Pattern Recognition Letters 10 (1989) 17–23
North-Holland

July 1989

Fuzzy skeletonization of an image

Sankar K. PAL
Electronics & Communication Sciences Unit, Indian Statistical Institute, Calcutta 700035, India

Received 11 October 1988

Abstract: Algorithms based on minimisation of fuzzy compactness are developed whereby it is possible to obtain automatically both fuzzy and nonfuzzy skeletons of an image. Compactness measure takes into account the fuzziness in geometry of regions in an image. The membership plane representing the subset 'Core line' is made gray level and position dependent such that its value decreases as one moves away towards the edges of object. Examples illustrating the performance of the techniques are given.

Key words: Image skeletonization, thinning, compactness, fuzzy sets.

1. Introduction

The problem of skeletonization or thinning plays a key role in image processing, analysis and recognition because of the simplicity of object representation it allows. Two-tone elongated objects can be thinned to arcs and lines, without changing their connectedness, by repeatedly deleting the border pixels whose absence do not affect the connectivity property of the object. Any gray level image for its thinning can be converted to a binary image by thresholding and the two-tone thinning algorithms can then be applied, but the selection of threshold has to be made judiciously.

When the regions in an image are ill-defined (fuzzy), it is natural and also appropriate to avoid committing ourselves to a specific segmentation/thresholding and hence skeletonization by allowing the segments and skeletons to be fuzzy subsets of the image. Fuzzy geometric properties (which are the generalization of those for ordinary regions) as defined by Rosenfeld [1] seem to provide a helpful tool for such analysis. These concepts were found recently, to provide an algorithm for both fuzzy and nonfuzzy segmentation of an image [2].

The present paper is an attempt to perform the task of skeleton extraction automatically with the help of fuzzy compactness [1] measure which takes into account fuzziness in the spatial domain, i.e. in the geometry of the image regions. The membership function for obtaining fuzzy skeleton (core line) plane is guided by three properties of a pixel, namely possessing maximum intensity, horizontally middle position and vertically middle position. Optimum fuzzy thinned versions are then extracted by minimising compactness of the fuzzy skeleton plane of the image.

From the optimum fuzzy skeleton thus produced, one may also obtain its nonfuzzy (crisp) single pixel width version using the algorithm of Pal et al. [3] which retains only those pixels which have strong skeleton-membership value compared to their neighbors.

2. Fuzzy geometry of image subsets [1, 2]

Image definition

An image X of size $M \times N$ and L levels can be considered as an array of fuzzy singletons, each having a value of membership denoting its degree of brightness relative to some brightness level l, $l = 0, 1, 2, \ldots, L-1$. In the notation of fuzzy sets, we may therefore write $X = \{\mu_X(x_{mn}) = \mu_{mn}/x_{mn};\ m = 1, 2, \ldots, M;\ n = 1, 2, \ldots, N\}$ where $\mu_X(x_{mn})$ or μ_{mn}/x_{mn} $(0 \leqslant \mu_{mn} \leqslant 1)$ denotes the grade of possessing some property μ_{mn} (e.g., brightness, edginess, smoothness etc.) by the (m,n)th pixel intensity x_{mn}.

Image geometry

Rosenfeld [1] extended the concepts of digital picture geometry to fuzzy subsets and generalized some of the standard geometric properties of and relationships among regions to fuzzy subsets. Among the extensions of the various properties, we only discuss here the area, perimeter and compactness of a fuzzy image subset, characterized by $\mu_X(x_{mn})$, which will be used in the following section for developing skeleton selection algorithms. In defining the above mentioned parameters we replace $\mu_X(x_{mn})$ by μ for simplicity.

The area of μ is defined as

$$a(\mu) \triangleq \int \mu \tag{1(a)}$$

where the integral is taken over any region outside which $\mu = 0$.

If μ is piecewise constant (for example, in a digital image) $a(\mu)$ is the weighted sum of the areas of the regions on which μ has constant values, weighted by these values. In other words, for an image $X = \{\mu_{mn}/x_{mn}\}$:

$$a(x) = \sum_m \sum_n \mu_{mn}, \qquad m = 1, 2, \ldots, M; \quad n = 1, 2, \ldots, N. \tag{1(b)}$$

For the piecewise constant case, the perimeter of μ is defined as

$$p(\mu) \triangleq \sum_{i,j} \sum_k |\mu_i - \mu_j|\,|A_{ijk}|, \qquad i,j = 1, 2, \ldots, r; \quad i < j; \qquad k = 1, 2, \ldots, r_{ij}. \tag{2(a)}$$

This is just the weighted sum of the length of the arcs A_{ijk} along which the ith and jth regions having constant μ values μ_i and μ_j respectively meet, weighted by the absolute difference of these values. For an image X, its perimeter will be

$$p(X) = \sum_{m=1}^{M} \sum_{n=1}^{N-1} |\mu_{mn} - \mu_{m,n+1}| + \sum_{n=1}^{N} \sum_{m=1}^{M-1} |\mu_{mn} - \mu_{m+1,n}| \tag{2(b)}$$

(excluding the frame of the image).

The compactness of μ is defined as

$$\mathrm{comp}(\mu) \triangleq a(\mu)/p^2(\mu). \tag{3}$$

For crisp sets, this is largest for a disk, where it is equal to $1/4\pi$. For a fuzzy disk where μ depends only on the distance from the origin (center), it can be shown that

$$a(\mu)/p^2(\mu) \geqslant 1/4\pi. \tag{4}$$

In other words, of all possible fuzzy disks, the compactness is smallest for its crisp version. For this reason, minimization (rather than maximization) of compactness is used here as a criterion for optimum skeleton extraction.

3. Skeleton extraction

Fuzzy segmentation [2]

To have an initial guess regarding the extent of the objects in an image X, it is desirable to extract its fuzzy segmented version without committing to a specific thresholding. Various algorithms in this context are reported by Pal and Rosenfeld [2] minimising ambiguity both in grayness and in spatial geometry.

The algorithms first of all extract different 'bright image' membership planes $\{\mu_{mn}\}$ using Zadeh's standard S function [4] with varying cross-over point. Among them, the one having minimum spatial and intensity fuzziness as measured by entropy, index of fuzziness and compactness measures is regarded as fuzzy segmented version of X. The results of the algorithms on blurred and noisy images are also available in [2].

Coreline membership plane

After obtaining a fuzzy segmented version of the input image X, the membership function of a pixel denoting the degree of its belonging to the subset 'Core line' (skeleton) is determined by three factors. These include the properties of possessing maximum intensity, occupying vertically and horizontally middle positions from the edges (pixels beyond which the membership value in the fuzzy segmented image is zero) of the object.

Let $x_{\max}$ be the maximum pixel intensity in the image and $p_0(x_{mn})$ be the function which assigns the degree of possessing maximum brightness to the (m,n)th pixel. Then the simplest way to define $p_0(x_{mn})$ is

$$p_0(x_{mn}) = x_{mn}/x_{\max}. \tag{5}$$

It is to be mentioned here that one may use other monotonically nondecreasing functions (including Zadeh's standard S function [4]) available in [5, 6] to define $p_0(x)$ with a flexibility of varying cross-over point. Equation (5) is the simplest one with fixed cross-over point at $x_{\max}/2$.

Let x_1 and x_2 be the distances of x_{mn} from the left and right edges respectively. (The distance being measured by the number of units separating the pixel under consideration from the first background pixel along that direction). Then $p_h(x_{mn})$ denoting the degree of occupying the horizontally central position in the object is defined as

$$\begin{aligned} p_h(x_{mn}) &= \frac{x_1}{x_2} && \text{if } d(x_1, x_2) \leqslant 1 \text{ and } x_1 \leqslant x_2, \\ &= \frac{x_2}{x_1} && \text{if } d(x_1,x_2) \leqslant 1 \text{ and } x_1 \geqslant x_2, \\ &= \frac{2x_1}{x_2(x_1+x_2)} && \text{if } d(x_1,x_2) > 1 \text{ and } x_1 < x_2, \\ &= \frac{2x_2}{x_1(x_1+x_2)} && \text{if } d(x_1,x_2) > 1 \text{ and } x_1 > x_2 \end{aligned} \tag{6}$$

where $d(x_1,x_2) = |x_1 - x_2|$.

Similarly, the vertical function is defined as

$$\begin{aligned} p_v(x_{mn}) &= \frac{y_1}{y_2} && \text{if } d(y_1,y_2) \leqslant 1 \text{ and } y_1 \leqslant y_2, \\ &= \frac{y_2}{y_1} && \text{if } d(y_1,y_2) \leqslant 1 \text{ and } y_1 \geqslant y_2, \\ &= \frac{2y_1}{y_2(y_1+y_2)} && \text{if } d(y_1,y_2) > 1 \text{ and } y_1 < y_2, \\ &= \frac{2y_2}{y_1(y_1+y_2)} && \text{if } d(y_1,y_2) > 1 \text{ and } y_1 > y_2. \end{aligned} \tag{7}$$

Equations (6–7) assign high values (≈ 1.0) for pixels near core and low values to pixels away from the core. The factor $(x_1 + x_2)$ or $(y_1 + y_2)$ in the denominator takes into consideration the extent of the object segment so that there is an appreciable amount of change in the property value for the pixels not belonging to the core.

These primary membership functions p_0, p_h and p_v may then be combined as either

$$\mu_C(x_{mn}) = \max\{\min(p_0,p_h), \min(p_0,p_v), \min(p_h,p_v)\} \tag{8}$$

or

$$\mu_C(x_{mn}) = w_1p_0 + w_2p_h + w_3p_v \tag{9a}$$

$$\text{with } w_1 + w_2 + w_3 = 1 \tag{9b}$$

to define the grade of belonging of x_{mn} to the subset 'Core line' of the image.

Equation (8) involves connective properties using max and min operators such that $\mu_C = 1$ when at least two of the three primary properties take values of unity. All the three primary membership values are given equal weight in computing the μ_C value. Equation (9), on the other hand, involves a weighted sum (weights being denoted by w_1, w_2 and w_3). Usually, one can consider the weight w_1 attributed to p_0 (property corresponding to pixel intensity) to be higher than the other two and $w_2 = w_3$.

Equation (8) or (9) therefore extracts (using both gray level and spatial information) the subset 'Core line' such that the membership value decreases as one moves away towards the edges (boundary) of object regions.

Figure 1. Input 'Shu' image

Optimum α-cut

Given the $\mu_C(x_{mn})$ plane developed in the previous stage with the pixels having been assigned values indicating their degree of membership to 'Core line', the optimum (in the sense of minimising ambiguity in geometry or in spatial domain) skeleton can be extracted from one of its α-cuts having minimum comp(μ) value (eq.(3)). The α-cut of $\mu_C(x_{mn})$ is defined as

$$\mu_{C_\alpha} = \{x_{mn} \in X \mid \mu_C(x_{mn}) \geqslant \alpha\}, \quad 0 < \alpha < 1. \tag{10}$$

Modification of α will therefore result in different fuzzy skeleton planes with varying comp(μ) value. As α increases, the comp(μ) value initially decreases to a certain minimum and then for a further increase in α, the comp(μ) measure increases.

The initial decrease in comp(μ) value can be explained by observing that for every value of α, the

Figure 2. Subset 'Core line' of Figure 1.

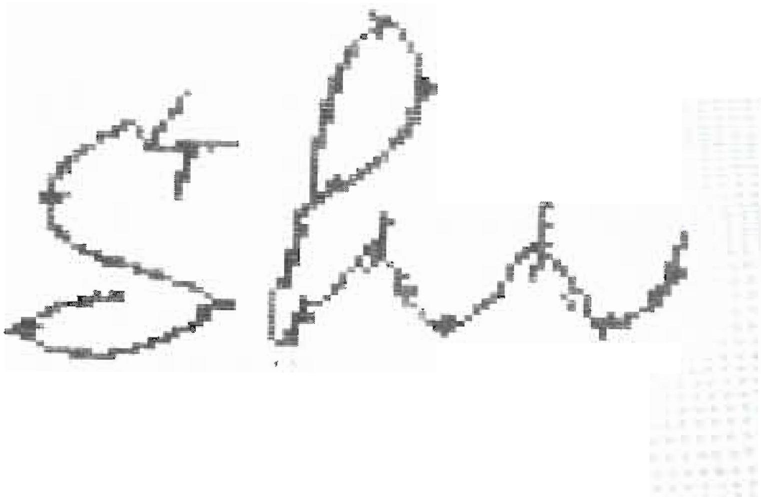

Figure 3. Fuzzy skeleton of Figure 1.

border pixels having μ-values less than α are not taken into consideration. So, both area (eq. (16)) and perimeter (eq. (26)) are less than those for the previous value of α. But the decrease in area is more than the decrease in its perimeter and hence the compactness (eq. (3)) decreases (initially) to a certain minimum corresponding to a value $\alpha = \alpha'$, say.

Further increase in α (i.e., for $\alpha > \alpha'$), results in a μ_{C_α} plane consisting of a number of disconnected regions (because majority of core line pixels being dropped out). As a result, decrease in perimeter here is more than the decrease in area and comp(μ) increases. The $\mu_{C_{\alpha'}}$ plane having minimum compactness value can be taken as an optimum fuzzy skeleton version of the image X. This is optimum in the sense that for any other selection of α (i.e., $\alpha \neq \alpha'$) the comp(μ) value would be greater.

Algorithm

Given an $M \times N$, L level image $X = \{x_{mn}\}$.

Step 1: Extract fuzzy segmented versions of the regions in X.

Step 2: Construct 'Skeleton' membership plane $\mu_C(x_{mn})$ using equation (8) or (9).

Step 3: Produce α-cuts on $\mu_C(x_{mn})$ using equation (10).

Step 4: Compute compactness comp($\mu_{C_\alpha}(x_{mn})$) using equations (1)–(3).

Step 5: Vary α and select $\alpha = \alpha'$, say for which comp(μ) is a minimum.

$\mu_{C_{\alpha'}}$ can thus be regarded as a subset denoting the fuzzy skeleton of the image X. The $\mu_{C_{\alpha'}}$ plane is least

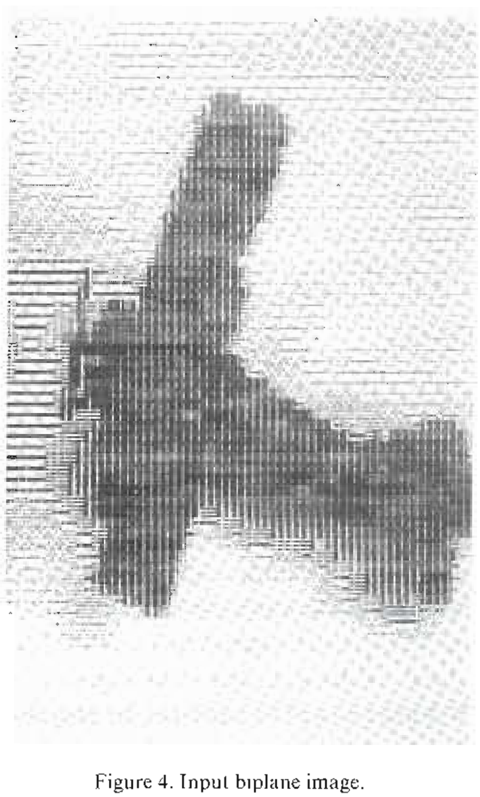

Figure 4. Input biplane image.

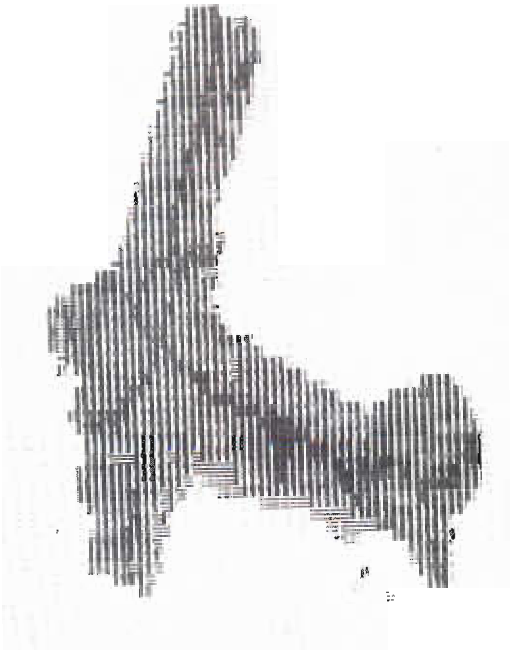

Figure 5. Subset 'Core line' of Figure 4.

Figure 6. Fuzzy skeleton of Figure 4.

compact (or more crisp) and has minimum spatial fuzziness as far as its core line extraction is concerned. If a nonfuzzy (crisp) single-pixel width skeleton is deserved, it can be obtained by a contour tracking algorithm [3] which takes into account the direction of contour, multiple crossing pixels, lost path due to spurious wiggles etc. based on octal chain code.

4. Implementation and results

Figure 1 shows a 96×99, 32 level image of handwritten 'Shu' data. The membership plane $\mu_C(x_{mn})$ representing its fuzzy subset 'Core line' is shown in Figure 2. Table 1 lists the values of compactness measure for different values of α. The minimum is found to correspond to $\alpha = 0.65$. The optimum fuzzy skeleton is shown in Figure 3. A dot matrix printer is used to represent the gray levels.

Figure 4 is a biplane image with dimension 64×64 and level 32. Figures 5 and 6 show its skeleton

Table 1
Compactness of the Shu data

α-Value	Area	Perimeter	Compactness
0.05	792.35	629.70	0.00199825
0.10	792.35	629.70	0.00199825
0.15	792.35	629.70	0.00199825
0.20	792.15	629.70	0.00199775
0.25	790.90	629.70	0.00199459
0.30	785.80	629.70	0.00198173
0.35	772.50	631.10	0.00193956
0.40	752.10	632.70	0.00187880
0.45	708.45	639.00	0.00173503
0.50	583.95	688.00	0.00123367
0.55	395.85	718.80	0.00076615
0.60	356.85	696.00	0.00073666
0.65	336.70	677.88	0.00073289*
0.70	214.20	521.00	0.00078912
0.75	65.70	216.50	0.00140169
0.80	58.50	192.50	0.00157869
0.85	57.65	189.10	0.00161220
0.90	18.05	70.30	0.00365230

* minimum

Table 2
Compactness values for the biplane data

α-Value	Area	Perimeter	Compactness
0.05	507.85	220.70	0.01042632
0.10	507.65	220.50	0.01044113
0.15	507.50	219.90	0.01049508
0.20	506.90	219.90	0.01048267
0.25	504.90	222.90	0.01034699
0.30	502.80	222.10	0.01019292
0.35	496.85	223.50	0.00994652
0.40	480.85	240.30	0.00832730
0.45	342.25	332.10	0.00310317
0.50	148.75	297.10	0.00168521
0.55	113.55	267.40	0.00158806*
0.60	94.95	230.20	0.00179179
0.65	9.45	34.80	0.00780321
0.70	2.70	10.80	0.02314815

* minimum

plane and the optimum version. The different compactness values for varying α are shown in Table 2 which shows that the plane corresponding to $\alpha = 0.55$ has minimum ambiguity as far as its skeleton extraction is concerned.

5. Discussion

Algorithms based on minimising fuzziness in geometry (namely, compactness) of a fuzzy set are formulated here whereby it is possible to extract both fuzzy and nonfuzzy skeletons of an input gray tone image. The algorithm does not lead one to commit to a specific segmentation as in the cases of conventional thinning operations.

The membership functions (involving three factors) mentioned here are not the exaustive ones. This is only an example and a way to demonstrate the possibility of obtaining fuzzy skeletonization of an image with optimizing its fuzzy geometry. For example, the upper circular part of the letter 'S' in Figure 1 is not seen to be properly reflected by its skeleton extracted in Figure 3. One can therefore definitely look for a better membership function involving other additional criteria for extracting a better shape of the skeleton. Furthermore, the connectivity [7] of the skeleton in the optimum version can be preserved, if necessary, by inserting pixels having intensity equal to the minimum of those of pairs of neighbors in the object. The single-pixel width, crisp (nonfuzzy) version removing spurious wiggles can also be obtained in a final stage using algorithms in [3].

Acknowledgement

The author wishes to thank Ms. C.B. Renuka for doing the computer programming, Mr. M.K. Kundu for his valuable discussion, Prof. D. Dutta Majumder for his interest in this work and Mr. J. Gupta for typing the manuscript.

References

[1] Rosenfeld, A. (1984). The fuzzy geometry of image subsets. *Patt. Recog. Lett.* 2, 311–317.

[2] Pal, S.K. and A. Rosenfeld (1988). Image enhancement and thresholding by optimization of fuzzy compactness. *Patt. Recog. Lett.* 7, 77–86.

[3] Pal, S.K., R.A. King and A.A. Hashim (1983). Image description and primitive extraction using fuzzy sets. *IEEE Trans. Syst. Man Cybern* 13, 94–100.

[4] Zadeh, L.A. (1975). Calculus of fuzzy restriction. In: L.A. Zadeh et al., Eds., *Fuzzy Sets and Their Applications in Cognitive and Decision Process.* Academic Press, London, 1–39.

[5] Kaufmann, A. (1975). *Introduction to the Theory of Fuzzy Subsets – Fundamental Theoretical Elements*, Vol 1. Academic Press, New York.

[6] Pal, S.K. and D. Dutta Majumder (1986). *Fuzzy Mathematical Approach to Pattern Recognition.* Wiley (Halsted Press), New York.

[7] Rosenfeld, A. (1983). On connectivity properties of grey scale pictures, *Patt. Recog.* 16, 47–50.