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Bounds for Membership Functions: A Correlation-Based Approach

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ABSTRACT

The paper describes a mathematical formulation for defining bounds for S-type membership functions using correlation [1]. These bounds impose restrictions on variation in the values of membership functions. The significance of these bounds in the image processing/analysis problems has been extensively studied on various images. It has been observed that membership functions satisfying the bounds can provide desirable results. It is also found that Zadeh's standard S-function [2] satisfies these bounds.

1. INTRODUCTION

Zadeh [2, 3] introduced the concept of S- and π-type membership functions. He stated that more or less any fuzzy subset of the real line can be represented by S- and π-type functions and their complements. It was also noted that the representation of the functions is semantic in nature. For instance, let us consider the example of a fuzzy set "tall." This is represented by an S function that is a nondecreasing function of height. Now, the question is, "Can any such nondecreasing function be taken to represent this fuzzy set?". Intuitively the answer is "no." An attempt is made in this paper to provide a quantitative answer to this problem using the measure "correlation" [1] between membership functions.

Correlation (c_{f_1, f_2}) between two fuzzy membership functions f_1 and f_2

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has been defined by Murthy et al. [1]. The basis for the definition is the following properties P_1, P_2, and P_3.

P_1: If for higher values of f_1, f_2 takes higher values and for lower values of f_1, f_2 also takes lower values then $c_{f_1,f_2} > 0$.
P_2: If $f_1 \uparrow$ and $f_2 \uparrow$ then $c_{f_1,f_2} > 0$.
P_3: If $f_1 \uparrow$ and $f_2 \downarrow$ then $c_{f_1,f_2} < 0$.

($\uparrow$ denotes increases and $\downarrow$ denotes decreases.)

In this paper the properties P_4, P_2, and P_3 are studied in depth. It has been stated conclusively that P_2 and P_3 should not be considered in isolation of P_1. Had this been considered, one can cite several examples where $f_1 \uparrow$ and $f_2 \uparrow$ but $c_{f_1,f_2} < 0$, and $f_1 \uparrow$ and $f_2 \downarrow$ but $c_{f_1,f_2} > 0$. It has been found from those examples that variation in membership function values is related to correlation. It has also been noted that some restrictions need to be imposed on the membership function for representing a fuzzy set like "tall." Subsequently, bound functions are derived.

The definition of correlation is given in Section 2. Section 3 deals with some features of correlation in the context of P_1. Variation in membership function values along with the interpretation of restrictions is discussed in Section 4. Section 5 constitutes the derivation of bound functions. Significance of the proposed bound functions in the image segmentation problem is extensively discussed in Section 6.

2. CORRELATION BETWEEN MEMBERSHIP FUNCTIONS

Correlation $[C_{f_1,f_2}]$ between membership functions $[f_1, f_2]$ has been defined by Murthy et al. [1]. The definitions of domain and correlation are given here for membership functions f_1 and f_2 defined on R.

DEFINITION 1. Let S be a closed interval in R. Let the membership functions f_1 and f_2 be such that

(a) f_i: $S \to [0, 1]$ is continuous and onto for $i = 1, 2$ and
(b) $\forall x \in S^c$, $f_i(x) = 0$ or 1 or undefined for $i = 1, 2$.

It is clear from these assumptions on f_1, f_2, and S that the domain is not unique. To get a unique domain, let

$$\beta = \big(S: S \text{ satisfies properties (a) and (b)}\big) \text{ and}$$

$$\Omega = \bigcap_{S \in \beta} S.$$

Such an Ω, as defined here, is unique.

Let

$$X_1 = \int_\Omega (2f_1 - 1)^2 \quad \text{and} \quad X_2 = \int_\Omega (2f_2 - 1)^2.$$

Then

$$c_{f_1, f_2} = 1 - \frac{4}{X_1 + X_2} \int_\Omega (f_1 - f_2)^2. \qquad \blacksquare$$

In the next section, various types of membership functions are considered and correlations are calculated in the context of P_1.

3. SALIENT FEATURES OF CORRELATION IN THE CONTEXT OF P_1

Some examples along with a few results are stated in this section. The examples bring out the salient features of correlation in relation to P_1.

PROPOSITION 3.1. *Let f and g be two membership functions defined on Ω. Then $C_{f,g} \geqslant 0 \Leftrightarrow \int_\Omega (f + g - 1)^2 \geqslant \int_\Omega (f - g)^2$.*

Proof.

$$C_{f,g} = 1 - \frac{4}{X_1 + X_2} \int_\Omega (f - g)^2$$

where

$$X_1 = \int_\Omega (2f - 1)^2 \quad \text{and} \quad X_2 = \int_\Omega (2g - 1)^2.$$

After a few calculations, it can be shown that

$$X_1 + X_2 = 2\left[\int_\Omega (f + g - 1)^2 + \int_\Omega (f - g)^2\right].$$

So

$$C_{f,g} \geqslant 0 \Leftrightarrow 1 - \frac{2}{\int_\Omega (f + g - 1)^2 + \int_\Omega (f - g)^2} \int_\Omega (f - g)^2 \geqslant 0$$

$$\Leftrightarrow \int_\Omega (f + g - 1)^2 \geqslant \int_\Omega (f - g)^2. \qquad \blacksquare$$

PROPOSITION 3.2. $C_{f,g} \geqslant 0 \Leftrightarrow \int_\Omega (2f - 1)(2g - 1) \geqslant 0$.

Proof. It is trivial from Proposition 3.1.

Note 3.1. From Proposition 3.1 it is clear that the difference between f and $1-g$ (complement of g) should be greater than the difference between f and g to make $C_{f,g} > 0$. This result is consistent with P_1. In Example 3.1, it is demonstrated that consideration of the property P_2 alone will not satisfy Proposition 3.1.

EXAMPLE 3.1. Let $f_k(x) = x^k$ and $g_k(x) = x^{1/k}$ for $x \in [0, 1]$ and $k \geqslant 1$. Observe that both f_k and g_k are increasing functions. As $k \to \infty$, $f_k(x) \to f(x)$ and $g_k(x) \to g(x)$ where

$$\begin{aligned} f(x) &= 0 \quad \text{for } x \in [0, 1), \\ &= 1 \quad \text{at } x = 1 \end{aligned}$$

and

$$\begin{aligned} g(x) &= 0 \quad \text{at } x = 0, \\ &= 1 \quad \text{for } x \in (0, 1], \end{aligned}$$

i.e., $f(x) = 1 - g(x)$ $\forall x \in (0, 1)$. So $C_{f,g} = -1$. But, according to P_2, $C_{f_k, g_k} \geqslant 0$ for $k \geqslant 1$. Now,

$$\begin{aligned} C_{f_k, g_k} \geqslant 0 &\Leftrightarrow \int_\Omega (2f_k - 1)(2g_k - 1) \geqslant 0 \qquad \text{[Proposition 3.2]} \\ &\Leftrightarrow \int_0^1 (2x^k - 1)(2x^{1/k} - 1)\, dx \geqslant 0. \end{aligned}$$

After calculations it is found that $C_{f_k, g_k} \geqslant 0 \Leftrightarrow k^2 - 3k + 1 \leqslant 0$. But $k^2 - 3k + 1 \geqslant 0$ for $k \geqslant (3 + \sqrt{5})/2$. So for all $k \geqslant 3$, $C_{f_k, g_k} < 0$.

The plot of $f_3(x)$ and $g_3(x)$ is shown in Figure 1. Observe that the property P_1 is not satisfied by f_3 and g_3 (Figure 1). For lower values of f_3, g_3 takes higher values. The correlation between f_3 and g_3 is around -0.248. As the value of k increases the correlation tends to (-1). The basic reason for negative correlation is that P_1 is not satisfied although f_k and g_k are increasing functions in x for every k. In other words, some conditions need to be imposed on membership functions to make them satisfy the property P_2.

Note 3.2. Observe that as $k \to \infty$, the variation in the values of $f_k(x)$ and $g_k(x)$ (Example 3.1) becomes negligible. In Figure 2, the functions x^6

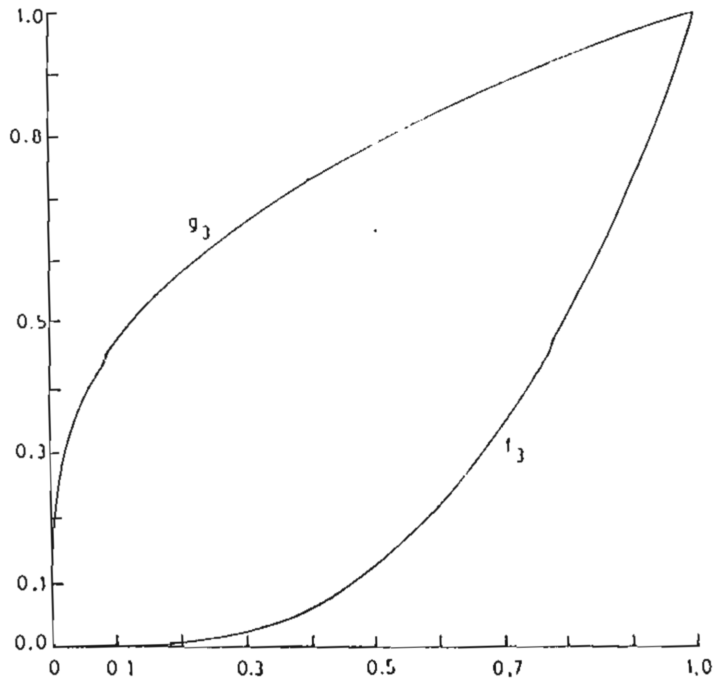


Fig. 1. Membership functions f_3 and g_3 of Example 3.1.

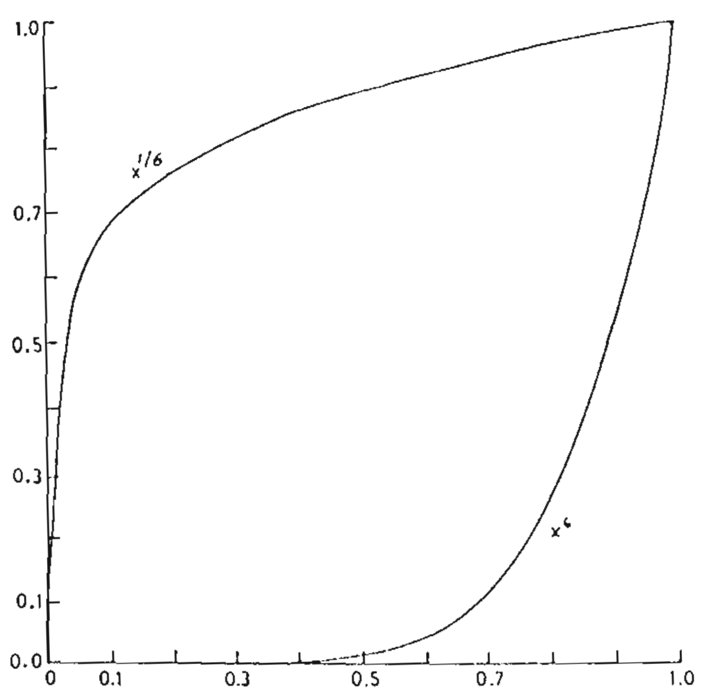


Fig. 2. Membership functions x^6 and $x^{1/6}$.

and $x^{1/6}$ are plotted. In the interval $[0, 0.6]$, x^6 increases from 0 to 0.1, but it increases from 0.1 to 1 in the interval $[0.6, 1]$. Similarly $x^{1/6}$ increases from 0 to 0.68 in the interval $[0, 0.1]$ and the variation in the rest is only 0.32. That is, most of the variation is concentrated in a small interval of $[0, 1]$ for both the functions. This point will be further elaborated in Section 4.

Another example of two such functions where one of them is increasing and the other decreasing but the correlation is positive is given next. Here also it is seen that a few constraints on membership functions are necessary to make the correlation negative.

EXAMPLE 3.2. Let $f_k(x) = (1-x)^k$ and $g_k(x) = x^k$ for $x \in [0, 1]$ and $k \geqslant 1$. $g_k(x)$ increases with x, whereas $f_k(x)$ decreases with x. So according to P_3, C_{f_k, g_k} should be negative for all k. But $f_k(x) \to f(x)$ as $k \to \infty$ where

$$f(x) = 1 \quad \text{at } x = 0$$
$$= 0 \quad \text{for } x \in (0, 1]$$

and $g_k(x) \to g(x)$ as $k \to \infty$ where

$$g(x) = 0 \quad \text{for } x \in [0, 1)$$
$$= 1 \quad \text{at } x = 1.$$

So $C_{f, g} = 1$.

$$\text{Now } C_{f_k, g_k} \geqslant 0 \Leftrightarrow \int_0^1 (2f_k - 1)(2g_k - 1) \geqslant 0. \qquad [\text{Proposition 3.2}]$$

After some calculations it can be seen that

$$C_{f_k, g_k} \geqslant 0 \Leftrightarrow 1 - \frac{4}{k+1} + \frac{4\Gamma^2(k+1)}{\Gamma(2k+2)} \geqslant 0$$

$$\left[\Gamma(m) = \int_0^\infty e^{-x} x^{m-1}\, dx; \qquad m > 0\right]$$

$$\Leftrightarrow 1 - \frac{4}{k+1} + \frac{4k!k!}{(2k+1)!} \geqslant 0 \quad \text{for } k \text{ being an integer.}$$

Observe that $k!k!/(2k+1)! \to 0$ as $k \to \infty$. So $C_{f_k, g_k} \to 1$ as $k \to \infty$. Note that C_{f_k, g_k} is positive for $k \geq 3$. But according to P_3, C_{f_k, g_k} should be less than zero for every $k \geq 0$.

The plot of $f_3(x)$ and $g_3(x)$ is shown in Figure 3. It is found that $C_{f_3, g_3} = 0.05$, which is positive though small. It was just shown that $C_{f_k, g_k} \to 1$ as $k \to \infty$. This behavior is because f_k and g_k become close to each other as $k \to \infty$. That is, for lower values of f_k, g_k also takes lower values as $k \to \infty$.

Note 3.3. Observe that Note 3.2 is valid in Example 3.2 also. Most of the variation in f_k and g_k of Example 3.2 is again seen to be concentrated (as in Example 3.1) in a small part of the interval [0, 1]. Therefore, some constraints need to be imposed on membership functions so as to avoid this phenomenon.

In the next section the implications of the restrictions are discussed.

4. VARIATION IN MEMBERSHIP FUNCTION AND INTERPRETATION OF RESTRICTION

Before discussing the variation in membership function values a note is given about the definition of domain Ω.

Note 4.1. Ω has been defined as the intersection of some intervals in R [Definition 1]. Observe that different Ω's can be obtained for the same

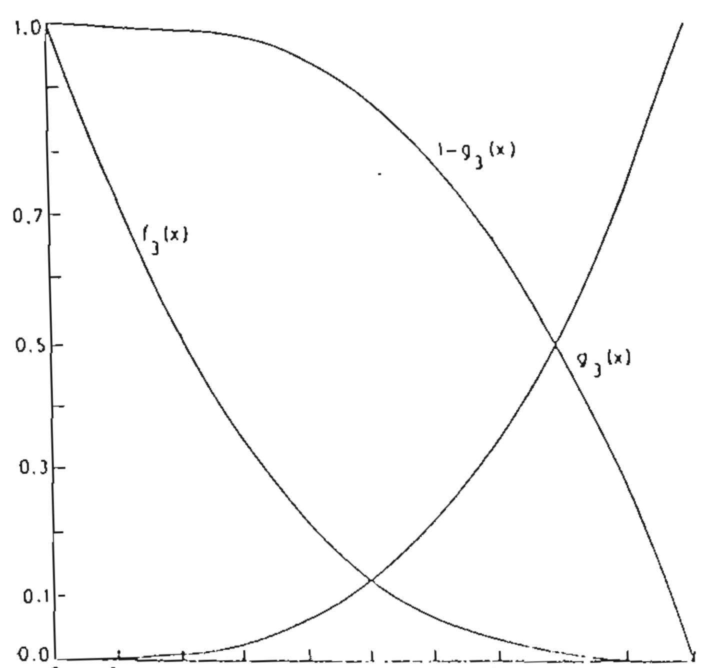


Fig. 3. Membership functions f_3 and g_3 of Example 3.2.

fuzzy set. For instance, there are various domains like [120 cm, 210 cm], [90 cm, 240 cm], etc. for "tall." To standardize it, intersection of some intervals is taken to be the domain. In this section, variation in membership function values is discussed with respect to this domain.

To discuss variation in membership functions values, let us consider, for example, the fuzzy set "tall" with the interval [120 cm, 210 cm] as domain. Five types of membership functional forms are shown in Figure 4 for this domain. The function forms II–V, shown in Figure 4, do not seem to be appealing because they do not reflect the intuition behind "tallness." The functional form I, on the other hand, is acceptable.

Most of the variation in the values of functions II–V is seen to be concentrated in a small subset of [120, 210]. The functions II–V result in an abrupt change from nonmembership to membership and thus make the fuzzy set crisp. The form IV can also be viewed as a highly contrast-intensified version attempting to make the function I crisp. Function V is meaningless in the sense that it attempts to give, more or less, the same value to most of the points in (0, 1).

It is to be noted that the same conclusion can be drawn for any other Ω of "tall."

Similar is the case with other fuzzy sets like "old," "brightness," "large," etc., which are characterized by S functions. In all these examples, most of the

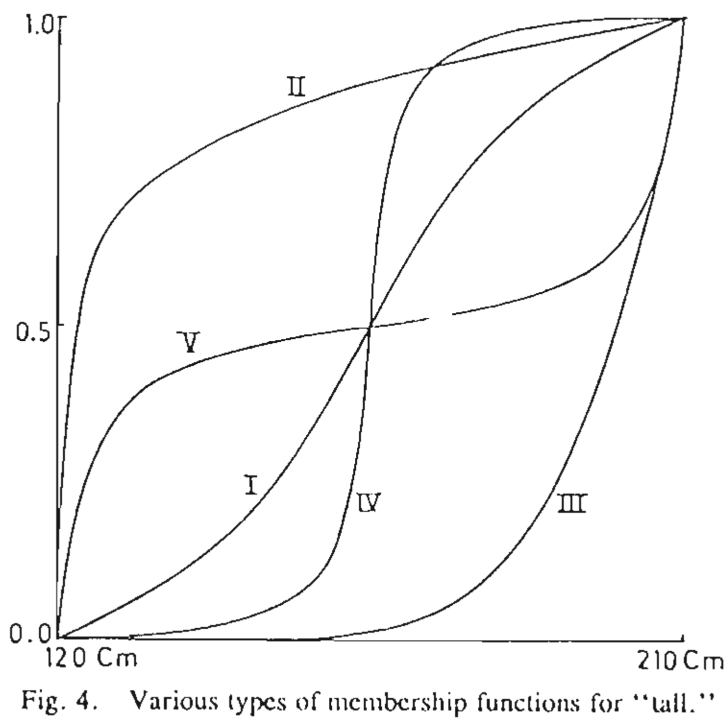


Fig. 4. Various types of membership functions for "tall."

variation in the membership functions is concentrated in a large subinterval of the respective domains.

In Section 3, some relation between properties of correlation and variation in the values of membership functions was discovered. It was concluded that some restrictions need to be imposed on the membership functions for restricting the variation in their values so that property P_2 (or property P_3) can be satisfied.

The implication of restricting the variation in membership functions is to essentially make the function similar to those of the previously mentioned fuzzy sets "tall," "old," etc. Since the correlation measure seems to be related to the variation in the functional values, it may therefore be used as a guiding tool for imposing restrictions.

Note that the restrictions will be dependent on the domain. Note also that some S and $(1-S)$ functional forms as shown in Figures 5–12 would automatically be sidetracked in this process.

An intuitive way of imposing restrictions on the variation in a membership function h is to define two functions f and g as shown in Figure 13 so that h satisfies the property $f \leqslant h \leqslant g$. f and g may be termed as *bound functions*, or simply *bounds*, for h. Observe that these restrictions hold for any nondecreasing function and in particular for S functions. The process of deriving these restrictions is stated in the next section.

5. BOUNDS FOR S-TYPE MEMBERSHIP FUNCTIONS

In this section, bounds for S-type functions are obtained using correlation. A few theorems are proved in this context.

PROPOSITION 5.1. *Let* $\Omega = [0, 1]$ *and* $0 < \epsilon \leqslant 0.5$. *Let*

$$f_\epsilon(x) = 0; \qquad 0 \leqslant x \leqslant \epsilon$$
$$= x - \epsilon; \; \epsilon \leqslant x \leqslant 1$$

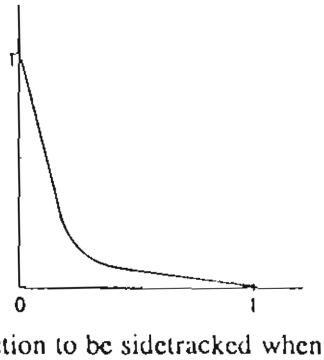


Fig. 5. Membership function to be sidetracked when restrictions are imposed.

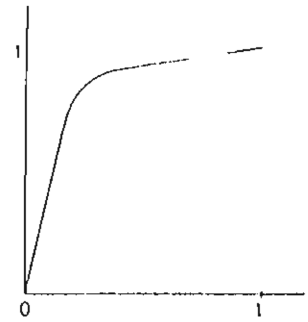


Fig. 6. Membership function to be sidetracked when restrictions are imposed.

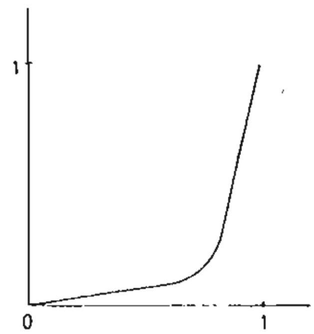


Fig. 7. Membership function to be sidetracked when restrictions are imposed.

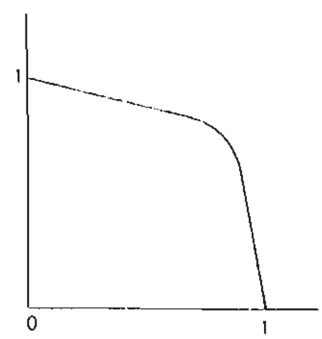


Fig. 8. Membership function to be sidetracked when restrictions are imposed.

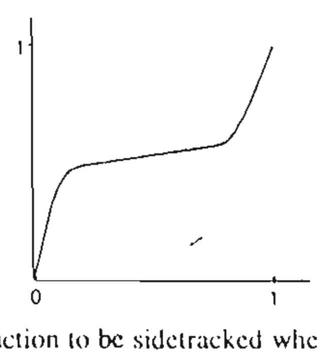


Fig. 9. Membership function to be sidetracked when restrictions are imposed.

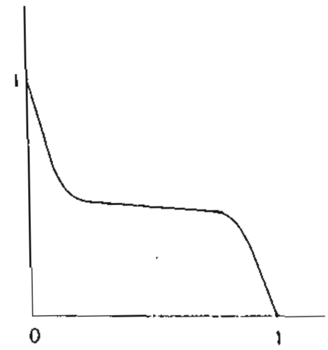


Fig. 10. Membership function to be sidetracked when restrictions are imposed.

and

$$g_\epsilon(x) = x + \epsilon; \qquad 0 \leqslant x \leqslant 1 - \epsilon$$

$$= 1; \qquad 1 - \epsilon \leqslant x \leqslant 1.$$

Then $C_{f_\epsilon}, g_\epsilon \geqslant 0$ *for* $\epsilon \leqslant 0.27$.

Proof.

$$C_{f_\epsilon}, g_\epsilon \geqslant 0 \Leftrightarrow I_\epsilon = \int_\Omega (2f_\epsilon - 1)(2g_\epsilon - 1) \geqslant 0$$

$$\Leftrightarrow \int_0^\epsilon [-(2x + 2\epsilon - 1)]\, dx + \int_\epsilon^{1-\epsilon} (2x - 2\epsilon - 1)(2x + 2\epsilon - 1)\, dx$$

$$+ \int_{1-\epsilon}^1 (2x - 2\epsilon - 1)\, dx \geqslant 0.$$

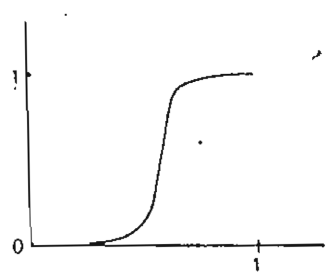


Fig. 11. Membership function to be sidetracked when restrictions are imposed.

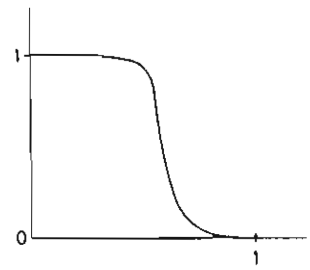


Fig. 12. Membership function to be sidetracked when restrictions are imposed.

After calculations, it can be seen that

$$I_\epsilon \geqslant 0 \Leftrightarrow 16\epsilon^3 - 18\epsilon^2 + 1 \geqslant 0.$$

It can also be seen that

$$I_\epsilon < 0 \quad \text{if } \epsilon \geqslant 0.271$$

and

$$I_\epsilon > 0 \quad \text{if } \epsilon \leqslant 0.27$$

Hence the proposition. ■

THEOREM 1. *Let f_ϵ and g_ϵ be as defined in Proposition* 5.1. *Let f_1 be a membership function defined on* [0.1] *such that $f_\epsilon(x) \leqslant f_1(x) \leqslant g_\epsilon(x)$ for $x \in [0, 1]$. Then $C_{g_\epsilon, f_\epsilon} \geqslant 0$ if $\epsilon \leqslant 0.25$.*

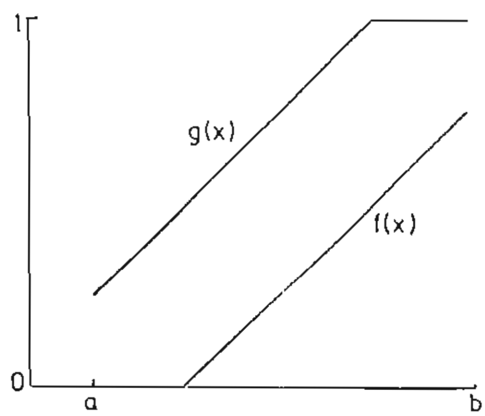


Fig. 13. Possible bound functions.

Proof. It suffices to show that

$$J_\epsilon = \int_0^1 (2g_\epsilon - 1)(2f_1 - 1)\, dx \geqslant 0 \quad \text{for } \epsilon \leqslant 0.25$$

Now

$$J_\epsilon = \int_0^{0.5-\epsilon} (2g_\epsilon - 1)(2f_1 - 1)\, dx + \int_{0.5-\epsilon}^{0.5+\epsilon} (2g_\epsilon - 1)(2f_1 - 1)\, dx$$

$$+ \int_{0.5+\epsilon}^{1} (2g_\epsilon - 1)(2f_1 - 1)\, dx$$

$$\geqslant \int_0^{0.5-\epsilon} (2g_\epsilon - 1)(2f_1 - 1)\, dx + \int_{0.5-\epsilon}^{0.5+\epsilon} (2g_\epsilon - 1)(2f_\epsilon - 1)\, dx$$

$$+ \int_{0.5+\epsilon}^{1} (2g_\epsilon - 1)(2f_\epsilon - 1)\, dx$$

$$\geqslant I_\epsilon = \int_0^{0.5-\epsilon} (2g_\epsilon - 1)^2\, dx + \int_{0.5-\epsilon}^{0.5+\epsilon} (2g_\epsilon - 1)(2f_\epsilon - 1)\, dx$$

$$+ \int_{0.5+\epsilon}^{1} (2g_\epsilon - 1)(2f_\epsilon - 1)\, dx.$$

Let $\epsilon \leqslant 0.25$. After the evaluation of the previously given integrals, it can be seen that

$$I_\epsilon \geqslant 0 \Leftrightarrow -4\epsilon^3 - 3\epsilon^2 - 3\epsilon + 1 \geqslant 0.$$

Observe that $1 - 3\epsilon - 3\epsilon^2 - 4\epsilon^3 = 0$ at $\epsilon = 0.25$ and $\geqslant 0$ for $\epsilon \leqslant 0.25$

Hence the theorem. ■

Similarly the following theorem can be shown.

THEOREM 2. *Let* f_1, f_ϵ, *and* g_ϵ *be as defined in Theorem* 1. *Then* $C_{f_1, f_\epsilon} \geqslant 0$ *for* $\epsilon \leqslant 0.25$.

Proof.

$$\int_0^1 (2f_1-1)(2f_\epsilon-1) = \int_0^{0.5-\epsilon} (2f_1-1)(2f_\epsilon-1)\,dx$$
$$+\int_{0.5-\epsilon}^{0.5+\epsilon} (2f_1-1)(2f_\epsilon-1)\,dx$$
$$+\int_{0.5+\epsilon}^{1} (2f_1-1)(2f_\epsilon-1)\,dx$$
$$\geqslant I = \int_0^{0.5-\epsilon} (2g_\epsilon-1)(2f_\epsilon-1)\,dx$$
$$+\int_{0.5-\epsilon}^{0.5+\epsilon} (2g\epsilon-1)(2f_\epsilon-1)$$
$$+\int_{0.5+\epsilon}^{1} (2f_\epsilon-1)^2.$$

Let $\epsilon \leqslant 0.25$. Then

$$I \geqslant I_1 = -\int_0^{\epsilon} (2x+2\epsilon-1)\,dx + \int_{\epsilon}^{0.5+\epsilon} (2g_\epsilon-1)(2f_\epsilon-1)\,dx$$
$$+\int_{0.5+\epsilon}^{1} (2f_\epsilon-1)^2\,dx.$$

After evaluating the integrals in I_1, it can be seen that

$$I_1 \geqslant 0 \Leftrightarrow -4\epsilon^3-3\epsilon^2-3\epsilon+1 \geqslant 0.$$

Observe that $-4\epsilon^3-3\epsilon^2-3\epsilon+1=0$ for $\epsilon=0.25$ and $-4\epsilon^3-3\epsilon^2-3\epsilon+1 \geqslant 0$ for $\epsilon \leqslant 0.25$.

Hence the theorem. ■

Note 5.1. Observe that $-4\epsilon^3-3\epsilon^2-3\epsilon+1$ has appeared in Theorem 1 as well as in Theorem 2. It reflects the symmetric nature of f_ϵ and g_ϵ in comparison to f_1. Observe also that only when $\epsilon \leqslant 0.25$, the proofs of

Theorems 1 and 2 are valid. Similar steps can be observed in the following theorem.

THEOREM 3. *Let* $0 \leqslant \epsilon \leqslant 0.5$. *Let* $f_\epsilon(x)$ *and* $g_\epsilon(x)$ *be as defined in Proposition* 5.1. *Let* f_1 *and* f_2 *be two membership functions such that* $f_\epsilon \leqslant f_1, f_2 \leqslant g_\epsilon$. *Then*

$$C_{f_1, f_2} \geqslant 0 \quad \text{for } \epsilon \leqslant 0.221.$$

Proof.

$$C_{f_1, f_2} \geqslant 0 \Leftrightarrow I = \int_0^1 (2f_1 - 1)(2f_2 - 1)\, dx \geqslant 0.$$

Let $\epsilon \leqslant 0.25$. Then

$$I = \int_0^{0.5-\epsilon} (2f_1 - 1)(2f_2 - 1)\, dx + \int_{0.5-\epsilon}^{0.5+\epsilon} (2f_1 - 1)(2f_2 - 1)\, dx$$

$$+ \int_{0.5+\epsilon}^{1} (2f_1 - 1)(2f_2 - 1)\, dx$$

$$\geqslant I_1 = \int_0^{0.5-\epsilon} (2g_\epsilon - 1)^2\, dx + \int_{0.5-\epsilon}^{0.5+\epsilon} (2g_\epsilon - 1)(2f_\epsilon - 1)\, dx$$

$$+ \int_{0.5+\epsilon}^{1} (2f_\epsilon - 1)^2\, dx.$$

After the calculation of integrals in I_1, it can be seen that

$$I_1 \geqslant 0 \Leftrightarrow 1 - 6\epsilon + 12\epsilon^2 - 24\epsilon^3 \geqslant 0.$$

Observe that $1 - 6\epsilon + 12\epsilon^2 - 24\epsilon^3 \geqslant 0$ for $\epsilon \leqslant 0.221$ and < 0 for $\epsilon > 0.222$.
Hence the theorem. ■

5.1 INTERPRETATION:

(i) In Proposition 5.1, two functions f_ϵ and g_ϵ are defined. Observe that they do not fall into the category of functions shown in Figures 5-12. Variations in f_ϵ and g_ϵ over the domain [0, 1] are uniform excepting towards the endpoints of [0, 1]. f_ϵ and g_ϵ are nondecreasing functions.

(ii) According to P_1, correlations between two functions f_1 and f_2 should be greater than zero if for higher values of f_1, f_2 also takes higher values and for lower values of f_1, f_2 also takes takes lower values. Proposition 5.1 reflects the maximum possible difference between f_ϵ and g_ϵ so as to make $C_{f_\epsilon, g_\epsilon} > 0$.

(iii) The importance of Proposition 5.1 is reflected in Theorems 1, 2, and 3. The functions f_1 and f_2 considered in those theorems also do not fall into the category of functions shown in Figures 5–12. The maximum amount of variation in f_1 and f_2 is prescribed by bound functions f_ϵ and g_ϵ.

(iv) Observe that if an S type membership function is such that it is not of the form shown in Figures 5–12 and its variation throughout the domain is significant, then it must lie between the bound functions f_ϵ and g_ϵ. The upper bound for ϵ is given by Theorems 1, 2, and 3.

(v) Observe also that any membership function satisfying the bounds need not possess the shape "S."

Remarks:

(1) Note that the domain of f_1 and f_2 has been considered to be [0, 1] for the derivation of bounds f_ϵ and g_ϵ. For any arbitrary domain $[a, b]$, the corresponding f_ϵ and g_ϵ will accordingly be changed.

(2) Similar bounds can be defined for π-type membership functions since they are combinations of S and $(1 - S)$ functions.

(3) If only one nondecreasing function is to be considered, then ϵ can be taken to be 0.25 [From Theorems 1 and 2].

In the next section, the utility of the bounds in image segmentation problems is dealt with.

6. APPLICATION OF BOUNDS TO IMAGE PROCESSING

The field of image processing and vision is considered here, as an example, for describing the significance of bounds in the selection of membership functions.

In the previous section, bounds (f, g) for membership functions have been defined. It has also been stated that the bounds vary proportionately according to the length of the interval. In the context of image processing, if we are dealing with, say, a gray-level image of 32 levels, the domain would be [0,31]. If we now select an S-type function f_1 that characterizes a fuzzy subset "bright image," then its corresponding bound functions f and g are as follows.

$$f(x) = 0: \qquad 0 \leqslant x \leqslant 31/4$$

$$= \frac{x}{31} - 0.25: \qquad \frac{31}{4} \leqslant x \leqslant 31$$

$$g(x) = (x/31) + 0.25: \qquad 0 \leqslant x \leqslant 31(3/4)$$

$$= 1: \qquad 31(3/4) \leqslant x \leqslant 31.$$

These bounds are shown in Figure 14.

EXAMPLE 6.1. Let us consider Zadeh's standard S-function [2], which is widely used in image processing and computer vision problems [4]. It is defined here for the interval $[a, c]$.

$$f_1(x) = 2[(x-a)/(c-a)]^2: \qquad a \leqslant x \leqslant b$$

$$= 1 - 2[(x-c)/(c-a)]^2: \qquad b \leqslant x \leqslant c$$

where

$$b = (a+c)/2.$$

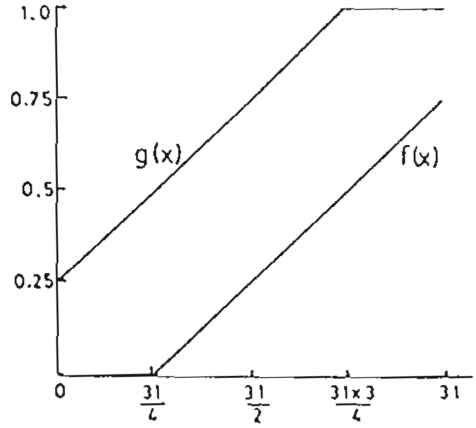


Fig. 14. Bound functions for the fuzzy set "bright image" for 32 gray levels.

By making a translation, the preceding function can be written as

$$f_2(x) = 2(x/2y)^2: \qquad 0 \leqslant x \leqslant y$$

$$= 1 - 2[(2y - x)/2y]^2: \quad y \leqslant x \leqslant 2y$$

where

$$y = b - a.$$

The corresponding bound functions $f(x)$ and $g(x)$ are given next.

$$f(x) = 0: \qquad 0 \leqslant x \leqslant y/2$$

$$= (x/2y) - 0.25: \quad y/2 \leqslant x \leqslant 2y$$

$$g(x) = (x/2y) + 0.25: \quad 0 \leqslant x \leqslant 3y/2$$

$$= 1: \qquad 3y/2 \leqslant x \leqslant 2y.$$

It can be easily shown that $f(x) \leqslant f_2(x) \leqslant g(x)\ \forall x \in [0, 2y]$. Thus Zadeh's standard S function satisfies the proposed bounds.

6.1. *INTERPRETATION OF BOUNDS FOR GRAY LEVEL THRESHOLDING*

Histogram thresholding using fuzzy measures has been well documented in literature [5, 6, 7]. Here we illustrate the procedure in brief and show the significance of the bound functions in this context.

Algorithm. Let X be an image of $L+1$ levels, M rows and N columns. Let μ be a membership function on $[0, c]$ such that

(i) μ is monotonic, continuous,
(ii) $\mu(0) = 0$, $\mu(c) = 1$ and
(iii) $\mu(x_0) = \frac{1}{2}$

where c is the length of the window, $c < L$ and x_0 is the cross-over point.

Let μ_{1q}: $[0, L] \rightarrow [0, 1]$ such that

(i) $\mu_{1q}(a) = 0$: $\quad a \leqslant q$
(ii) $\mu_{1q}(a) = 1$: $\quad a \geqslant q + c$ and
(iii) $\mu_{1q}(a) = \mu(a - q)$: $\quad q \leqslant a \leqslant q + c$

where $q \leqslant L - c$.

So $\mu_{1q}(q + x_0) = \frac{1}{2}\ \forall q \in [0, L - c]$.

Now, for a particular c and for a particular form of μ, μ_{1q} is moved from $q = 0$ to $q = L - c$ and various ambiguity measures, like index of fuzziness [8] and entropy [9], are calculated for each q. The values of the ambiguity measure ($\nu_1(q)$) are plotted against q (Figure 15). Let q_0 be a valley point. Then $q_0 + x_0$ is taken to be a threshold for classification of gray levels (i.e., for histogram thresholding or for image segmentation).

In the preceding algorithm, two decisions are to be made by the user. These are (i) selection of c and (ii) selection of the form of membership function μ. The problem of selection of c has been discussed in literature [5, 7]. In [5], the optimum values for c were determined experimentally. It was reported that if c is greater than the distance between the modes, the corresponding valley point may be lost. The mathematical justification of this finding has been reported recently [7].

In this section, the importance of bound functions is demonstrated in selecting the appropriate membership function μ. For this purpose two images are considered. These are (a) biplane (Figure 16) and (b) Lincoln

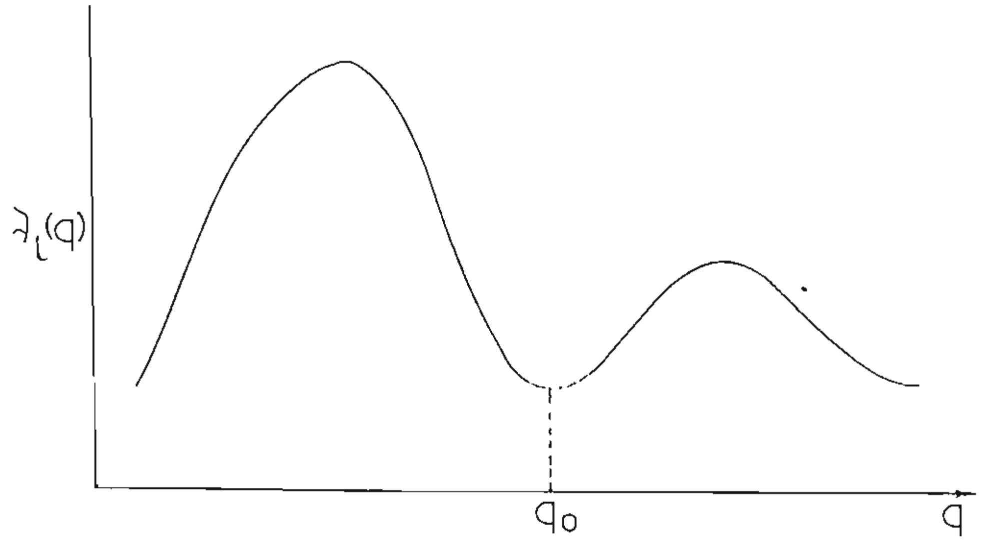
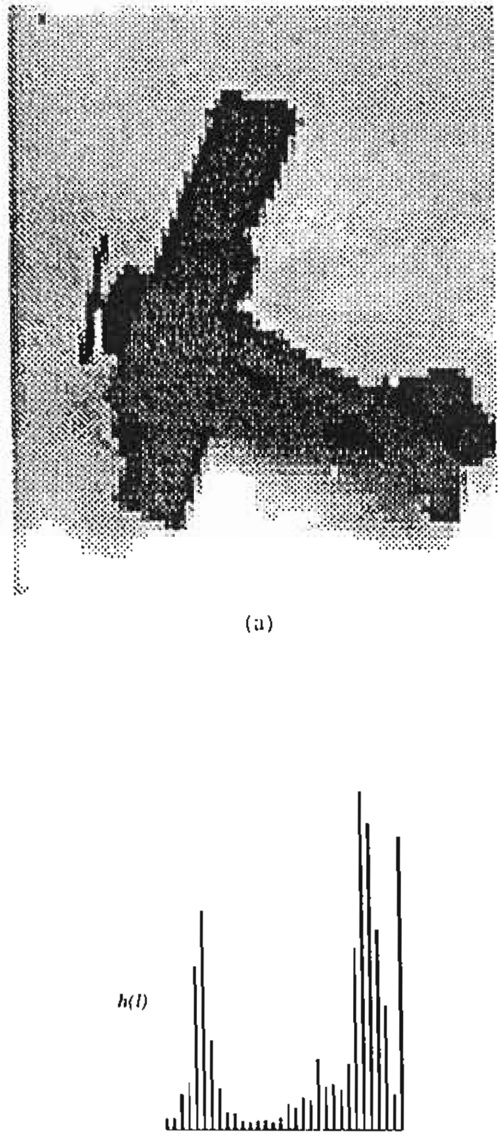


Fig. 15. Plot of ambiguity measure values versus starting points of the window. (Algorithm of Section 6.1.)



(a)

(b)

Fig. 16. "Biplane" image and its histogram.

(Figure 17). Two types of membership functions representing the forms II and III of Figure 4 are considered. These are

$$\text{(a)} \quad h_1(x) = (x/c)^k: \qquad 0 \leqslant x \leqslant c \tag{1}$$

and

$$\text{(b)} \quad h_2(x) = 1 - (1 - (x/c))^k: \quad 0 \leqslant x \leqslant c \tag{2}$$

where $k \geqslant 1$. Two types of measures of ambiguity are considered. These are (a) linear index of fuzziness and (b) entropy.* The selection of c for the preceding images has been guided by the findings in [7]. For the Lincoln image, it appears that there are three main valleys separating four regions. Accordingly, the value of c is taken to be less than or equal to 6 to detect three valleys. In the case of the biplane image, the histogram seems to have a strong valley in the range 10 to 15 and a weak valley around 23. So, to detect the strong valley, the value of c comes to be less than or equal to 14 whereas it is ≈ 5 for the detection of the weaker one also. The results are provided for the values of c lying in [6, 12].

Regarding the choice of membership function, observe that for larger values of k, most of the variation in h_1 and h_2 is not concentrated in the middle portion of the interval $[0, c]$. It has been proved in the appendix that both h_1 and h_2 satisfy the bounds for $k \in [1, 2]$ and do not lie within the bounds for $k \geqslant 3$.

Tables 1–7 show the thresholds detected for the Lincoln and Biplane images using the previously mentioned values of c. It is seen from the tables that the desired thresholds have always been found to be detected when h_1 and h_2 lie between the bound functions. Otherwise, it is very likely that some spurious (undesirable) valleys will also be detected.

Let us now consider the functional form IV of Figure 4. A generalized expression for this functional form is given next.

$$h_k(x) = \left[2^k/2\right](x/c)^k: \qquad 0 \leqslant x \leqslant c/2$$

$$= 1 - \left[2^k/2\right]\left((c-x)/c\right)^k: \quad c/2 \leqslant x \leqslant c \qquad (3)$$

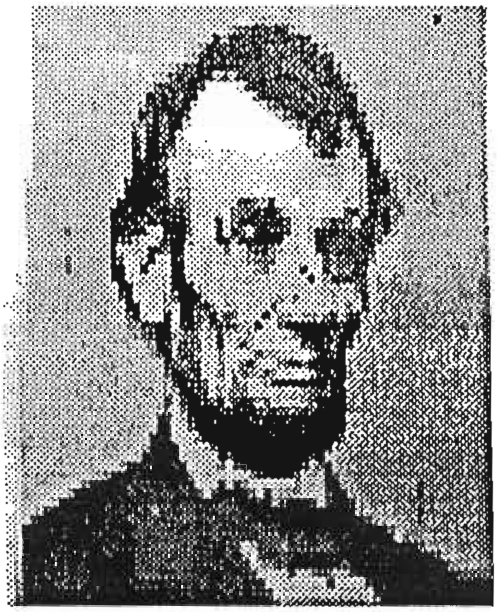
where $k \geqslant 1$.

---

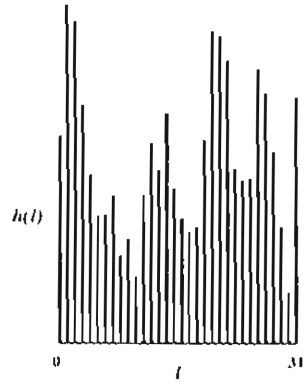
*Let X be a gray tone image with M rows, N columns, and $L+1$ levels. Let μ be a membership function on grey levels. Let x_{mn} represent the grey level at (m, n)th pixel, $m = 1, \ldots M$; $n = 1, \ldots N$. Then linear index of fuzziness

$$\nu(x) = \frac{2}{MN} \sum_m \sum_n \operatorname{Min}\left[\mu(x_{mn}), (1 - \mu(x_{mn}))\right] \quad \text{and} \quad \text{entropy}$$

$$H(x) = \frac{1}{MN \ln 2} \sum_m \sum_n \left[-\mu(x_{mn}) \log\left[(\mu(x_{mn})\right] - (1 - \mu(x_{mn})) \times \log(1 - \mu(x_{mn}))\right].$$



(a)



(b)

Fig. 17. "Lincoln" image and its histogram.

Table 8 shows the threshold values detected for the Lincoln image using Equation (3). It is seen that the number of thresholds exceeded three (the desirable number) when h_k is not within the bounds.

Zadeh's S function has been found (Example 6.1) to satisfy the bound functions. Observe that Equation (3) reduces to Zadeh's function when $k = 2$ and it is able to detect the desirable thresholds. There are several other experiments [4, 5, 6] where this function was found to be successful.

TABLE 1

Thresholds for Lincoln Image Using Equation (1) and Entropy

Window size c	Values for k 1	2	3	4	5	6
4	10,18,25	7,11,19 26	7,11,19	7,11,14, 19,26	7,9,11, 14,18,26	8,10,12,15, 19,27
5	10,17,25	11,19,26	7,11,19, 26	7,11,19, 26	7,11,19, 26	7,11,14, 19,26
6	10,18,25	10,18,25	11,19,26	11,19,26	7,11,19, 26	7,11,19, 26

TABLE 2

Thresholds for Lincoln Image Using Equation (2) and Entropy

Window size c	Values for k 1	2	3	4	5	6
4	10,18,25	9,17,24	6,10,14, 18,24	6,9,11 14,18,25	6,9,11, 14,18,25	5,8,10, 13,14,24
5	10,17,25	9,17,24	9,17,24	6,9,14, 17,24	6,9,11, 14,18,25	6,9,11, 14,18,25
6	10,18,25	10,17,25	9,17,24	9,17,24	6,9,14, 17,24	6,9,11, 14,18,24

TABLE 3

Thresholds for Lincoln Image Using Equation (1) and Index of Fuzziness

Window size c	Values for k 1	2	3	4	5	6
4	11,18,25	7,11,19 26	7,9,11, 14,18,26	7,9,11, 14,18,26	7,9,11, 14,18,26	8,10,12, 15,19,26
5	10,17,25	11,19,26	7,11,19 26	7,11,14, 19,26	7,9,11, 14,18,26	7,9,11, 14,18,26
6	10,18,25	10,18,25	7,11,19, 26	7,11,19, 26	7,11,14, 19,26	7,9,11, 14,18,26

TABLE 4

Thresholds for Lincoln Image Using Equation (2) and Index of Fuzziness

Window size c	Values for k					
	1	2	3	4	5	6
4	11,18,25	6,10,14, 71,24	6,9,11, 14,18,25	6,9,11, 14,18,25	6,9,11, 14,18,25	5,8,10, 13,17,24
5	10,17,25	9,17,24	9,9,11, 14,17,24	6,9,11, 14,18,25	6,9,11, 14,18,25	6,9,11, 14,18,25
6	10,18,25	10,18,25	9,17,24	6,9,11, 14,17,24	6,9,11, 14,18,25	6,9,11, 14,18,25

TABLE 5

Thresholds for Biplane Image Using Equation (1) and Index of Fuzziness

Window size c	Values for k					
	1	2	3	4	5	6
6	13,22	14,23	13,15,24	12,15,24	12,15,24	12,15,18, 20,24
7	13	14,23	15,24	13,15,24	13,15,24	12,15,24
8	13	14,23	14,23	15,24	13,15,24	13,15,24
9	12	14	14,23	15,24	13,15,24	13,15,24
10	13	14	15,23	14,23	15,24	13,15,24
11	13	14	15	14,23	15,24	15,24
12	13	14	15	15,23	14,23	15,24

TABLE 6

Thresholds for Biplane Image Using Equation (1) and Entropy

Window size c	Values for k					
	1	2	3	4	5	6
6	13	14,23	15,24	13,15,24	13,15,24	12,15,24
7	13	14	15,24	15,24	13,15,24	13,15,24
8	13	15	14,23	15,24	15,24	13,15,24
9	13	14	14	15,24	15,24	15,24
10	13	14	15	14,23	15,24	15,24
11	13	15	15	15	15,24	15,24
12	13	14	15	15	15,23	15,24

TABLE 7
Thresholds for Biplane Image Using Equation (2) and Entropy

Window Size	Values for k					
c	1	2	3	4	5	6
6	13	12	12,22	12,22	12,22	12,22
7	13	12	11	12,22	12,22	12,22
8	13	12	12	11	12,22	12,22
9	12	13	12	11	11,22	12,22
10	13	12	12	12	11	11,22
11	13	12	12	12	11	11
12	14	12	11	12	12	11

In the preceding discussion, we have considered several membership functions for representing the functional forms II–IV of Figure 4 and have shown the utility of bounds. The form V of Figure 4, as mentioned earlier, is meaningless because it leads to de-enhancement (i.e., making the image blurred) instead of enhancement [10].

7. DISCUSSION AND CONCLUSIONS

The purpose of this paper is to provide bounds for certain types of S functions and show the utility of these bounds in the context of image processing. A relationship is discovered between correlation and variation in membership function values which leads to the definition of bound functions. It is shown that the widely used Zadeh's function satisfies the proposed bounds.

Note that the domain Ω (as in Definition 1) plays as crucial role in the definition of bounds. For different such Ω's, different bound functions can be obtained.

The significance of the bounds is successfully demonstrated on bimodal and multimodal images. Figure 14 shows the bound functions. The crossover point of any membership function satisfying the bounds belongs to the middle half of the domain. In the image segmentation algorithm, the crossover point of a membership function has been considered to be the threshold for segmenting (classifying) two regions. From Tables 1–8, the thresholds for object background classification are found to be 10 and 11 for the Lincoln image and 12, 13, and 14 for the biplane image. These values lie in the middle half (i.e., in $[31/4, 31*3/4]$) of $[0, 31]$. This further implies that, the membership function that best represents the fuzziness in gray levels of an image usually has values ≈ 0.5 over the ambiguous gray levels (around 16).

TABLE 8

Thresholds for Lincoln Image Using Equation (3) and Entropy

Window size c	Values for k								
	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
4	10,18,25	10,18,25	10,18,25	11,18,25	11,18,25	7,11,18, 25	7,11,14, 18,25	7.11.14. 18.25	7,9.11, 14,18,25
5	10,18.25	10,18,25	10,18,25	10,18,25	11,18,25	11,18,25	11,18,25	7,11.18, 25	7,11,14, 18,25

Although the significance of the bounds is studied in the case of image analysis, they may also be applicable to any field where fuzzy sets are used. Similar bounds for the $1 - S$, π, and $1 - \pi$ functions can be derived because the S function is a primary function that gives rise to the said functions.

APPENDIX

PROPOSITION A1. *Let $c > 0$. Let the lower bound f: $[0, c] \to [0, 1]$ be such that*

$$f(x) = 0: \qquad x \leqslant c/4$$

$$= (x/4) - 1/4: x \geqslant c/4$$

Let the upper bound g: $[0, c] \to [0, 1]$ be such that

$$g(x) = (x/c) + 1/4: \qquad x \leqslant 3c/4$$

$$= 1: \qquad x \geqslant 3c/4.$$

Let h_{1k}: $[0, c] \to [0, 1]$ be such that $h_{1k}(x) = (x/c)^k$ for $k \geqslant 1$. Let h_{2k}: $[0, c] \to [0, 1]$ be such that $h_{2k}(x) = 1 - (1 - (x/c))^k$ for $k \geqslant 1$. Then $f(x) \leqslant h_{1k}(x) \leqslant g(x)$ is not satisfied for $k \geqslant 3$ and $f(x) \leqslant h_{2k}(x) \leqslant g(x)$ is not satisfied for $k \geqslant 3$.

Proof.

$$(x/c)^k \geqslant (x/c) - 1/4$$

$$\Leftrightarrow y^k \geqslant y - 1/4, \qquad (y = x/c \quad \& \quad 1/4 \leqslant y \leqslant 1)$$

$$\Leftrightarrow 4y^k - 4y + 1 \geqslant 0.$$

Let $y = 1/3$. Then

$$4y^k - 4y + 1 = 4/3^k - 1/3 \geqslant 0$$

$$\Leftrightarrow 4 - 3^{k-1} \geqslant 0$$

$$\Leftrightarrow 3^{k-1} \leqslant 4.$$

For $k \geqslant 3$, $3^{k-1} > 4$.

So $f(x) \leqslant h_{1k}(x)$ for $x = c/3$ and $k \geqslant 3$. Similarly it can be shown that

$$h_{2k}(x) > g(x) \quad \text{for } x = 2c/3 \quad \text{and} \quad k \geqslant 3.$$

Hence the proposition.

PROPOSITION A2. *Let f, g, h_{1k} and h_{2k} be as defined in Proposition A1. Then*

$$f(x) \leqslant h_{1k}(x) \leqslant g(x) \quad \text{and}$$
$$f(x) \leqslant h_{2k}(x) \leqslant g(x) \quad \text{for } x \in [0, c] \quad \text{and} \quad k \in [1, 2]$$

Proof.

$$h_{1k}(x) \geqslant 0 \quad \forall x \in [0, c]. \text{ So}$$
$$h_{1k}(x) \geqslant f(x) \quad \forall x [0, c/4]. \text{ Now}$$
$$h_{1k}(x) \geqslant f(x) \quad \text{for } x \in [c/4, c]$$
$$\Leftrightarrow (x/c)^k \geqslant (x/c) - 1/4.$$
$$\Leftrightarrow y^k \geqslant y - 1/4, \ (y = x/c \quad \text{and} \quad 1/4 \leqslant y \leqslant 1)$$
$$\Leftrightarrow 4y^k - 4y + 1 \geqslant 0.$$

Observe that $4y^k \geqslant 4y^2$ for $y \in [1/4, 1]$ and $k \in [1, 2]$. So $4y^k - 4y + 1 \geqslant 4y^2 - 4y + 1 = (2y-1)^2 \geqslant 0$. Hence $f(x) \leqslant h_{1k}(x)$ $\forall x \in [0, c]$.

Now, to show that $g(x) \geqslant h_{1k}(x)$, we shall show that

$$g(x) \geqslant x/c \geqslant h_{1k}(x) \quad \text{for } k \in [1, 2] \quad \text{and} \quad x \in [0, c].$$
$$h_{1k}(x) = (x/c)^k \leqslant x/c \quad \text{for } x \in [0, c] \quad \text{and} \quad k \geqslant 1.$$

Now

$$(x/c) + 1/4 \geqslant x/c, \qquad x \in [0, 3c/4]$$

and

$$1 \geqslant x/c \quad \text{for } x \in [3c/4, c].$$

So $g(x) \geqslant x/c$ $\forall x \in [0, c]$ and $h_{1k}(x) \leqslant g(x)$ $\forall x [0, c]$.

Hence $f(x) \leqslant h_{1k}(x) \leqslant g(x)\ \forall x \in [0, c]$. Similarly, it can be shown that

$$f(x) \leqslant h_{2k}(x) \leqslant g(x) \quad \forall x \in [0, c]$$

Hence the proposition.

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