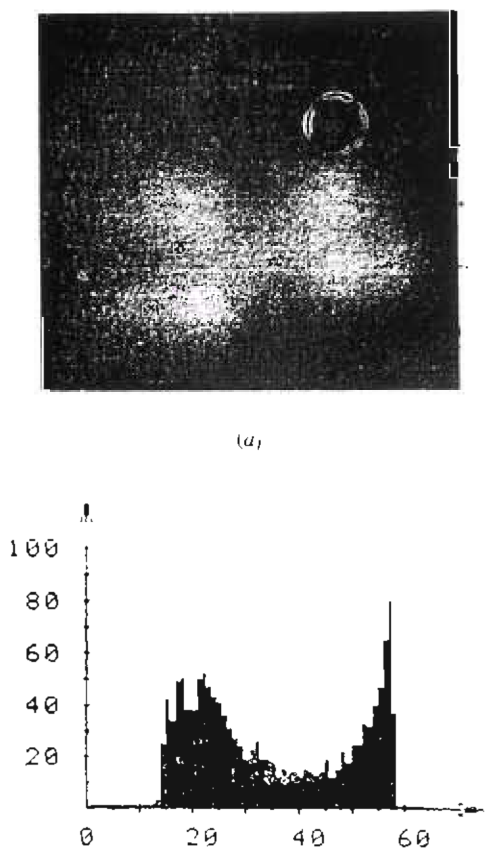
Example

Figure 1 shows a 64 × 64, 64 level image of a blurred chromosome with $l_{min} = 12$ and $l_{max} = 59$. The levels at which minima obtained for $r(X)$, comp(μ) and θ_l when Δb of the S-function is considered to be 4 are as follows:

$l = 40$ for $r(X)$

$l = 33, 48*$ for comp(μ)

$l = 42, 45, 53$ for θ_{l_i}



(*a*)

(*b*)

Fig 1 (*a*) Chromosome Image (*b*) Histogram

Here * denotes the level corresponding to global minimum.

It is thus seen that the compactness measure usually results in more minima as compared to index of fuzziness. The index of fuzziness basically sharpens the histogram and it detects a single threshold in the valley region.

Comp(μ), on the other hand, detects a higher-valued threshold (global minimum) which results in better segmentation (or enhancement) of the chromosome as far as its shape is concerned.

The advantage of the compactness measure over the index value is that it takes fuzziness in the spatial domain (*ie*, the geometry of the object) into consideration in extracting thresholds. $r(X)$, on the other hand, incorporates fuzziness only in grey level.

The enhanced version of the chromosome corresponding to these thresholds are shown in Fig 2.

Other examples showing the effectiveness of the algorithm are available in [12].

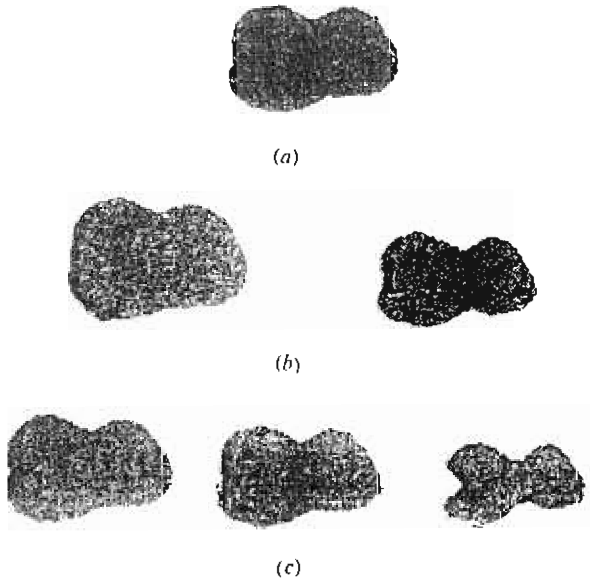


Fig 2 Enchanced/thresholded versions of chromosome for $\Delta b=4$ (a) $r(X)$ (b) Comp(μ) (c) θ_{l_1}

Fuzzy transforms to grey tone image

Let us construct the 'bright image' membership μ_X from X with

$$\mu_X(l) = S(l;a,b,c) \quad (23)$$

$$l_{min} \leqslant l \leqslant l_{max}$$

Let us now generate three more fuzzy sets

$$U = \bar{X} \,|\!-\!|\, X \quad (24a)$$

$$V = 2(X \cap \bar{X}) \quad (24b)$$

$$\text{and } W = X \,|\!-\!|\, \bar{X} \quad (24c)$$

such that

$$\mu_U(l) = \max(0,\ 1-2\ \mu_X(l)) \quad (25)$$

$$\mu_V(l) = 2 \min(\mu_X(l),\ 1-\mu_X(l)) \quad (26)$$

$$\text{and } \mu_W(l) = \max(0,\ 2\ \mu_X(l)-1) \quad (27)$$

$|-|$ denotes bounded difference which is defined for the two sets A and B as

$$A \,|\!-\!|\, B => \max(0, \mu_A(x) - \mu_B(x)) \text{ for all } x \quad (28)$$

U represents the fuzzy set of pixels which belong to $\bar{X}$ (dark) more than to X (bright) and the reverse is the true for W. The set V represents, on the other hand, the set of levels which are semibright *ie*, $\mu_X(l) \approx 0.5$.

Figure 3 shows the transformation. Thus we can represent each pixel intensity as a 3-dimensional vector

$$\mu_X(l) = \begin{bmatrix} \mu_U(l) \\ \mu_V(l) \\ \mu_W(l) \end{bmatrix} \quad (29)$$

$$\text{with} \quad \mu_U(l) + \mu_V(l) + \mu_W(l) = 1 \quad (30)$$

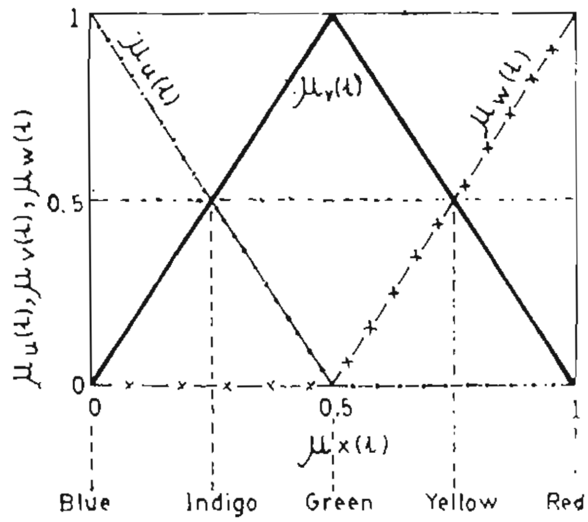


Fig 3 μ_U, μ_V and μ_W Transformations

whose components denote the degree to which the level l may be considered as bright, semibright and dark. It is also seen from Fig 3 that at most two of the components may have non-zero value.

It is to be mentioned here that the similar property of μ_U, μ_V and μ_W can also be obtained with $(1-S)$ function in [0, 0.5], π function in [0,1] and S function in [0.5, 1] respectively.

The supporting intensities in the respective sets U, V and W can thus be obtained with S^{-1} (inverse S function). For example, the transformed level in U corresponding to the level l in X can be obtained with

$$l'_U = S^{-1}(\mu_U(l)) \quad (31)$$

Now if we use l'_U, l'_V and l'_W as the values of three basic colours blue, green and red respectively for each level l in the original image X, we can obtain a pseudo-colour image of the original gray level image X. The change of gray level from lower to higher is viewed as the change of colour from blue to red with spectral continuity in natural way.

The transform described by equations (25)-(27) is very simple and can easily be implemented with a very low computational cost.

Let us now consider the case of classifying or segmenting only the pixels whose intensities lie in a specific range say, $[L_a, L_b]$.

To perform the above task it is preferable to consider

$$l_{max} = L_b, l_{min} = L_a \text{ and } l = l_i - L_a$$
$$l_i \in [L_a, L_b]$$

Then any shade in gray level image can be enhanced and displayed as different colours.

Example

Let us consider a 32 level ($l_{min} = 0, l_{max} = 31$) image and for simplicity, assume

$$c(l) = \frac{1}{l_{max} - l_{min}} = \frac{1}{31}$$

Then for levels 14 and 17 we have

$$\mu_U = \frac{3}{31}, 0$$

$$\mu_V = \frac{28}{31}, \frac{28}{31}$$

and $$\mu_W = 0, \frac{3}{31}$$

Thus the levels 14 and 17 will be displayed as slightly greenish indigo and greenish yellow.

On the other hand, if we consider $L_a = 10$ and $L_b = 21$ then we have for 14 and 17

$$\mu_U = \frac{3}{11}, 0$$

$$\mu_V = \frac{8}{11}, \frac{8}{11}$$

$$\mu_W = 0, \frac{3}{11}$$

As a result, the levels 14 and 17 will be more discriminated as being indigo and yellow respectively.

Fuzzy transforms to colour image

Let us now consider the case of colour image whose pixels are represented by the intensity values of their blue, green and red components (B,G,R). Similar to the case of gray level image, we can write a colour pixel P as

$$P = \begin{bmatrix} \mu_B \\ \mu_G \\ \mu_R \end{bmatrix}, \mu_B, \mu_G, \mu_R \in [0\ 1] \qquad (32)$$

where the elements μ_i, i = B,G,R denote the degree to which the components B,G,R belong to the fuzzy sets 'Blue', 'Green' and 'Red' respectively.

Let us now define eight fuzzy colour sets [14]

$$1 \,|\!-\!|\, (B \cup G \cup R) = \bar{B}\bar{G}\bar{R}, \text{ say} \qquad (33a)$$
$$R \,|\!-\!|\, (B \cup G) = \bar{B}\bar{G}R, \text{ say} \qquad (33b)$$
$$(G \cap R) \,|\!-\!|\, B = \bar{B}\,G\,R, \text{ say} \qquad (33c)$$
$$G \,|\!-\!|\, (B \cup R) = \bar{B}\,G\,\bar{R}, \text{ say} \qquad (33d)$$
$$(B \cap G) \,|\!-\!|\, R = B\,G\,\bar{R}, \text{ say} \qquad (33e)$$
$$B \,|\!-\!|\, (G \cup R) = B\,\bar{G}\,\bar{R}, \text{ say} \qquad (33f)$$
$$(B \cap R) \,|\!-\!|\, G = B\,\bar{G}\,R, \text{ say} \qquad (33g)$$
$$B \cap G \cap R = B\,G\,R, \text{ say} \qquad (33h)$$

characterised by the membership functions

$$T_0 = \max(0, 1 - \max(\mu_B, \mu_G, \mu_R)) \qquad (34a)$$
$$T_1 = \max(0, \mu_R - \max(\mu_B, \mu_G)) \qquad (34b)$$
$$T_2 = \max(0, \min(\mu_G, \mu_R) - \mu_B) \qquad (34c)$$
$$T_3 = \max(0, \mu_G - \max(\mu_B, \mu_R)) \qquad (34d)$$
$$T_4 = \max(0, \min(\mu_B, \mu_G) - \mu_R)) \qquad (34e)$$
$$T_5 = \max(0, \mu_B - \max(\mu_G, \mu_R)) \qquad (34f)$$
$$T_6 = \max(0, \min(\mu_B, \mu_R) - \mu_G)) \qquad (34g)$$
$$\text{and } T_7 = \min(\mu_B, \mu_G, \mu_R). \qquad (34h)$$

It is therefore seen that the sets characterised by $T_1/T_3/T_5$ represent the set of pixels which belong more to red/green/blue than to blue or green/blue or red/green or red. Similarly, $T_2/T_4/T_6$ represent the sets of pixels which belong equally and more to green and red/blue and green/blue and red than to blue/red/green. T_7 represents the set of pixels which belong equality to blue, green and red, and the reverse is the true for T_0.

The value of each T_i, i = 0,1,2,......7 can therefore be interpreted as the degree of membership (Co-efficients) that the pixel P belongs to the corresponding one of the eight colours namely, white and black, three primary colours (blue, green and red) and three secondary colours (yellow, indigo and purple (or violet)).

Again,

$$\sum_i T_i = \max\{\mu_B, \mu_G, \mu_R\} \qquad (35)$$

$$i = 0,1,2,\ldots\ldots 7.$$

For any pixel P, at most three of T_i's (i=1,2,......7) can have non-zero value. These non-zero values correspond to the three vertices of a sector in the chroma graph Fig (4) in which P lies.

Any colour vector (B,G,R) can be mapped to a point $P(\theta,r)$ within the unit circle shown in Fig 4 θ and r are

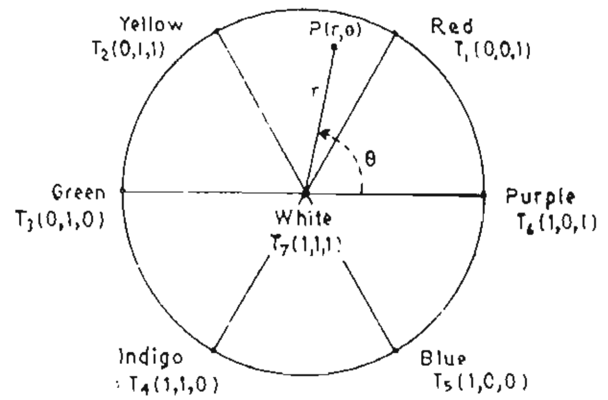


Fig 4 Chromaticity graph

defined as

$$\theta = \frac{\pi}{3}\,\frac{i\,T_i + j\,T_j}{T_i + T_j} \qquad (36)$$

$$r = (T_i + T_j)/(T_i + T_j + T_7) \qquad (37)$$

$$T_i,\ T_j \in \{T_1,\ T_2 \ldots T_6\}$$

where θ is modulo 2π, and θ and r represent the colour tone (or spectral hue) and relative saturation respectively.

Example

Let $P = (\mu_B, \mu_G, \mu_R) = (0.2, 0.5, 0.9)$

We have

$T_1 = \max(0, 0.9 - \max(0.2, 0.5)) = 0.4$

$T_2 = \max(0, \min(0.5, 0.9) - 0.2) = 0.3$

$T_0 = T_3 = T_4 = T_5 = T_6 = 0$

$T_7 = \min(0.2, 0.5, 0.9) = 0.2$

$$\theta = \frac{\pi}{3}\,\frac{0.4 + 0.6}{0.4 + 0.3} = \frac{\pi}{0.21} = \frac{10\pi}{21}$$

$$r = \frac{0.7}{0.9} = \frac{7}{9}$$

$P(\theta,r)$ is shown in Fig 4 and is located in the sector of which three vertices are white (T_7), Red (T_1) and Yellow (T_2).

Using the notation of Colour vectors we may write

$$P = (\mu_B,\mu_G,\mu_R) = T_1(0,0,0) + T_2(0,0,1) + T_3(0,1,1) + T_4(0,1,0) + T_5(1,1,0) + T_6(1,0,0) + T_7(1,0,1) + T_8(1,1,1) \qquad (38)$$

Since for any $P = (B,G,R)$, at most three terms in the right hand side of equation (38) can be non-zero corresponding to three vertices of a certain sector in Fig 4, we can say that any colour can be decomposed into linear combination of white (1,1,1), one of the primary colours (0,0,1), (0,1,0), (1,0,0) and one of the secondary colours (0,1,1), (1,1,0), (1,0,1). The combination co-efficients are represented by the corresponding T_i values. In other words, we can write

$$P = (\mu_B, \mu_G, \mu_R) = (\mathrm{Pr}) + T_j(\mathrm{Se}) + T_7(\mathrm{Wh}) \qquad (39)$$

where (Wh) = (1,1,1)

T_7(Wh) represents the signal for white

T_i(Pr) represents the signal for one of the primary colours

T_j(Se) represents the signal for one of the secondary colours

$$T_i \in \{T_1, T_3, T_5\}$$

$$T_j \in \{T_2, T_4, T_6\}$$

$$T_i + T_j + T_7 = \max\{\mu_B, \mu_G, \mu_R\} \qquad (40)$$

Example

Let $P = (\mu_B,\mu_G,\mu_R) = (0.2, 0.7, 0.4)$

Then we can write using vector operations

$$(0.2,0.7,0.4) = 0(0,0,0) + 0(0,0,1) + 0.2(0,1,1) + 0.3(0,1,0) + 0(1,1,0) + 0(1,0,0) + 0(1,0,1) + 0.2(1,1,1)$$

This means that

$$0.2|_{blue} + 0.7|_{green} + 0.4|_{red} = 0.2|_{yellow} + 0.3|_{green} + 0.2|_{white}$$

To prove the validity of this equation we break the right hand side which is

$$0.2|_{red} + 0.2|_{green} + 0.3|_{green} + 0.2|_{blue} + 0.2|_{red} + 0.2|_{green} = 0.2|_{blue} + 0.7|_{green} + 0.4|_{red} = \text{left hand side}$$

Colour recognition

In addition to the parameters θ and r in Fig 4 let us define another parameter

$$z = (T_i + T_j + T_7)/3 \qquad (41)$$

$$T_i,\ T_j \in \{T_1, T_2 \ldots T_6\}$$

which represents the relative chroma-luminance of (B,G,R). Therefore, the equations (36), (37) and (41) perform a one-to-one mapping from (μ_B,μ_G,μ_R) space which is a unit cube to (θ,r,Z) psycological colour space (a cylinder with unit radius and unit height).

It is to be noted that composition of a couple of complementary colours (*ie*, $|\theta_1 - \theta_2| = \pi$) will yield white if $r_1 z_1 = r_2 z_2$. Again, it can be shown that the (r,θ) value of a colour P remains same when B,G,R is multiplied by a constant. This relative invariance property may be helpful for colour recognition by machine. For example, for two objects with the same colour but under different lighting conditions, the colour sensor produced two different colour vectors with

$$(\mu_{B_1}, \mu_{G_1}, \mu_{R_1}) = k\,(\mu_{B_2}, \mu_{G_2}, \mu_{R_2}) \qquad (42)$$

where k is a real number.

Then these two vectors will be mapped to the same point in (r,θ) chroma space. In other words, the results of recognition will not be affected by changes in lighting condition. The exactness of recognition will therefore be greatly improved and it will be more convenient to recognise objects in $2D$ (θ,r) chroma space than in 3D (B,G,R) colour space

Example

Let $P_1 = (0.1, 0.2, 0.3)$

and $P_2 = 2P_1 = (0.2, 0.4, 0.6)$

Then for P_1, $T_1 = 0.1$, $T_2 = 0.1$, $T_7 = 0.1$

$$T_0 = T_3 = T_4 = T_5 = T_6 = 0$$

$$\theta = \frac{\pi}{3} \cdot \frac{0.1 + 0.2}{0.2} = \frac{\pi}{2}$$

$$r = \frac{0.2}{0.3} = \frac{2}{3}$$

for P_2, $T_1 = 0.2$, $T_2 = 0.2$, $T_7 = 0.2$

$$T_0 = T_3 = T_4 = T_5 = T_6 = 0$$

$$\theta = \frac{\pi}{3} \cdot \frac{0.2 + 0.4}{0.4} = \frac{\pi}{2}$$

$$r = \frac{0.4}{0.6} = \frac{2}{3}$$

Similarly, $P = (0, 0.1, 0.1) = (0, 0.5, 0.5) = (0,1,1)$ and will correspond to yellow with $r = 1$ and $\theta = 2\pi/3$.

Colour enhancement and segmentation

Since θ represents colour tone or spectral hue of a colour, its value may be applied for classifying pixels into different classes based on their spectral characteristics. We can therefore compute histogram of θ in an image and, as in the case of gray tone image processing, we can determine thresholds automatically [10,12] by sharpening the histogram, without referring to the histogram or by any method based on transition or co-occurrence matrix.

Let us now explain the method of enhancement of θ between two colour pixels. The underlying principle is that, for all pixels lying in a sector with vertices T_i, T_j and T_7, if we replace the original values μ_B, μ_G, μ_R with T_i, T_j and T_7 (the correspondence between (T_i, T_j, T_7) and (μ_B, μ_G, μ_R) can be arbitrary but should make the resultant image more natural) then we can obtain an image which is segmented by classifying its pixels on the basis of chromaticities. That is, the task is to give a transform of the form

$$\begin{bmatrix} \mu'_B \\ \mu'_G \\ \mu'_R \end{bmatrix} = D \begin{bmatrix} T_i \\ T_j \\ T_7 \end{bmatrix} \qquad (43)$$

where D is a unit matrix or its transforms defining the correspondence between (B,G,R) and (T_i, T_j, T_7).

Example [14]

Let $P_1 = (0, 0.6, 0.4)$

$P_2 = (0.3, 0.8, 0.4)$

Then for P_1 we have $T_2 = 0.4$, $T_3 = 0.2$, $T_7 = 0$

for P_2 we have $T_2 = 0.1$, $T_3 = 0.4$, $T_7 = 0.3$

P_1 and P_2 are therefore mapped to points

$P'_1\left(\frac{7}{9}\pi, 1\right)$ and $P'_2\left(\frac{15}{15}\pi, \frac{5}{8}\right)$ respectively laying in the sector yellow, green and white

$$\text{Let } D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

It means we replace original (μ_B, μ_G, μ_R) by $(T_7, T_3$ and $T_2)$. As a result, the new colours of P_1 and P_2 will be displayed as saturated orange and semi-saturated indigo with (θ,r) value equal to $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{11}{9}\pi, \frac{3}{4}\right)$ respectively.

It is now therefore no longer difficult to segment such regions based on their new θ values.

If we consider, on the other hand,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ie, $(\mu_B, \mu_G, \mu_R) = (T_2, T_3, T_7)$ then the points P_1 and P_2 will be mapped to $\left(\frac{3}{2}\pi, 1\right)$ and $\left(\frac{7}{9}\pi, \frac{3}{4}\right)$ respectively with distinct colour. That is, the difference $|\theta_1 - \theta_2|$ is enhanced using that transformation.

EXPERT SYSTEM

An expert system can be viewed as a rule based AI application program having mainly three-fold task, namely, the front-end compiler, the inference engine and the question-answer program. The front end compiler compiles the input data received from the user or any other connected software. The purpose of compilation is to transform the data into a format understandable to the inference engine. The inference engine then uses this formated data in conjunction with a knowledge base which is in the form of production rules. Based on the conclusions reached during the inference phase and the list of compiled data, the question-answer program traces a certain conclusion and shows the user how this conclusion was reached *ie*, what rules and data took part in drawing that conclusion.

Let us now explain the uncertainty (fuzziness) involved in different parts of an expert system and the ways in which the fuzzy tools may be able to manage them.

Fuzziness in facts and rules of expert system

The knowledge base of an expert system consists of a collection of propositions which represent the facts and a collection of conditional propositions which constitute the rules. For example, the facts may be:

(*i*) Ashim is an M. Tech. student

(*ii*) The population of Calcutta is approximately 10,000000

(*iii*) The age of John is more or less 20

(*iv*) Indian Statistical Institute is an internationally reputed institution

(*v*) IEEE Transactions are usually very good journals

and the rules taken from assessment of skeletal maturity [15] and MYCIN [16] may be:

(*i*) *If* (*a*) the maximum diameter of the epiphysis is half or more the width of the metaphysis,

(*b*) the epiphysis has broadened chiefly at its lateral side, so that this portion is thicker and more rounded, the medial portion more tapering, and

(*c*) the centre third of the proximal surface is flat and slightly thickened and the gap between it and the radial metaphysis has narrowed to about a milli-meter,

then the stage of maturity is likely *D*.

(*ii*) *If* (*a*) the route of the administration of the penicillin is oral, and

(*b*) there is a gastrointestinal factor which may interfere with the absorption of the penicillin.

then there is suggestive evidence (certainty factor = 0.6) that the route of administration of the penicillin is not adequate.

The conventional approaches to the management of uncertainty in facts and rules of expert system and hence in the conclusions drawn therefrom are intrinsically inadequate because they fail to come to grips with the fact that much of the uncertainty is possibilistic rather than probabilistic in nature. The employment of fuzzy logic [17, 18] as a framework for the management of uncertainty in expert systems makes it possible to consider a number of issues which cannot be dealt with effectively or correctly by the conventional techniques. Some of the important issues are:

(*a*) The fuzziness of antecedents and/or consequents in rules such as—If X is small than Y is large with CF = 0.8.

(*b*) Partial match between the antecedent of a rule and a fact supplied by the user.

Since the number of rules in an expert system is usually relatively small, it is likely that a fact such as X is A^* may not match exactly the antecedent of any rule of the form 'If X is A then Y is B with CF = α'.

(*c*) The presence of fuzzifiers [10, 13] such as most, usually, small etc in the antecedent and/or the consequent of a rule.

Again, in the existing expert systems, the fuzziness of the knowledge base is ignored because neither predicate logic nor probability-based methods provide a systematic basis for dealing with them. As a result, the fuzzy rules and facts are treated as if they are nonfuzzy which leads to conclusions whose validity is questionable. For example, [19].

John has duodenal ulcer with certainty factor CF = 0.3. Here has duodenal ulcer is a fuzzy predicate. Therefore, John may have it to a degree, the meaning of the certainty becomes ambiguous. In other words, does CF = 0.3 mean that:

(*a*) John had duodenal ulcer to the degree 0.3

or (*b*) the probability of the fuzzy event 'John has duodenal ulcer' is 0.3?

In order to make the later interpretation meaningful, the definition of probability of fuzzy event [20] acts as an important tool.

Relevance of Fuzzy logic

The main features of fuzzy logic which are of relevance to the management of uncertainty in an expert system are explained below [19].

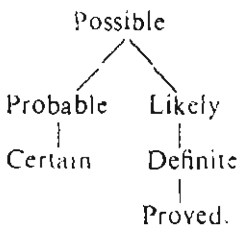
(*a*) Let T represent the unit interval. Then a truth value in fuzzy logic *eg*, very true, may be interpreted as a fuzzy subset of the unit interval which defines the possibility distribution [17] associated with the truth-value in question. Thus a fuzzy truth value may be viewed as an imprecise characterisation of an intermediate truth value.

(*b*) In fuzzy logic, the predicates may be crisp (*ie*, father of, mortal etc) or more generally, fuzzy *eg*, tired heavier, friend of etc.

(*c*) Fuzzy logic allows, in addition to two-valued and multi-valued logics, the use of fuzzy quantifiers exemplified by most, many, several, about, frequently etc. Such quantifiers may be interpreted as fuzzy numbers which provide an imprecise characterisation of the cardinality of one or more fuzzy or nonfuzzy predicate.

(d) In two-valued logical system, a proposition p may be qualified, principally, by associating with p a truth-value, true or false, a model operator such as possible or necessary and an intensional operator such as know, believe etc. In fuzzy logic, principal modes of qualification are of three types namely, (*a*) truth qualification expressed as p is τ, in which τ is a fuzzy truth value, (*b*) probability qualification expressed as p is λ, in which λ is a fuzzy probability and (*c*) possibility qualification expressed as p is π, in which π is a fuzzy possibility *eg*, quite possible, almost impossible etc.

A simple hierarchy showing the relationships between the terms such as possible, certain and definite is given



below. A term lower down on the hierarchy is stronger than one higher up. For example, what is possible may not be probable and what is improbable may not be impossible—Zadeh's consistency principle [21].

Let us discuss two examples for illustrating the above criteria in an expert system.

Example

Consider the problem of evaluation by compiler of a data [8]:

The age of Tom is not more or less than 20.

Assuming more or less of x means $x-10\%$ to $x+10\%$, we have the lower bound (LB) and upper bound (UB) as:

LB (more or less 20 years old) = 18 years *ie*, $\mu_1 = 0.18$, say.

UB (more or less 20 years old) = 22 years *ie*, $\mu_2 = 0.22$, say

For the adjective 'not more or less', we have

LB (not more or less 20 years old) = 1— UB (more or less 20 years old) = 1—0 22 = 0 78

UB (not more or less 20 years old) = 1— LB (more or less 20 years old) = 1—0.18 = 0.82

Therefore the compiled data may be stored on the black board (*ie*, the contents of the black board is accessable from any part of the system) as

age	Tom	0.78	0.82	1

where the last entry denotes the certainty factor (CF). CF = 1 is assumed for any data provided by the user.

Fuzzy expected value and fuzzy expected interval may be used for compiling incomplete or vague information.

Example [22]

Let us consider a fuzzy expert system to be constructed to evaluate possible design of an electronic device consisting of three components, connected in series. The failure of any component will therefore cause failure of the entire device. In order to improve the reliability of the device, back-up units are installed for each component. The data on the reliability, cost and weight of each component, as a function of the number of back-up units employed is shown in Table 1. All the data mentioned are fuzzy number.

TABLE 1 The data on the reliability cost and weight of each component

No. of back-ups	Reliability	Cost	Weight
0	α_{i0}	β_{i0}	γ_{i0}
1	α_{i1}	β_{i1}	γ_{i1}
2	α_{i2}	β_{i2}	γ_{i2}
3	α_{i3}	β_{i3}	γ_{i3}
.	.	.	.
.	.	.	.
.	.	.	.

For example, α_{20} = 'around' 0.70 for no back-ups,

α_{21} = 'approximately' 0.80 for one back-up.

α_{22} = 'about' 0.90 for two back-ups etc.

The vagueness of these statements are incorporated by an appropriate fuzzy number.

Given an input of the form $V_i = \{2,3,4\}$ (where 2,3,4 denote the number of back-ups employed for 1st, 2nd and 3rd components respectively), the system computers the fuzzy numbers Re, Cost, Wt representing the reliability, cost and weight of the electronic device as:

$$R_{e_l} = \alpha_{12} (.) \alpha_{23} (.) \alpha_{34} \quad (44)$$

$$Cost_l = \beta_{12} (+) \beta_{23} (+) \beta_{31} \quad (45)$$

$$Wt_l = \gamma_{12} (+) \gamma_{23} (+) \gamma_{34} \quad (46)$$

where (.) and (+) denote the fuzzy multiplication and addition [9] respectively.

Let one of the rules in the expert system be:

R_1 = if reliability is greater than or equal to high and cost is more or less acceptable and weight is not very high, *then* the design is good.

For the given input V_l, the value of this rule is:

$$\Gamma(R_1) = \min(\theta_1, \theta_2, \theta_3, \tau_{R_l}) \quad (47)$$

where

$$\theta_1 = R_{e_l}\ \overline{GTE} A_3$$

$$\theta_2 = Cost_l\ \overline{\overline{E}}\ B_2, \quad \overline{\overline{E}} \underline{\underline{\Delta}} \text{ more or less } \overline{E}$$

$$\theta_3 = Wt_l\ \overline{LT}\ C_3$$

$\overline{GTE}$, E and $\overline{LT}$ denote the fuzzy relations representing 'greater than or equal', 'equal' and 'less than' respectively, on the set of fuzzy numbers. A_3, B_2 and C_3 are the fuzzy numbers high, acceptable and very high corresponding to the descriptions reliability, cost and weight respectively. T_{R_1} is the prior degree of confidence in the rule R_1. $\Gamma(R_1)$ is then called the posterior degree of confidence in R_1 *ie*, it represents the degree of confidence in 'the design is good' (which is called action) of the rule R_1.

Let design quality = {poor, acceptable, good}. Then $\Gamma(R_1)$ denotes both the degree of confidence in 'good' and also the degree of membership of 'good' in the fuzzy subset 'design quality'.

Let us consider another rule say,

R_2 = if reliability is less than high and (cost is very high or weight is very high), then the design is poor.

The value of R_2 for V_l is then

$$\Gamma(R_2) = \min(\theta_1, \theta_2, \tau_{R_2}) \quad (48)$$

where

$$\theta_1 = R_{e_l}\ \overline{LT} A_3 \quad (49)$$

$$\theta_2 = \max(Cost_l\ \overline{E} B_3\ Wt_l\ \overline{E} C_3 \quad (49b)$$

$$B_3 \underline{\underline{\Delta}} \text{ very high} \quad (49c)$$

The output actions will produce a fuzzy set of design classifications for each input. Suppose we have a set of rules R with conclusion 'poor', then we define

$$\mu_{V_l}(\text{poor}) = \max\{\Gamma(R)\} \quad (50)$$

representing the membership value of the input corresponding to the action 'poor'. Here the maximum is taken over all rules having the same conclusion.

Example

Consider the problem of developing an expert system for determining stage of maturity, age and adult height from X-rays of hand and wrist [15]. With growth of a child, epiphysis continuous to grow larger and ultimately because of the styloid process, fusion of epiphysis and metaphysis begins.

The features for this structural development therefore include the contour, shape and orientation of the metaphysis and epiphysis including palmar and dorsal surfaees. Typical rule for describing one of the nine stages of maturity of radius as shown in Fig 5 is [15]:

D

Fig 5 Maturity stage D of radius

If (*i*) the maximum diameter of epiphysis is half or more the width of the metaphysis,

(*ii*) the epiphysis has broadened chiefly at its lateral side, so that this portion is thicker and more rounded, the medial portion more tapering, and

(*iii*) the centre third of the proximal surface is flat and slightly thickened and the gap between it and the radial metaphysis has narrowed to about a millimeter

then the stage of maturity is likely D.

From the above mentioned rule, the terms 'more rounded', 'tapering', 'slightly thickened' and 'flat' are seen to be fuzzy. For any curve b, the degree of arcness may be defined as [10]

$$\mu_{arc}(b) = \left(1 - \frac{l}{p}\right)^{F_e} \quad (51)$$

where l is the length of the line segment joining the two extreme points of the arc b, p is the length of the arc b and $F_e > 0$. When b is a line segment, we have $p = l$ and $\mu_{arc}(b) = 0$, whatever F_e may be. As the sharpness of b increases, μ_{arc} also increases and $\mu_{arc} \in (0,1)$. Let us now define

$$f(\mu_{arc}) = \left[1 + \left(\frac{|\mu_{arc} - \mu|}{F_d}\right)^{F_e}\right]^{-1} \quad (52)$$

which represents the set of sharp, fair and gentle curves corresponding to $\mu = 1$, 0.5 and 0. F_e and F_d control fuzziness in a set.

Therefore, the compiled string X representing the contour of epiphysis of the stage mentioned before may be represented as:

$$X = a^m \, b \, a^n \, b, \; m, n \geqslant 0$$

b denotes 'fair' or 'sharp', a denotes 'line segment'.

Similarly, the compiled string Y representing the interior of the epiphysis contour may be written as [45]:

$Y = L^* \, b \, L^* \, b$

b denotes 'not gentle'

$L^* = L, ML, LM, a^x M a^y, M a^x M a^y M,$
$a^x M a^y M$ or $M a^x M a^y$

$L = a^x, a^x M a^y M a^z$ or $a^x M M a^z$

$M = b$ or $\bar{b}$ (gentle)

$x, y, z > 0$

The string Y differentiates the stages C and D from E. Furthermore, $r = D_E/W_M \leqslant 0.5$ (D_E and W_M denote the maximum diameter of the epiphysis and the width of the metaphysis) discriminates the stage D from C.

CONCLUSIONS

Different fuzzy parameters which may be considered as useful tools for the management of uncertainty (difficulty) in problems of pattern recognition, image processing and vision, and expect system are enunciated. These are used (i) for segmenting image region when it is not justified to commit ourselves to a specific segmentation, (ii) for enhancing and segmenting different colour regions when the colours between regions are visually alike, and (iii) for designing expert system and analysing the transmission of uncertainty in information from knowledge base to the uncertainty in the validity of its conclusions when the rules and facts are neither totally certain nor precise and complete

The algorithms are simple, approximate and yet effective This approach is more general and natural, and a very useful supplement to the classical approaches. For example, the conventional segmentation algorithms implicitly assume that the clusters are disjoint, but in practice, the clusters in many cases are not completely disjoint, rather, the separation is a fuzzy notion. In such cases, it is thus not desirable to commit ourselves to a specific thresholding.

Let us now consider the case of designing an expert system. Fuzzy expected value and intervals are found to be very much effective in extracting information when the data and rules are imprecise, incomplete or not totally reliable. Fuzzy hedges and operations can be used to handle and to quantify the fuzzy numbers *eg*, very, much, almost, likely etc.

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