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Image Segmentation Using Fuzzy Correlation

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ABSTRACT

A concept of correlation between two properties (fuzzy representations) of an image is introduced. A set of algorithms for image segmentation (both fuzzy and nonfuzzy) has been formulated. The spatial information is taken care of by the following measures: transitional correlation and within-class correlation. A relation between the correlation coefficient and the index of fuzziness is theoretically established and experimentally verified. The effectiveness of the algorithms is illustrated on images having different types of histograms.

I. INTRODUCTION

Image segmentation plays a key role in image-processing and computer-vision problems. It can be done by gray-level thresholding as well as by pixel classification (region growing). There exist a number of approaches (both classical- and fuzzy-mathematical) to the problem [1, 5–8]. A recent fuzzy-set-theoretic algorithm [7] used both gray-level ambiguity and fuzzy compactness measures in order to take into account global and spatial information about an image. It provides fuzzy (and nonfuzzy as a special case) segmented output in order to avoid committing oneself to a specific thresholding for ill-defined input regions. The compactness measure incorporates the shape of regions of an image.

The present work is an attempt to demonstrate another application of fuzzy set theory to image segmentation based on correlation between two properties

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of an image. The properties may be brightness, edginess, texture, color, blurredness, etc. The measure correlation between two membership functions as introduced by Murthy, Pal, and Dutta Majumder [2] has been used here in providing such measures for images. It is to be mentioned here that the correlation due to Murthy et al. [2] considered neither the number of supports of a set, nor the dependencies of their occurrence. Both these factors have been taken into account here in implementing this correlation measure on an image

The present work consists of three parts. The correlation between two properties of an image is defined in the first part (Sections II and III). As a special case, the correlation between a fuzzy property and its nearest two-tone property is then computed (Section III.A). The second part (Sections IV–VI) describes a few algorithms for image segmentation. The spatial information has been considered in the transitional correlation and within-class correlation measures. The measures have been maximized to obtain optimal segmented output.

Finally (Sections VII and VIII), a relation between correlation coefficient and index of fuzziness [1] has been established. The effectiveness of the algorithms has been demonstrated on a number of images (Section VIII).

II. CORRELATION

A. *CORRELATION AS A STATISTICAL CONCEPT [2]*

Let (x_i, y_i), $i := 1, 2, \dots, n$, be n sample points, with mean $\underline{x}$ and $\underline{y}$ respectively. Let

$$S_x^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \underline{x})^2$$

and

$$S_y^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \underline{y})^2$$

and define the covariance between x and y,

$$\mathrm{Cov}(x, y) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \underline{x})(y_i - \underline{y}).$$

Then the correlation coefficient between x and y as defined in statistics is

$$r_{xy} = \frac{\mathrm{Cov}(x, y)}{S_x S_y}.$$

Here $r_{xy} = r_{yx}$ and $-1 \leqslant r_{xy} \leqslant 1$.

B. *CORRELATION AS A FUZZY-SET-THEORETIC CONCEPT [2]*

A fuzzy set A with its finite number (say n) of supports $x_1, x_2, \dots, x_n$ in the universe of discourse U is formally defined as

$$A = \{\mu_A(x_i), x_i)\}, \qquad i = 1, 2, \dots, n,$$

where the characteristic function $\mu_A(x_i)$, known as the membership function and having a positive value in the interval [0, 1], denotes the degree to which an event x_i is a member of A. Any event x_i for which $\mu_A(x_i) > 0$ is termed a support of A.

In real-life phenomena we come across many characteristics and attributes which are similar in nature, e.g., tall and very tall, glamorous and beautiful. The distinguishing factor between the membership functions of tall and very tall is the degree of tallness. But glamorous and beautiful are two distinct characteristics. Nevertheless, if the value of the membership function for one is high, that for the other one cannot be very low. Correlation provides a measure of such a relationship between two fuzzy membership functions.

Let Ω be a closed interval in R. Let $\mu_1 : \Omega \rightarrow [0, 1]$ and $\mu_2 : \Omega \rightarrow [0, 1]$ be two continuous fuzzy membership functions. Let the correlation between the fuzzy membership functions μ_1 and μ_2 (defined on the same domain) be represented by $C(\mu_1, \mu_2)$. Then the mathematical formula for $C(\mu_1, \mu_2)$ as defined in [2] is

$$C(\mu_1, \mu_2) = 1 - \frac{4 \int (\mu_1 - \mu_2)^2 \, dx}{X_1 + X_2}, \tag{1}$$

where $X_1 = \int (2\mu_1 - 1)^2 \, dx$ and $X_2 = \int (2\mu_2 - 1)^2 \, dx$.

Now $C(\mu_1, \mu_2)$ gives a measure of the relation between the natures of μ_1 and μ_2, i.e., what happens to μ_1 and μ_2 with change of x. $C(\mu_1, \mu_2)$ satisfies a number of properties [2]. Some of them are:

(a) If for higher values of $\mu_1(x)$, $\mu_2(x)$ takes higher values and vice versa, then $C(\mu_1, \mu_2)$ must be high.

(b) If with increase of x both μ_1 and μ_2 increase, then $C(\mu_1, \mu_2) > 0$.
(c) If, with increase of x, μ_1 increases and μ_2 decreases or vice versa, then $C(\mu_1, \mu_2) < 0$.
(d) $C(\mu_1, \mu_1) = 1$.
(e) $C(\mu_1, \mu_1) \geqslant C(\mu_1, \mu_2)$.
(f) $C(\mu_1, 1 - \mu_1) = -1$.
(g) $C(\mu_1, \mu_2) = C(\mu_2, \mu_1)$.
(h) $-1 \leqslant C(\mu_1, \mu_2) \leqslant 1$.
(i) $C(\mu_1, \mu_2) = -C(1 - \mu_1, \mu_2)$.
(j) $C(\mu_1, \mu_2) = C(1 - \mu_1, 1 - \mu_2)$.

If the functions are discrete, the integration will simply be replaced by summation. In this case the expression takes the form

$$C(\mu_1, \mu_2) = 1 - \frac{4\sum_x(\mu_1 - \mu_2)^2}{X_1 + X_2}, \quad \text{if } X_1 + X_2 \neq 0$$
$$= 1 \quad , \quad \text{if } X_1 + X_2 = 0 \qquad (2)$$

where $X_1 = \sum_x(2\mu_1 - 1)^2$ and $X_2 = \sum_x(2\mu_2 - 1)^2$.

Note that the Equation (1) or (2) defines the correlation between two functions representing two different fuzzy sets. The expressions do not take into account the number of occurrences of supports in a set.

For the rest of the paper, we shall skip the second part of Equation (2); i.e., $C(\mu_1, \mu_2) = 1$, if $X_1 + X_2 = 0$ because it is of no practical use.

III. CORRELATION BETWEEN TWO FUZZY REPRESENTATIONS (PROPERTIES) OF AN IMAGE

An L-level image X $(M \times N)$ can be considered as an array of fuzzy singletons, each having a value of membership denoting its degree of possessing some property (e.g. brightness, darkness, edginess, blurredness, texture). In the notation of fuzzy sets one may therefore write $X = \{\mu_X(x_{mn})\}$, $m = 1, 2, \ldots, M$, $n = 1, 2, \ldots, N$, where $\mu_X(x_{mn})$ denotes the grade of possessing the property μ by the pixel (m, n). Let μ_1 and μ_2 denote two such properties of X.

As pointed out in an earlier section, Equation (1) or (2) does not consider the number of supports in a set. In an image a particular gray level occurs many times. Such an image is denoted as an array of singletons (supports). The correlation between two representations characterized by μ_1 and μ_2 may be defined as follows:

Case 1: Correlation in terms of pixel intensity. Let $h(i)$ be the frequency of occurrence of the gray level i. Let f and g be two functions such that

$$f(i) = \mu_1(i)h(i),$$
$$g(i) = \mu_2(i)h(i), \qquad (3)$$

where $\mu(i)$ represents the membership value of possessing some property μ by the ith gray level. Now the correlation between f and g will be given by the expression

$$C(f, g) = 1 - \frac{4\sum_{i=1}^{L}\{f(i) - g(i)\}^2}{X_1 + X_2}$$
$$= 1 - \frac{4\sum_{i=1}^{L}[\mu_1(i) - \mu_2(i)]^2[h(i)]^2}{X_1 + X_2}, \qquad (4)$$

where $X_1 = \sum_{i=1}^{L}\{2f(i) - h(i)\}^2$ and $X_2 = \sum_{i=1}^{L}\{2g(i) - h(i)\}^2$.

Case 2: Correlation in terms of pixel position. As above, μ_1 and μ_2 denote two different properties of X. The correlation between μ_1 and μ_2 may also be expressed as

$$C(\mu_1, \mu_2) = 1 - \frac{4\sum_m\sum_n\{\mu_1(m, n) - \mu_2(m, n)\}^2}{X_1 + X_2} \qquad (5)$$

with $X_1 = \sum_m\sum_n\{2\mu_1(m, n) - 1\}^2$ and $X_2 = \sum_m\sum_n\{2\mu_2(m, n) - 1\}^2$, where $\mu_1(m, n)$ and $\mu_2(m, n)$ denote the degrees of possessing the property μ_1 and μ_2 respectively by the (m, n)th pixel. Let the values of μ_1 and μ_2 in Equation (5) depend only on the gray level of the pixels. If the (m, n)th pixel has intensity i, then $\mu_1(m, n) = \mu_1(i)$ for a particular choice of i $(i = 1, 2, \ldots, L)$. In that case, summing over all the pixels is nothing but summing over the gray levels after multiplying by the number of occurrences. Hence we can write

$$C(\mu_1, \mu_2) = 1 - \frac{4\sum_{i=1}^{L}\{\mu_1(i) - \mu_2(i)\}^2 h(i)}{X_1 + X_2} \qquad (6)$$

with $X_1 = \sum_{i=1}^{L}[2\mu_1(i) - 1]^2 h(i)$ and $X_2 = \sum_{i=1}^{L}[2\mu_2(i) - 1]^2 h(i)$.

Note that the expression in (4) differs from that in (6) by a factor of $h(i)$.

A. CORRELATION BETWEEN A GRAY-TONE IMAGE AND ITS TWO-TONE VERSION

Let μ_2 be the nearest two-tone version of μ_1:

$$\mu_2(x) = \begin{cases} 0 & \text{if } \mu_1(x) \leqslant 0.5, \\ 1 & \text{otherwise.} \end{cases} \tag{7}$$

Let μ_1 denote a fuzzy *bright-image* plane of X having crossover point at s, say, and be dependent only on the gray level. Then μ_2 represents its closest two-tone version thresholded at s. In this case, from (4),

$$\begin{aligned} C(f, g) &= 1 - \frac{4}{X_1 + X_2}\left(\sum_{i=1}^{s} [\mu_1(i) - 0]^2 [h(i)]^2 + \sum_{i=s+1}^{L} [\mu_1(i) - 1]^2 [h(i)]^2\right) \\ &= 1 - \frac{4}{X_1 + X_2}\left(\sum_{i=1}^{s} [\mu_1(i)]^2 [h(i)]^2 + \sum_{i=s+1}^{L} [1 - \mu_1(i)]^2 [h(i)]^2\right) \\ &= C(X_f, \underline{X}), \end{aligned} \tag{8a}$$

where X_f is the fuzzy representation (bright-image plane) of the image X, and $\underline{X}$ is its closest two-tone version, with

$$X_1 = \sum_{i=1}^{L} [2\mu_1(i) - 1]^2 [h(i)]^2$$

and

$$X_2 = \sum_{i=1}^{L} [2\mu_2(i) - 1]^2 [h(i)]^2$$

$$= \sum_{i=1}^{L} [h(i)]^2 = \text{constant, since } \mu_2 = 0 \text{ or } 1.$$

If the membership value is expressed in terms of pixel position, a similar expression can also be obtained with the exception that instead of i, the summation will be taken over (m, n). Then, as above [Equation (6)], we get the expressions as

$$C(X_f, \underline{X}) = 1 - \frac{4}{X_1 + X_2}\left(\sum_{i=1}^{s} [\mu_1(i)]^2 h(i) + \sum_{i=s+1}^{l} [1 - \mu_1(i)]^2 h(i)\right) \tag{8b}$$

with

$$X_1 = \sum_{i=1}^{L} [2\mu_1(i) - 1]^2 h(i)$$

and

$$X_2 = \sum_{i=1}^{L} [2\mu_2(i) - 1]^2 h(i)$$

$$= \sum_{i=1}^{L} h(i) = MN.$$

In Section IV we will show the way to use (8) as a criterion of optimum segmentation of an image.

B. STATEMENT OF RESULT

PROPOSITION 1. *The correlation coefficient between a fuzzy representation X_f of an image X and the corresponding two-tone version $\underline{X}$ satisfies the relation $0 \leqslant C(X_f, \underline{X}) \leqslant 1$.*

The proof is given in Appendix 1.

NOTE 1. In the case of membership functions expressed in terms of pixel position, Proposition 1 can also be shown to be valid.

IV. MAXIMIZATION OF CORRELATION AND HISTOGRAM THRESHOLDING

Let us construct, say, a fuzzy-subset bright image characterized by a membership function μ_1 using the standard S function of Zadeh [3] as

$$\mu_1(x) = S(x, b, w) = \begin{cases} 0 & \text{if } x \leqslant a, \\ 2\left(\frac{x-a}{c-a}\right)^2 & \text{if } a \leqslant x \leqslant b, \\ 1-2\left(\frac{x-c}{c-a}\right)^2 & \text{if } b \leqslant x \leqslant c, \\ 1 & \text{if } x \geqslant c, \end{cases} \tag{9a}$$

where $1 \leqslant x \leqslant L$, $a = b - w/2$, $c = b + w/2$, and $w = c - a$. Here w is the window size and b is the crossover point of the membership function. Let

$$\mu_2(x) = \begin{cases} 0 & \text{if } \mu_1 \leqslant 0.5, \\ 1 & \text{otherwise}. \end{cases} \tag{9b}$$

The function μ_1 is spread over the dynamic range of gray levels (of the image), but is discrete, i.e., it has membership values at discrete values of x only. Note that $\mu_1(x)$ [and hence $\mu_2(x)$] is dependent only on the gray level.

From the properties of correlation we notice that if the two functions μ_1 and μ_2 are very close, then $C(\mu_1, \mu_2)$ is very high, whereas $C(\mu_1, \mu_2)$ is least when $\mu_2 = 1 - \mu_1$. Since μ_2 is the closest two-tone version of μ_1, $C(X_f, \underline{X})$ [Equation (8)] gives a measure of the closeness of the two images X_f (a fuzzy bright-image plane of the image X) and $\underline{X}$ (the corresponding two-tone version).

Now if we vary the crossover point (keeping w fixed), both X_f and $\underline{X}$ will change, and we will get different values of $C(X_f, \underline{X})$. Among them the image X_f for which $C(X_f, \underline{X})$ is maximum can then be viewed as the optimum fuzzy segmented version of the image X [optimum in the sense that for any other selection of the crossover point of $\mu_1(x)$, the value of $C(X_f, \underline{X})$ will be lower]. The corresponding $\underline{X}$ can be taken as the nonfuzzy (crisp) segmented output.

This can further be interpreted in terms of Equation (8a). Let the crossover point of the membership function be b. Here the quantity

$$T = 4\left(\sum_{i=1}^{b} [\mu_1(i)]^2 [h(i)]^2 + \sum_{i=b+1}^{L} [1 - \mu_1(i)]^2 [h(i)]^2\right)$$

decreases as b moves towards a valley of the histogram of an image. X_2 is constant. $X_1 + X_2$ is a large positive quantity, and the rate of decrease of $X_1 + X_2$ is less than that of T. Therefore $C(X_f, \underline{X})$ will increase as b moves towards a valley, and it will attain a maximum when b corresponds to the appropriate boundary between two regions.

For a multimodal image, the algorithm will result in a set of maxima corresponding to different boundaries. Of them the global one corresponds to the boundary between object and background.

These ideas are summarized below in algorithmic form.

ALGORITHM 1.

Step 1. Construct the fuzzy membership planes $X_f = \{\mu_1(x)\}$ and $\underline{X} = \{\mu_2(x)\}$ using Equation (9).

Step 2. Find the correlation coefficient $C(X_f, \underline{X})$ using Equation (8) for a fixed b, $1 \leqslant b \leqslant L$.

Step 3. Vary b, and select those b for which $C(X_f, \underline{X})$ has local maxima.

Among the local maxima, let the global one correspond to X_f^g and $\underline{X}^g$. X_f^g then can be viewed as a fuzzy segmented version of the image X such that $\mu_1(x)$ denotes the degree of possessing brightness (or the degree of belonging to an object, say) of the pixel intensity x. On the other hand, $\underline{X}$ denotes the nonfuzzy (crisp) segmented output.

V. CORRELATION USING LOCAL INFORMATION

In the previous section, the membership values were computed based on global information, i.e. individual gray levels. Equation (9) was used in this context to represent a fuzzy-subset bright image. In this section, we are going to explain how the local information in X can be incorporated in computing the correlation. For example, the properties edginess, blurredness, and texture involve local (spatial) information about the image for their computation.

Let μ_1 and μ_2 be two such properties of the image. The correlation between two properties can be expressed in the following ways:

Case 1: In terms of gray level. Let X $(M \times N)$ be an L-level image, and $C(i, j)$ the frequency of occurrence of the gray level i followed by j, i.e., C is the cooccurrence matrix [5]. Let f and g be two functions such that

$$\begin{aligned} f(i, j) &= \mu_1(i, j)C(i, j), \\ g(i, j) &= \mu_2(i, j)C(i, j), \end{aligned} \tag{10}$$

where $\mu(i, j)$ represents the grade of possessing some property by the gray level i followed by j in a specific fashion. Then

$$C(f,g) = 1 - \frac{4\sum_{i=1}^{L}\sum_{j=1}^{L}\{f(i,j)-g(i,j)\}^2}{X_1+X_2}$$

$$= 1 - \frac{4\sum_{i=1}^{L}\sum_{j=1}^{L}\{\mu_1(i,j)-\mu_2(i,j)\}^2\{C(i,j)\}^2}{X_1+X_2}, \qquad (11)$$

where $X_1 = \sum_{i=1}^{L}\sum_{j=1}^{L}\{2f(i,j)-C(i,j)\}^2$ and $X_2 = \sum_{i=1}^{L}\sum_{j=1}^{L}\{2g(i,j)-C(i,j)\}^2$.

Case 2: In terms of pixel position. Let $\mu_1(m,n)$ and $\mu_2(m,n)$ denote the grade of possessing two local properties μ_1 and μ_2 (respectively) by the (m,n)th pixel. Then

$$C(\mu_1,\mu_2) = 1 - \frac{4\sum_m\sum_n\{\mu_1(m,n)-\mu_2(m,n)\}^2}{X_1+X_2} \qquad (12)$$

with $X_1 = \sum_m\sum_n\{2\mu_1(m,n)-1\}^2$ and $X_2 = \sum_m\sum_n\{2\mu_2(m,n)-1\}^2$. Let the local properties possessed by the pixels depend on their neighboring gray values. If the (m,n)th pixel has intensity i and one of its neighboring intensities is j, then $\mu_1(m,n) = \mu_1(i,j)$, and the Equation (12) then takes the form

$$C(\mu_1,\mu_2) = 1 - \frac{4\sum_{i=1}^{L}\sum_{j=1}^{L}\left\{\left[\mu_1(i,j)-\mu_2(i,j)\right]^2 C(i,j)\right\}}{X_1+X_2} \qquad (13)$$

with

$$X_1 = \sum_{i=1}^{L}\sum_{j=1}^{L}\left[2\mu_1(i,j)-1\right]^2 C(i,j)$$

and

$$X_2 = \sum_{i=1}^{L}\sum_{j=1}^{L}\left[2\mu_2(i,j)-1\right]^2 C(i,j).$$

Note that, as in the case of $C(f,g)$ and $C(\mu_1,\mu_2)$ in Section III, the Equation (13) differs from (11) by a factor of $C(i,j)$.

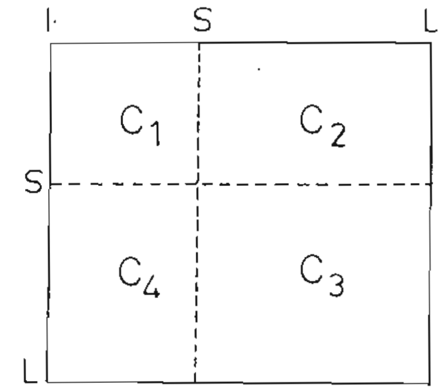


Fig. 1. Four submatrices (quadrants) of the cooccurrence matrix of an image.

A. CORRELATION BETWEEN TWO PROPERTIES OF A SEGMENTED IMAGE

Suppose an image has two distinct regions, the object and the background. Let the background consist of the levels $[1, s]$ and the object $[s+1, L]$. Then the cooccurrence matrix C can be divided into four quadrants (Figure 1) c_1, c_2, c_3, and c_4, where

entries in c_1 reflect the number of occurrences of background pixels followed by background pixels,
entries in c_2 reflect the number of occurrences of background pixels followed by object pixels,
entires in c_3 reflect the number of occurrences of object pixels followed by object pixels,
entries in c_4 reflect the number of occurrences of object pixels followed by background pixels.

The correlation between any two properties of X computed over the entries in c_1, c_3, c_2, and c_4 may be termed the background correlation $C(f,g)_B$, object correlation $C(f,g)_O$, transitional (background-to-object) correlation $C(f,g)_{B/O}$, and transitional (object-to-background) correlation $C(f,g)_{O/B}$ respectively. They may be computed from the following equations:

$$C(f,g)_B = 1 - \frac{4}{X_1+X_2}\sum_{i=1}^{s}\sum_{j=1}^{s}\left\{\left[\mu_1(i,j)-\mu_2(i,j)\right] C(i,j)\right\}^2 \qquad (14)$$

with

$$X_1 = \sum_{i=1}^{s} \sum_{j=1}^{s} \{[2\mu_1(i,j)-1]C(i,j)\}^2$$

and

$$X_2 = \sum_{i=1}^{s} \sum_{j=1}^{s} \{[2\mu_2(i,j)-1]C(i,j)\}^2;$$

$$C(f,g)_O = 1 - \frac{4}{X_1+X_2} \sum_{i=s+1}^{L} \sum_{j=s+1}^{L} \{[\mu_1(i,j)-\mu_2(i,j)]C(i,j)\}^2 \tag{15}$$

with

$$X_1 = \sum_{i=s+1}^{L} \sum_{j=s+1}^{L} \{[2\mu_1(i,j)-1]C(i,j)\}^2$$

and

$$X_2 = \sum_{i=s+1}^{L} \sum_{j=s+1}^{L} \{[2\mu_2(i,j)-1]C(i,j)\}^2;$$

$$C(f,g)_{B/O} = 1 - \frac{4}{X_1+X_2} \sum_{i=1}^{s} \sum_{j=s+1}^{L} \{[\mu_1(i,j)-\mu_2(i,j)]C(i,j)\}^2 \tag{16}$$

with

$$X_1 = \sum_{i=1}^{s} \sum_{j=s+1}^{L} \{[2\mu_1(i,j)-1]C(i,j)\}^2$$

and

$$X_2 = \sum_{i=1}^{s} \sum_{j=s+1}^{L} \{[2\mu_2(i,j)-1]C(i,j)\}^2;$$

$$C(f,g)_{O/B} = 1 - \frac{4}{X_1+X_2} \sum_{i=s+1}^{L} \sum_{j=1}^{s} \{[\mu_1(i,j)-\mu_2(i,j)]C(i,j)\}^2 \tag{17}$$

with

$$X_1 = \sum_{i=s+1}^{L} \sum_{j=1}^{s} \{[2\mu_1(i,j)-1]C(i,j)\}^2$$

and

$$X_2 = \sum_{i=s+1}^{L} \sum_{j=1}^{s} \{[2\mu_2(i,j)-1]C(i,j)\}^2.$$

Note that $C(f,g)_O + C(f,g)_B$ gives the total within-class (object-background) correlation, and $C(f,g)_{O/B} + C(f,g)_{B/O}$ gives the total transitional correlation. It can also be noticed that the sum of the above four quantities [Equations (14)–(17)] gives the same expression as Equation (11).

Similar definitions can also be provided in terms of pixel coordinates.

Let us consider μ_2 as follows:

$$\mu_2(x) = \begin{cases} 0 & \text{if } \mu_1 \leqslant 0.5, \\ 1 & \text{otherwise.} \end{cases}$$

Then Equations (14)–(17) provide the correlation between a gray-tone image and the nearest two-tone version using local information (computed over different quadrants).

NOTE 2. Proposition 1 is also valid when we use Equation (11) as the expression for the correlation. Proof is given in Appendix 2.

The proposition is also valid for the correlation expressions in Equations (14)–(17).

VI. MAXIMIZATION OF CORRELATION AND HISTOGRAM THRESHOLDING

A. *MEMBERSHIP-FUNCTION SELECTION*

Equation (9a) is suitable for one-dimensional input only (i.e., $x \in R$), and it considers only the global information about X. To take the local information into account, the function has been modified by considering $x \in R^n$. From

Equation (9a) when $a \leqslant x \leqslant b$ and $x \in R$, we can write

$$S(x, b, w) = 2\left(\frac{x-a}{w}\right)^2 = 2\left(\frac{x-b+w/2}{w}\right)^2$$
$$= 2\left(\frac{1}{2} + \frac{x-b}{w}\right)^2 = 2\left(\frac{1}{2} - \frac{b-x}{w}\right)^2.$$

When $x \in R^n$ the above expression can be written in the form

$$S(x, b, w) = 2\left(\frac{1}{2} - \frac{\|b-x\|}{w}\right)^2 \quad \text{if} \quad \|a\| \leqslant \|x\| \leqslant \|b\|,$$

where $\|b-x\|$ is the euclidian distance between b and x, and w is the euclidian distance between c and a, i.e., $w = \|c-a\|$. Similarly, one can find

$$S(x, b, w) = 1 - 2\left(\frac{\frac{1}{2} - \|x-b\|}{w}\right)^2 \quad \text{if} \quad \|b\| \leqslant \|x\| \leqslant \|c\|.$$

Therefore, when $x \in R^n$,

$$S(x, b, w) = \begin{cases} 0 & \text{if} \quad \|x\| \leqslant \|a\|, \\ 2\left(\frac{1}{2} - \|b-x\|/w\right)^2 & \text{if} \quad \|a\| \leqslant \|x\| \leqslant \|b\|, \\ 1 - 2\left(\frac{1}{2} - \|x-b\|/w\right)^2 & \text{if} \quad \|b\| \leqslant \|x\| \leqslant \|c\|, \\ 1 & \text{if} \quad \|x\| \geqslant \|c\| \end{cases} \tag{18}$$

with b as the crossover point and $w = \|c-a\|$.

B. *IMAGE SEGMENTATION*

Let us consider μ_2 as follows:

$$\mu_2(x) = \begin{cases} 0 & \text{if} \quad \mu_1 \leqslant 0.5, \\ 1 & \text{otherwise}. \end{cases}$$

Let us consider equation (11). It is shown in Appendix 2 that $C(X_f, \underline{X}) \geqslant 0$. So maximization of $C(X_f, \underline{X})$ is equivalent to minimization of [from Equation (11)]

$$T = \frac{4\sum_{i=1}^{L}\sum_{j=1}^{L}\{[\mu_1(i, j) - \mu_2(i, j)]C(i, j)\}^2}{X_1 + X_2}.$$

Now X_2 is constant and $X_1 + X_2$ is very large. The rate of change of $X_1 + X_2$ is much less. So the above expression decreases with decreasing $|\mu_1(i, j) - \mu_2(i, j)|$, and so does the corresponding $C(i, j)$. When the threshold(s) corresponds to the appropriate boundary between two regions, the value of $C(i, j)$ for $i \leqslant s$, $j \leqslant s$ or for $i > s$, $j > s$ (i.e. a pixel from one region followed by another pixel from the same region) will be more, whereas the value of $C(i, j)$ for $i \leqslant s$, $j > s$ or for $i > s$, $j \leqslant s$ (i.e. a pixel from one region followed by a pixel from another region) will be less. So the number of cooccurrences of levels, $C(i, j)$ with low absolute value $|\mu_1(i, j) - \mu_2(i, j)|$ will be more, and with higher $|\mu_1(i, j) - \mu_2(i, j)|$ will be less. Therefore there will be less contribution to T.

Hence the minimization of T—i.e. maximization of $C(X_f, \underline{X})$—may be viewed as a criterion for detecting the thresholds between regions of X.

Although maximization of Equation (11) has been shown above to be a criterion of thresholding, one can also establish

$$C(X_f, \underline{X})_O + C(X_f, \underline{X})_B \tag{19a}$$

or

$$C(X_f, \underline{X})_{O/B} + C(X_f, \underline{X})_{B/O} \tag{19b}$$

as a criterion for thresholding. The algorithm for their computation is same as Algorithm 1.

The algorithm using Equation (11) as criterion will be referred to as Algorithm 2. Similarly, Equations (19a) and (19b) correspond to Algorithm 3 and Algorithm 4 respectively.

If the global maximum value of correlation obtained is $C(X_f^g, \underline{X}^g)$, then X_f^g and $\underline{X}^g$ can be taken as a fuzzy and the nonfuzzy segmented version of the image respectively.

Use of local information is expected to yield the boundary between regions even when the corresponding valley is not present in the histogram of the image.

VII. (DIS)SIMILARITY BETWEEN CORRELATION COEFFICIENT AND QUADRATIC INDEX OF FUZZINESS

The quadratic index of fuzziness of an image X ($M \times N$) reflects the average amount of ambiguity (fuzziness) present in it by measuring the distance (quadratic) between its fuzzy property plane μ_X and the nearest two-level property $\underline{\mu}_X$—in other words, the distance between the gray-tone image and its nearest two-tone version. Its mathematical expression is [1]

$$\tau_q(X) = \frac{2\left[\sum_{m=1}^{M} \sum_{n=1}^{N} \{\mu_X(x_{mn}) - \mu_{\underline{X}}(x_{mn})\}^2\right]^{1/2}}{\sqrt{MN}}$$

$$= \frac{2\left[\sum_{i=1}^{L} \{\mu_X(i) - \mu_{\underline{X}}(i)\}^2 h(i)\right]^{1/2}}{\sqrt{MN}}, \qquad (20a)$$

where $\mu_X(x_{mn})$ [or $\mu(i)$] represents the grade of possessing some property μ by the (m, n)th pixel (ith gray level), and $\mu_{\underline{X}}(x_{mn})$ [or $\mu_{\underline{X}}(i)$] denotes the nearest two-level version of X.

If we consider spatial information in the membership function, then the expression for $\tau_q(X)$ takes the form

$$\tau_q(X) = \frac{2\left[\sum_{m=1}^{M} \sum_{n=1}^{N} \{\mu_{mn}(i, j) - \underline{\mu}_{mn}(i, j)\}^2\right]^{1/2}}{\sqrt{MN}}$$

$$= \frac{2\left[\sum_{i=1}^{L} \sum_{j=1}^{L} \{\mu(i, j) - \underline{\mu}(i, j)\}^2 C(i, j)\right]^{1/2}}{\sqrt{MN}} \qquad (20b)$$

The quadratic index of fuzziness of an image can also be computed by considering the various quadrants c_1, c_2, c_3, and c_4 of the cooccurrence matrix.

In this section we will establish a relation between $\tau_q(X)$ and $C(X_f, \underline{X})$.

Let s be the crossover point of the membership function $\mu(i)$, $i = 1, 2, \ldots, L$, i.e., $\mu(i) \leqslant 0.5$ for $i \leqslant s$ and $\mu(i) \geqslant 0.5$ for $i \geqslant s$. From Equation (20a)

$$\tau_q(X) = \frac{2}{\sqrt{MN}} \left(\sum_{i=1}^{s} [\mu(i)]^2 h(i) + \sum_{i=s+1}^{L} [1 - \mu(i)]^2 h(i) \right)^{1/2}$$

$$\tau_q(X)^2 = \frac{4}{MN} \left(\sum_{i=1}^{s} [\mu(i)]^2 h(i) + \sum_{i=s+1}^{L} [1 - \mu(i)]^2 h(i) \right),$$

$$(MN)\tau_q(X)^2 = 4\left(\sum_{i=1}^{s} [\mu(i)]^2 h(i) + \sum_{i=s+1}^{L} [1 - \mu(i)]^2 h(i) \right). \qquad (21)$$

Now from Equation (8b) we get

$$1 - C(X_f, \underline{X}) = \frac{4}{X_1 + X_2} \left(\sum_{i=1}^{s} [\mu(i)]^2 h(i) + \sum_{i=s+1}^{L} [1 - \mu(i)]^2 h(i) \right) \qquad (22)$$

$$= \frac{MN\tau_q(X)^2}{X_1 + X_2} \qquad \text{[using Equation (21)]}$$

$$= \tau_q(X)^2 \frac{X_2}{X_1 + X_2} \qquad [\text{since } X_2 = MN],$$

or

$$C(X_f, \underline{X}) = 1 - \tau_q(X)^2 \frac{X_2}{X_1 + X_2}.$$

Now $X_2/(X_1 + X_2) \leqslant 1$. Therefore

$$C(X_f, \underline{X}) \geqslant 1 - \tau_q(X)^2. \qquad (23)$$

NOTE. If the window size w of the membership function is very small, then $X_2/(X_1 + X_2) \approx \frac{1}{2}$, as $X_1 \approx MN$. Hence

$$\tau_q(X)^2 \approx 2[1 - C(X_f, \underline{X})]. \qquad (24)$$

In the limiting case as $w \to 0$ [i.e., $\mu(i)$ becomes a step function], $C(X_f, \underline{X}) \to 1$ and $\tau_q(X) \to 0$.

From the above discussion it is evident that the thresholds obtained by minimization of $1 - C(X_f, \underline{X})$—i.e., maximization of $C(X_f, \underline{X})$—will be more or less the same as those obtained by minimization of τ. This has been experimentally verified and is shown in Tables 1 to 4.

Similar relations can also be established using local information in the expressions for the membership function and $C(i, j)$.

VIII. IMPLEMENTATION AND RESULTS

The proposed algorithms have been implemented on two different images, namely, an image of a biplane [Figure 2(a)] and one of Lincoln [Figure 3(a)].

TABLE 1

Thresholds Based on Algorithm 1

	Thresholds			
	Biplane		Lincoln	
w	Alg. 1	Fuzz. min.	Alg. 1	Fuzz. min.
6	12[a] 15 23 31	12[a] 15 22 31	7 11[a] 14 18 25 31	7 11[a] 18 25 31
8	12[a] 23 31	12[a] 23 31	11[a] 18 25 31	11[a] 18 25 31
10	12[a] 22 31	12[a] 22 31	10[a] 18 25 31	10[a] 18 25 31
12	13[a] 22 31	13[a] 22 31	10[a] 18 25 31	10[a] 18 25 31

[a] Global maxima/minima.

TABLE 2

Thresholds Based on Algorithm 2

	Thresholds			
	Biplane		Lincoln	
w	Alg. 2	Fuzz. min.	Alg. 2	Fuzz. min.
12	13[a] 23 30	13[a] 15 23 31	10[a] 14 18 25 30	10[a] 18 25 31
16	13[a] 22 30	13[a] 22 31	10[a] 18 25 30	10[a] 18 25 31
20	13[a] 31	13[a] 31	9[a] 18 25 30	10[a] 18 25 31
24	14[a] 31	14[a]	9[a] 17 25 30	10[a] 17 25

[a] Global maxima/minima.

TABLE 3

Thresholds Based on Algorithm 3

	Thresholds			
	Biplane		Lincoln	
w	Alg. 3	Fuzz. min.	Alg. 3	Fuzz. min.
12	13[a] 23 30	13[a] 15 23 31	10[a] 14 18 25 30	9[a] 14 18 25 31
16	13[a] 22 30	13[a] 31	9[a] 14 18 25 30	9[a] 18 25 31
20	13[a] 31	13[a] 31	9[a] 18 25 30	9[a] 18 25 31
24	14[a] 31	13[a] 31	9[a] 17 25 30	9[a] 17 26 31

[a] Global maxima/minima.

TABLE 4

Thresholds Based on Algorithm 4

	Thresholds			
	Biplane		Lincoln	
w	Alg. 4	Fuzz. min.	Alg. 4	Fuzz. min.
12	11[a] 14 19 22 30	11[a] 14 22 30	6 10[a] 13 18 24 31	6 10[a] 13 18 24 31
16	11[a] 14 22 30	11[a] 14 22 30	6 10[a] 13 18 24 31	6 10[a] 13 18 24 31
20	11[a] 14 22 30	11[a] 14 22	6 10[a] 13 18 24 31	6 10[a] 13 18 24 31
24	11[a] 14 22 30	11[a] 14 22	6 10[a] 13 18 24 31	6 10[a] 13 18 24 31

[a] Global maxima/minima.

The images are of size 64×64 and have 32 levels. The corresponding histograms are shown in Figures 2(b) and 3(b). The histogram of the biplane is bimodal, and that of Lincoln is multimodal.

For the computation of the correlation and the quadratic index of fuzziness (with local information), the membership value for a particular combination (i, j) of the cooccurrence matrix was assigned as follows:

(i) For a particular threshold s, any combination of pixel intensities s was assigned the value 0.5, since (s, s) is the most ambiguous point, i.e., the boundary.

(ii) If one object pixel is followed by another object pixel (i.e. for the entries of quadrant c_3), then its degree of belonging to the object (membership value) is greater than 0.5. The membership value increases with increase in pixel intensity.

(iii) For quadrants c_2 and c_4, where one object pixel is followed by one background pixel or vice versa, the membership value is less than or equal to 0.5, depending on the deviation from the boundary point (s, s).

(iv) If one background pixel is followed by another background pixel (i.e. for the entries in c_1), then its degree of belonging to the object (membership value) is less than 0.5. The membership value decreases with decrease of pixel intensity.

The thresholds obtained for different window sizes by the proposed algorithms are given in Tables 1 and 4. Table 1 shows the thresholds obtained by Algorithm 1, which considers only global information. Tables 2, 3, and 4 contain the thresholds obtained by Algorithm 2, Algorithm 3, and Algorithm 4 respectively. The quadratic index of fuzziness of an image (with both global and local information) was calculated. The thresholds obtained by its minimization are also included in the tables for comparison. It is to be noticed here that

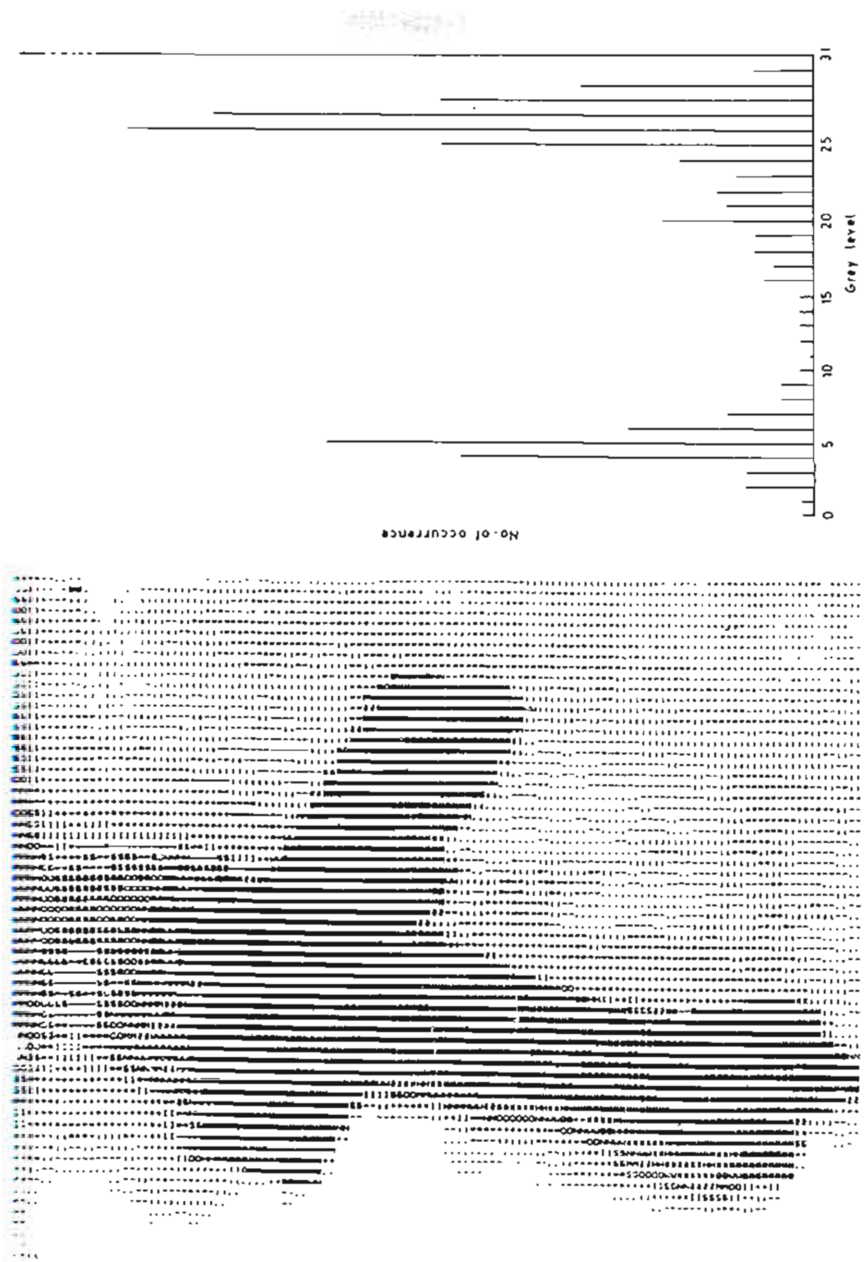
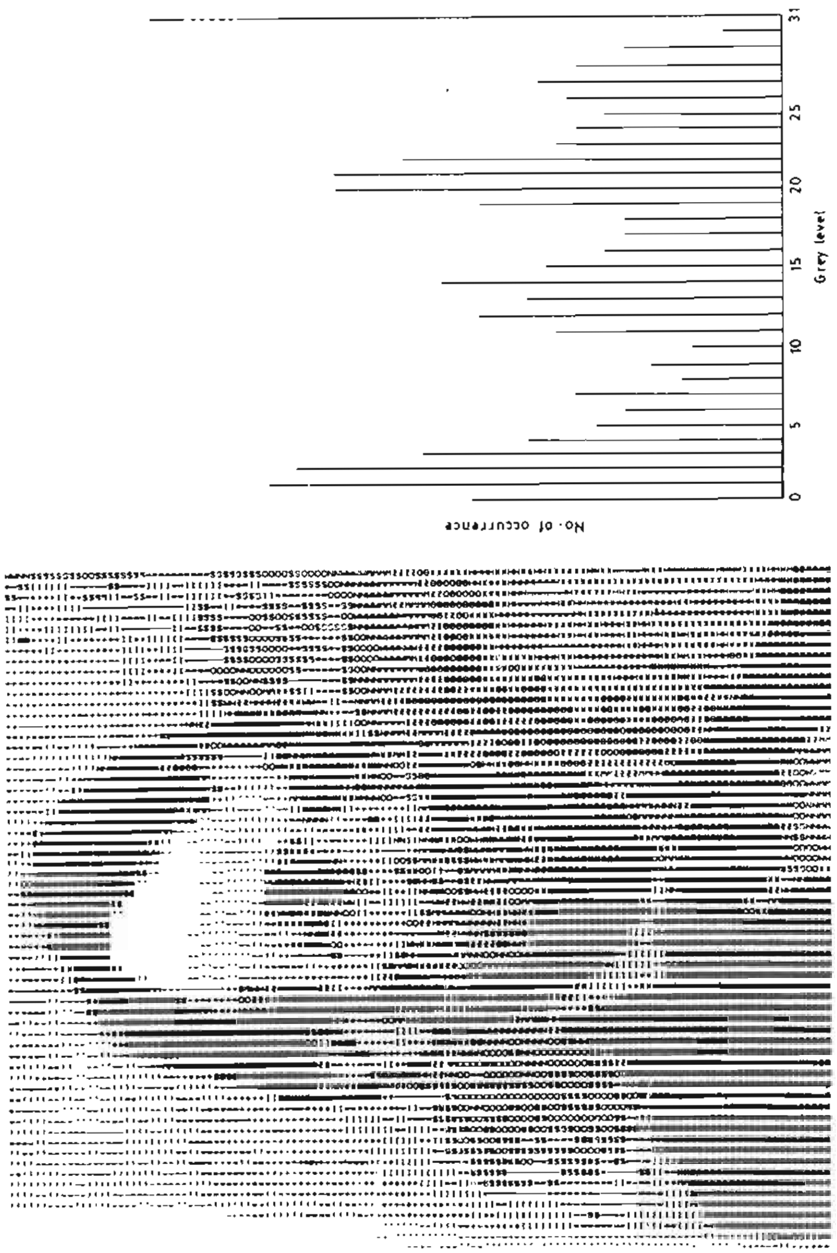


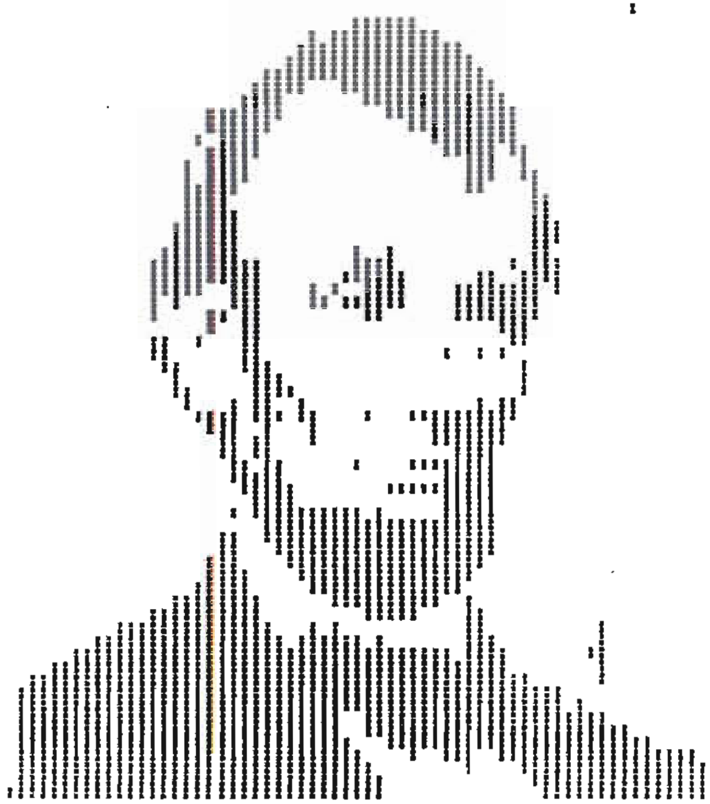
Fig. 2 Biplane image. (a) Input. (b) Histogram. (c) Segmented at the threshold 12. (d) Segmented at the threshold 13.

Fig. 2—*Continued*



(a) (b)

Fig. 3. Lincoln image: (a) Input. (b) Histogram. (c) Segmented at the threshold 10.



(c)

Fig. 3—*Continued*

the thresholds obtained by index-of-fuzziness minimization, as expected, are more or less the same as those obtained by correlation maximization. A curve (Figure 4) showing the variation of the correlation coefficient with crossover point (using global information) for a fixed window size (6) of the image of Lincoln is given. From Figure 4 it is evident that the graph is complementary to the "sharpened" version of the histogram. For visual inspection, nonfuzzy segmented versions of the two images [corresponding to the global maxima of $C(X_f, \underline{X})$] for a fixed window size ($w = 10$ for global information, $w = 16$ for local information) are given in Figures 2(c) and 3(c) and in Figures 2(d) and 3(c) respectively.

The histogram of the image of the biplane has a sharp valley in the gray-level range 11–16, a weak valley in the range 21–26, and a spurious valley at 31. From the results (thresholds) obtained by the proposed algo-

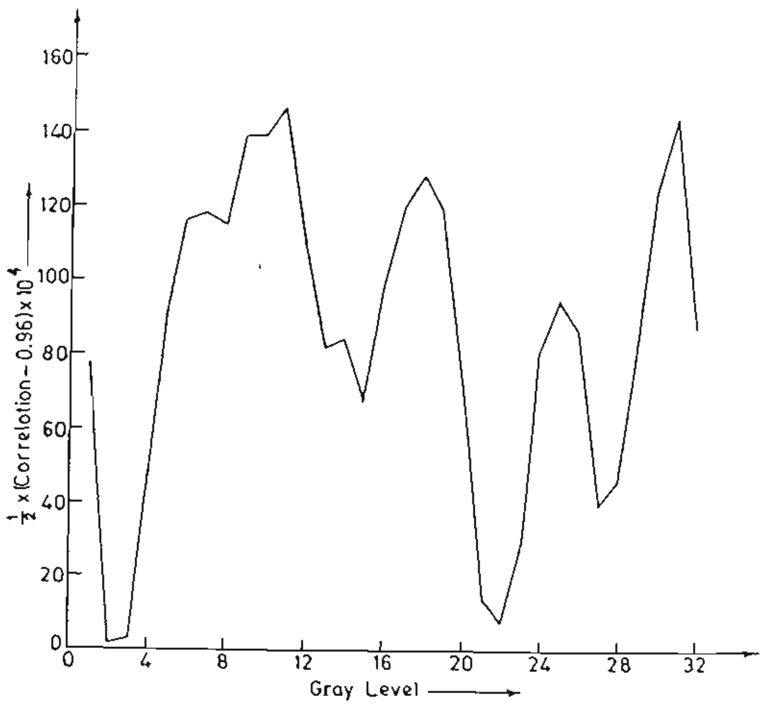


Fig. 4. Curve showing the variation of correlation coefficient with gray levels (using global information) for the image of Lincoln and a window size w of 6.

rithms, it is evident that for all window sizes most of the valleys are detected. For smaller window sizes, more valleys are detected in the range 11–16, which is in agreement with the earlier findings of Murthy and Pal [4]. The valley (12-13) corresponding to the maximum (global) correlation is seen to be the appropriate threshold for object background separation.

The histogram of Lincoln's image has four sharp valleys and one or two weak valleys. Here also it is noticed that for lower window sizes more valleys are detected and for higher window sizes less valleys are detected. The global threshold obtained here is also seen to be appropriate for object-background separation.

The thresholds obtained using local information agree well with the existing cooccurrence-matrix-based algorithms [5–7]. A comparative study of the thresholds obtained with three different local-information-based correlation measures (Algorithms 2, 3, and 4) show that the transitional correlation (using the submatrices C_2 and C_4) is most effective in determining the boundaries between regions in an image, which also is in agreement with the earlier findings of Pal and Pal [5]. This further establishes that interset distance

(contrast between regions) is more important than the within-class distance (homogeneity) in separating regions.

Furthermore, it is evident from the results that incorporation of local information makes the algorithm more effective in detecting boundaries between regions. For instance, let us consider the thresholds of Lincoln's image detected by the transitional correlation (Table 4). Here the boundaries (valleys of the histogram) at gray levels 6 and 13 were detected as local maxima for almost all window sizes. This is not the case with the global-information-based algorithm (for most of the window sizes).

IX. DISCUSSION AND CONCLUSIONS

An attempt has been made here to apply the concept of correlation between two fuzzy membership functions in image segmentation (and thereby object-background separation) problems. In this connection four algorithms have been proposed, involving global and local information respectively.

It is to be mentioned here that the concept of correlation between two membership functions as introduced by Murthy and Pal [2] neither included the number of supports in a set nor considered their dependencies of occurrence. In the present work on image-processing problems, both the number of occurrences of levels and spatial relationships among them have been taken into account (in defining the correlation between two properties of an image). A relation between correlation and index of fuzziness of an image has also been established.

The proposed image segmentation algorithms are based on the correlation computed between a fuzzy segmented image and its corresponding two-tone version. Their efficiencies have been compared with those of the existing algorithms. Incorporation of spatial information, as expected, makes the algorithms more efficient in detecting boundaries. Transitional correlation is seen to provide better results. It has also been noticed that with variation of window size the changes of threshold values are insignificant. This makes the algorithms flexible.

APPENDIX 1

To prove $0 \leqslant C(X_f, \underline{X}) \leqslant 1$ (using global information).

Proof. Let s be the crossover point of the membership function $\mu_1(i)$ such that $\mu_1(i) \leqslant 0.5$ if $i \leqslant s$ and $\mu_1(i) > 0.5$ if $i > s$. Let

$$\mu_2(i) = \begin{cases} 0 & \text{if } \mu_1(i) \leqslant 0.5, \\ 1 & \text{if } \mu_1(i) > 0.5. \end{cases}$$

Then in the range [1, s], $\mu_1(i) \leqslant 0.5$ and hence $[\mu_1(i)]^2 \leqslant 0.25$, and in the range [$s+1$, L], $1-\mu_1(i) \leqslant 0.5$ and hence $[1-\mu_1(i)]^2 \leqslant 0.25$.

We have

$$T = 4\left(\sum_{i=1}^{s}\{[\mu_1(i)]^2[h(i)]^2\} + \sum_{i=s+1}^{L}\{[1-\mu_1(i)]^2[h(i)]^2\}\right)$$

$$\leqslant 4\left(\sum_{i=1}^{s}\{0.25[h(i)]^2\} + \sum_{i=s+1}^{L}\{0.25[h(i)]^2\}\right)$$

$$= \sum_{i=1}^{s}[h(i)]^2 + \sum_{i=s+1}^{L}[h(i)]^2$$

$$= \sum_{i}^{L}[h(i)]^2 = X_2 \qquad [\text{from Equation (8a)}]$$

Thus, $T \leq X_2$,

or

$$T/X_2 \leqslant 1.$$

Now $X_1 = \sum_{i=1}^{L}[2\mu_1(i)-1]^2[h(i)]^2$, which is a positive quantity. Hence

$$\frac{T}{X_1 + X_2} \leqslant 1,$$

and thus from Equation (8a) we get $C(X_f, \underline{X}) \geqslant 1 - 1 = 0$.

From the above it is evident that $T \geqslant 0$, i.e., $T/(X_1 + X_2) \geqslant 0$, so that

$$C(X_f, \underline{X}) \leqslant 1 - 0 = 1.$$

Hence

$$0 \leqslant C(X_f, \underline{X}) \leqslant 1.$$

APPENDIX 2

To prove $0 \leqslant C(X_f, \underline{X}) \leqslant 1$ (using local information).

Proof. Let μ_1, μ_2, and s be chosen as in Appendix 1.

For the entries of the quadrants c_1, c_2, and c_4, we have $\mu_1(i, j) \leqslant 0.5$ and hence $[\mu_1(i, j)]^2 \leqslant 0.25$, and for the entries of the quadrant c_3, $1 - \mu_1(i, j) \leqslant 0.5$ and hence $[1 - \mu_1(i, j)]^2 \leqslant 0.25$. Now

$$T = 4\left(\sum_{i=1}^{L}\sum_{j=1}^{s}\{\mu_1(i,j)C(i,j)\}^2 + \sum_{i=1}^{s}\sum_{j=s+1}^{L}\{\mu_1(i,j)C(i,j)\}^2 + \sum_{i=s+1}^{L}\sum_{j=s+1}^{L}\{[1-\mu_1(i,j)]C(i,j)\}^2\right)$$

$$\leqslant 4\left(\sum_{i=1}^{L}\sum_{j=1}^{s}\{0.5C(i,j)\}^2 + \sum_{i=1}^{s}\sum_{j=s+1}^{L}\{0.5C(i,j)\}^2 + \sum_{i=s+1}^{L}\sum_{j=s+1}^{L}\{0.5C(i,j)\}^2\right)$$

$$= \sum_{i=1}^{L}\sum_{j=1}^{L}\{C(i,j)\}^2 = X_2 \qquad [\text{from Equation (11)}].$$

Therefore $T \leqslant X_2$ and hence $T/X_2 \leqslant 1$.

Now $X_1 = \sum_{i=1}^{L}\sum_{j=1}^{L}\{[2\mu_1(i, j) - 1]C(i, j)\}^2$ is a positive quantity. Hence

$$\frac{T}{X_1 + X_2} \leqslant 1$$

So from Equation (11) we get $C(X_f, \underline{X}) \geqslant 1 - 1 = 0$. From the above it is evident that $T \geqslant 0$ and hence $T/(X_1 + X_2) \geqslant 0$. Then from Equation (11) we get $C(X_f, \underline{X}) \leqslant 1 - 0 = 1$. Hence

$$0 \leqslant C(X_f, \underline{X}) \leqslant 1.$$

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